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A new Numerical Integrator for the Solution of Problems Arising from Some Dynamical Systems

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ABSTRACT

In this paper, we developed a new Numerical Integrator for solving oscillatory and exponential problems that can be represented by Ordinary Differential Equation. This integrator was applied in solving some physical problems including the dynamic and aerospace routing problem. The results obtained showed that the numerical integrator is suitable for the simulation of these problems.

KEYWORDS: Dynamic Aerospace problem, Numerical Integrator, Algorithms, Oscillatory and Exponential problems. Dynamical systems.

Introduction

Differential equations originate from the mathematical formulation of a great variety of problems in Science and Engineering (Shepley ,1984). Such problems give rise to some of the types of ordinary differential equations . In literature, Some of these problems involved Orthogonal and Oblique Trajectories, Mechanics, Frictional Forces, Rate and Population among others. The usual approach to the study and analysis of such system is to solve the resulting equation and interpret the solution in terms of the quantities involved in the original problems. Unfortunately not all of these problems have solutions that can be expressed in explicit form hence we find approximate solution using numerical methods. In this paper we tried to use a new algorithm generated from non-polynomial integrating function to solve some dynamical problems including the dynamic and aerospace routine problems. (Gerald, 1981)

THE DERIVATION OF THE SCHEME

We assume that the theoretical solution $y(x)$ to the initial value problem

$y' = f(x,y),\ y(x_0) = y_0$ can be locally represented in the interval $[x_{n,}\ x_{n+1}]$, $n \geq 0$ by the polynomial interpolating function;

$$F(x) = (\alpha_1 + \alpha_2)e^{-2x} + \alpha_3 x^2 + \alpha_4 x + \alpha_5 \qquad (1)$$

Where $\alpha_1, \alpha_2, \alpha_3 \, and \, \alpha_4$ are real undetermined coefficients, and α_5 is a constant.

(Ayinde, et. al. 2015)

We shall assume y_n is a numerical estimate to the theoretical solution $y(x)$ and $f_n = f(x_n, y_n)$.

We define mesh points as follows: $x_n = a + nh, n = 0, 1, 2, \ldots$ (2)

We can impose the following constraints on the interpolating function (1) in order to get the undetermined coefficients.

a. The interpolating function must coincide with the theoretical solution at $x = x_n \, and \, x = x_{n+1}$. Hence we required that

$$F(x_n) = (\alpha_1 + \alpha_2)e^{-2x_n} + \alpha_3 x_n^2 + \alpha_4 x_n + \alpha_5 \quad (3)$$

$$F(x_{n+1}) = (\alpha_1 + \alpha_2)e^{-2x_{n+1}} + \alpha_3 x_{n+1}^2 + \alpha_4 x_{n+1} + \alpha_5 \quad (4)$$

b. Secondly, the derivatives of the interpolating function are required to coincide with the differential equation as well as its first, second, and third derivatives with respect to $x\ at\ x = x_n$

We denote the ith total derivatives of $f(x,y)$ with respect to x with $f^{(i)}$ such that

$$F^1(x_n) = f_n,\ F^2(x_n) = f_n^1,\ F^3(x_n) = f_n^2 \quad (5)$$

This implies that,

$$f_n = -2(\alpha_1 + \alpha_2)e^{-2x_n} + 2\alpha_3 x_n + \alpha_4 \quad (6)$$

$$f_n{}^1 = 4(\alpha_1 + \alpha_2)e^{-2x_n} + 2\alpha_3 \quad (7)$$

$$f_n{}^2 = -8(\alpha_1 + \alpha_2)e^{-2x_n} \quad (8)$$

Solving for $\alpha_1 + \alpha_2$ from equation (8), we have

$$(\alpha_1 + \alpha_2) = -\frac{1}{8} f_n{}^2 e^{2x_n} \quad (9)$$

Substituting (9) into (7), we have

$$\alpha_3 = \frac{1}{2}\left(f_n{}^1 + \frac{1}{2} f_n^2\right) \quad (10)$$

Substituting (10) and (9) into (6), we have

$$\alpha_4 = \left(f_n - \frac{1}{4} f_n^2\right) - \left(f_n^1 + \frac{1}{2} f_n^2\right)x_n \quad (11)$$

Since $F(x_{n+1}) \equiv y(x_{n+1})$ and $F(x_n) \equiv y(x_n)$

Implies that $y(x_{n+1}) = y_{n+1}$ and $y(x_n) = y_n$ (12)

$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n$ and

Then we shall have

$$y_{n+1} - y_n = (\alpha_1 + \alpha_2)[e^{-2x_{n+1}} - e^{-2x_n}] + \alpha_3[x_{n+1}^2 - x_n^2]$$
$$+\alpha_4[x_{n+1} - x_n] \quad (13)$$

Recall that $x_n = a + nh,\ x_{n+1} = a + (n+1)h$ with $n = 0, 1, 2 \ldots \ldots$

Substitute (9), (10) and (11) into (13), we have

$$y_{n+1} = y_n - \frac{1}{8} f_n^2 (e^{-2h} - 1) + \frac{1}{2}\left(f_n^1 + \frac{1}{2} f_n^2\right)h^2 + \left(f_n - \frac{1}{4} f_n^2\right)h \quad (14)$$

Hence we have a new scheme for the solution of first order ordinary differential equation.

IMPLEMENTATION AND APPLICATION OF THE DERIVED SCHEME

For the implementation of scheme (14), see (Ayinde, et. al. 2015). We shall consider some typical examples of applications of the scheme (14) to the first order differential equations on problems in physics and engineering.

Problem 1: (Dynamics and Aerospace problems). (Amos, 2011)

An Airplane uses a parachute and other means of braking as it shows down on the runway after landing. Its acceleration is given by

$$\boldsymbol{a = -0.0045v^2 - 3} \qquad (15)$$

Consider an airplane with a velocity of 300k*m/h* that opens its parachute and starts decelerating at $t = 0$. By solving the differential equation, determine the velocity as a function of time using numerical integration.

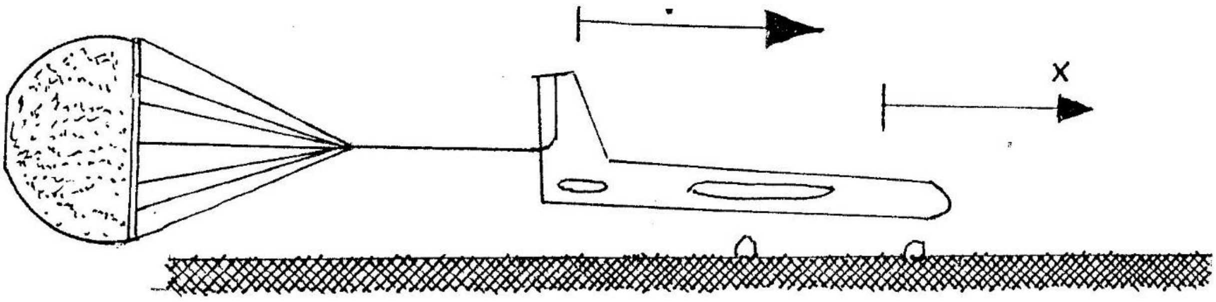

Fig 1

Formulation

Since the acceleration is the rate of change in velocity. We shall calculate the reduction in the speed of the aircraft as its decelerating.

Since $\boldsymbol{a = \frac{dv}{dt}}$,

Then $\boldsymbol{\frac{dv}{dt} = -0.0045v^2 - 3}$, and the analytical solution is

$$\boldsymbol{v(t) = \frac{20}{3} * 15^{\frac{1}{2}} * tan\,(\frac{1}{300}) * (-9 * t + 20 * 15^{\frac{1}{2}}) * atan\,(3 * 15^{\frac{1}{2}}))) * 15 * (1/2)).} \quad (16)$$

The initial condition will be v(0)=300. Then we can now obtain the different velocity in time as the airplane decelerate. The results using the scheme is compared with the analytic solution as shown below.

Table 1: Result from the problem of an airplane braking using parachute

t_n	Scheme(14) Result	Theoretical Result	Error of Deviation
0.000	300.00000000000000	300.0000000000000	0.000000000000000
0.100	263.99841227728026	264.0514381499183	0.053025872635715
1.000	125.93899221271901	126.0495820054564	0.110589792737329
2.000	78.266881760922431	78.37739396979794	0.110512208875534
3.000	55.537920929738412	55.64804898109734	0.110128051358892
4.000	41.945514001769929	42.05545828945492	0.109944287685053
5.000	32.695537431026430	32.80538980795871	0.109852376932281
6.000	25.834654601702653	25.94445671497935	0.109802113276732
7.000	20.414105921491149	20.52387821522128	0.109772293730089
8.000	15.913947704251353	16.02370097527729	0.109753271025877
9.000	12.021673942110606	12.13141420914740	0.109740267036804
10.00	8.5340968756101890	8.643827588705030	0.109730713094841
11.00	5.3088116283568750	5.418534731580277	0.109723103223403
12.00	2.2377468519998570	2.347463304790785	0.109716452790927
12.10	1.9355733376167090	2.045289148280883	0.109715810664174
12.20	1.6339555655536190	1.743670735455116	0.109715169901497
12.30	1.3328107738469600	1.442525303668002	0.109714529821042
12.40	1.0320567149976403	1.141770604739996	0.109713889742353

12.50	0.7316115655988028	0.841324814583711	0.109713248984883
12.60	0.4313938369430682	0.541106443810105	0.109712606866424
12.70	0.1313222864047087	0.241034249106394	0.109711962701607
12.80	-0.168684170620124	- 0.05897285481983	0.109711315800341

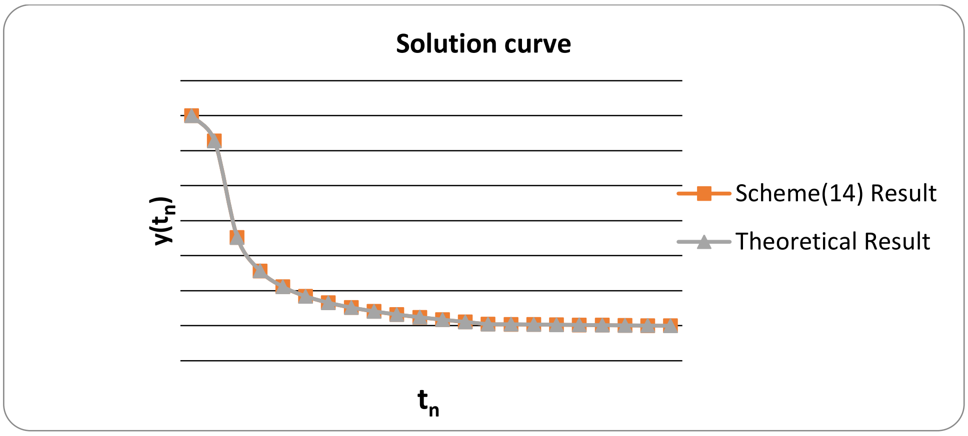

Fig 2: The solution curves of the Dynamic Aerospace problem

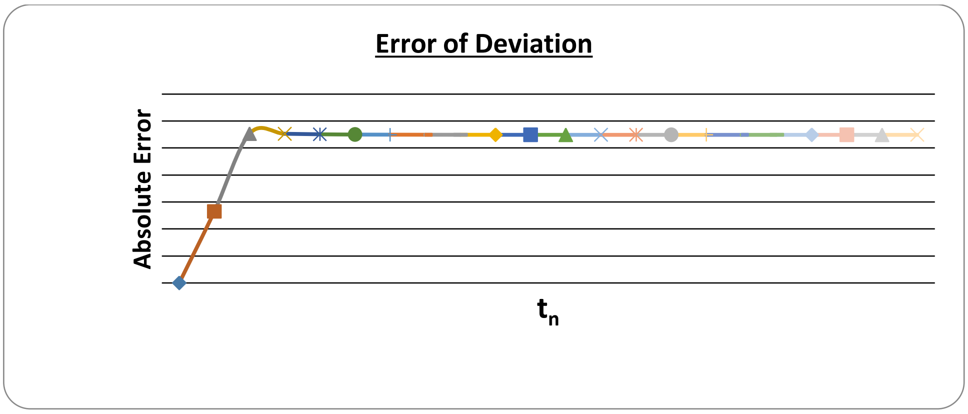

Fig 3: The Absolute Error of deviation of the scheme from the Analytic solution

NOTE: We can conclude from the result above that the velocity of the airplane is 0 between the interval of 12.70s and 12.80s. If we find the average we have 12.75s as the time the airplane finally came to a stop.

Problem 2: Problem of a falling body under air resistance (Shepley, 1984)

A body weighing 8kg falls from rest toward the earth from a great height. As it falls, air resistance act upon it, and we shall assume that this resistance is numerically equal to 2*v*, where *v* is the velocity. (meter per seconds). The problem is to find the velocity and distance fallen at time *t* seconds.

Formulation.

We choose the positive x axis vertically downward along the path of the body B and the origin as the point from which the body fell. The forces acting on the body are:

1. F_1, its weight, 8kg, which acts downward and hence, is positive.

2. F_2, the air resistance, numerically equal to $2v$, which acts upward and is a negative quantity $-2v$.

The figure below shows these forces as indicated.

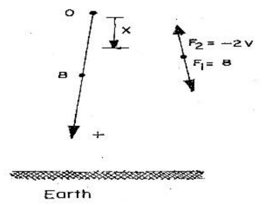

FIG. 1S : A BODY FALLS FROM REST TOWARDS THE EARTH.
SOURCE: Shepley Ross (1984) Problem 3·5 Page 80.

Fig 4

From the Newton's second law, $F = ma$, the force acting on the body can be represented thus

$$m\frac{dv}{dt} = F_1 + F_2 \tag{17}$$

taking $g = 32$ and using $m = {}^{w}/_{g} = {}^{8}/_{32} = {}^{1}/_{4}$,

Hence, $$\frac{1}{4}\frac{dv}{dt} = 8 - 2v \tag{18}$$

Since the body was initially at rest, we have the initial condition $v(0) = 0$

Analytic Solution.

From equation (18), we have $$\frac{dv}{8-2v} = 4dt \tag{19}$$

Integrating and applying the condition $v(0) = 0$ to (19), we have

The velocity at time t given by

$$v = 4(1 - e^{-8t}) \tag{20}$$

The result using scheme (14) and Analytic solution (20) is as shown below :

Table 2 : THE BODY FALLING PROBLEM

t_n	Scheme(14) Result	Theoretical Result	Error of Deviation
0.00	**0.000000000000000**	**0.000000000000000**	**0.00000000000000000000**
0.10	**2.202720091710861**	**2.202684143531114**	**3.594817974672537e-005**
0.20	**3.192466028759494**	**3.192413928021379**	**5.210073811534599e-005**
0.30	**3.637187545392783**	**3.637128186842350**	**5.935855043315286e-005**
0.40	**3.837013803782256**	**3.836951184086535**	**6.261969572030068e-005**
0.50	**3.926801529467819**	**3.926737444445063**	**6.408502275556671e-005**
0.60	**3.967145755240555**	**3.967081011803920**	**6.474343663454008e-005**
0.70	**3.985273584415130**	**3.985208545134068**	**6.503928106127432e-005**
0.80	**3.993418943119835**	**3.993353770907304**	**6.517221253110606e-005**
0.90	**3.997078888708984**	**3.997013656766493**	**6.523194249119513e-005**
1.00	**3.998723408269282**	**3.998658149488390**	**6.525878089203019e-005**

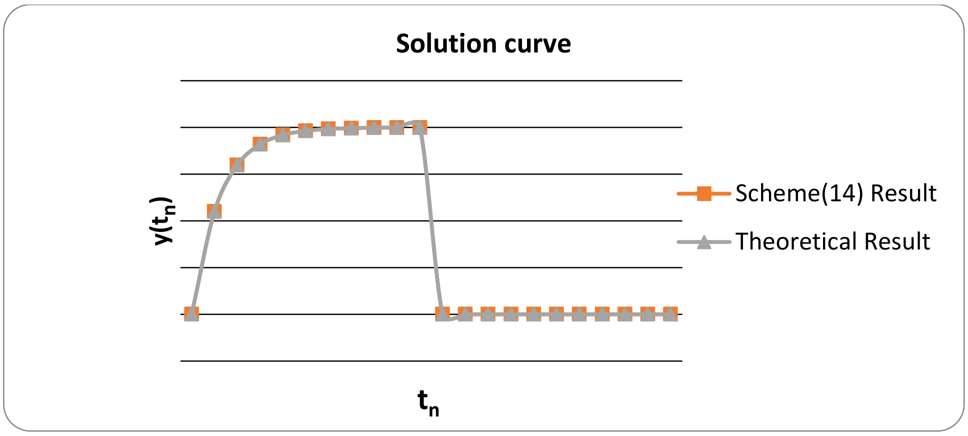

Fig 5: The solution curves of the Falling Body problem

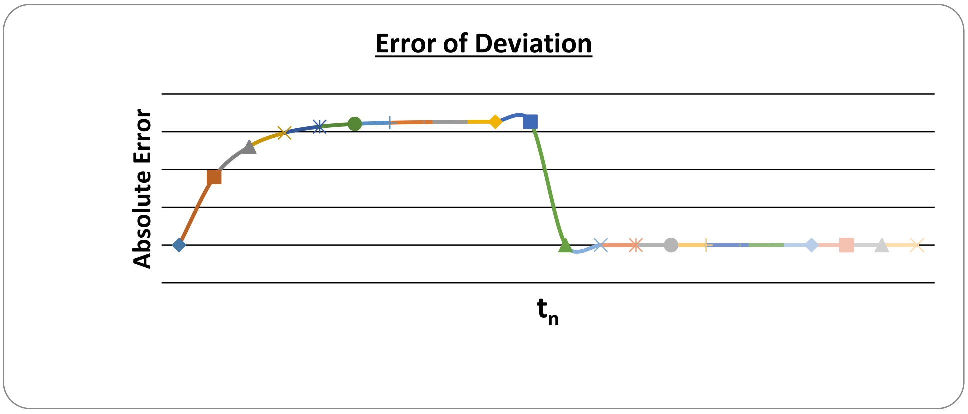

Fig 6: The Absolute Error of deviation of the scheme from the Analytic solution

In conclusion from table2 and Fig 5 we can see that as $t_n \to 1$, the velocity v approaches the limiting velocity (4m/sec). We also observe that this limiting velocity is approximately attained in a short time.

Furthermore, to determine the distance fallen at time t, (20) can be written as

$$\frac{dx}{dt} = 4(1 - e^{-8t}) \qquad (21)$$

By solving, we obtain

$$x = 4(t + \frac{1}{8}e^{-8t} - \frac{1}{8}) \qquad (22)$$

Again applying scheme (14) and comparing with Analytic solution (22) we have:

Table 3

A BODY FALLING TOWARDS THE EARTH (FINDING THE DISTANCE)

t_n	Scheme(14) Result	Theoretical Result	Error of Deviation
0.00	**0.000000000000000**	**0.000000000000000**	**0.000000000000000**
0.10	**0.119384098495418**	**0.124664482058611**	**0.005280383563193**
0.20	**0.393295246157544**	**0.400948258997328**	**0.007653012839783**
0.30	**0.736639872749838**	**0.745358976644706**	**0.008719103894868**
0.40	**1.111182972504879**	**1.120381101989183**	**0.009198129484304**
0.50	**1.499744449888177**	**1.509157819444367**	**0.009413369556191**
0.60	**1.894604790369782**	**1.904114873524510**	**0.009510083154729**
0.70	**2.292295392482466**	**2.301848931858242**	**0.009553539375776**

0.80	**2.691257713122024**	**2.700830778636587**	**0.009573065514563**
0.90	**3.090791453729911**	**3.100373292904188**	**0.009581839174277**
1.00	**3.490581949880243**	**3.500167731313951**	**0.009585781433708**

From Table (3), we can conclude that as x increases as t increases. Hence it implies that as the body reaches the earth's surface its motion ceased.

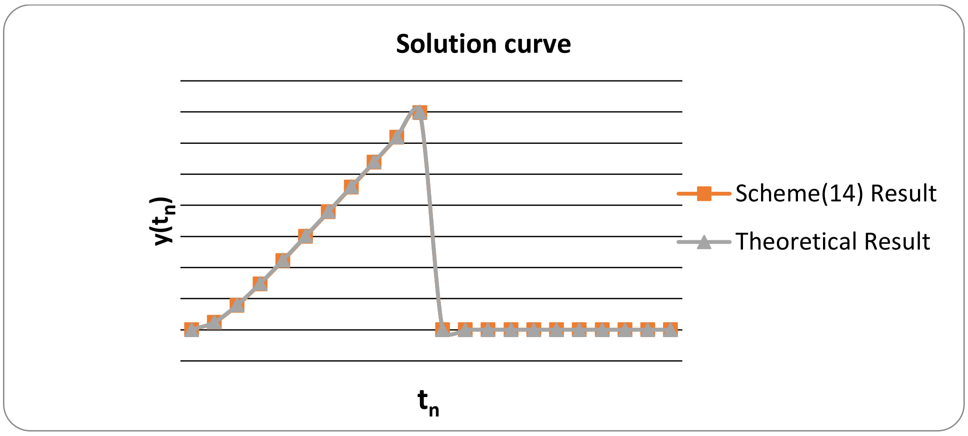

Fig 7: The solution curves of the distance travelled by the Falling Body

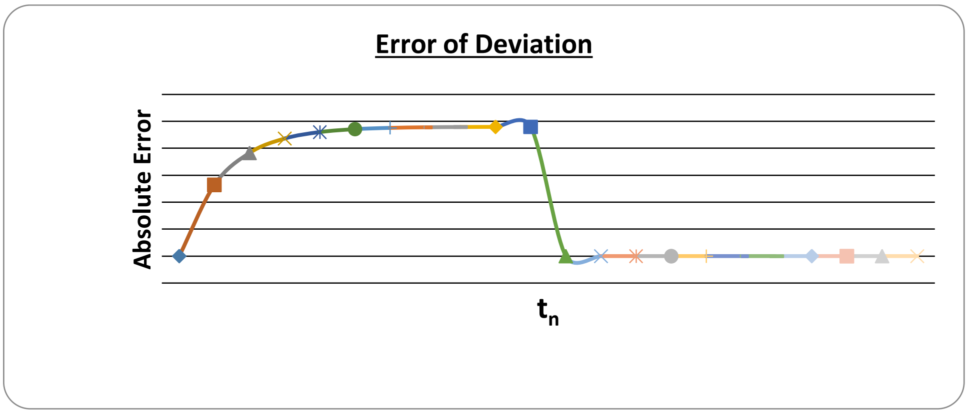

Fig 8: The Absolute Error of deviation of the scheme from the Analytic solution

CONCLUSION

The application of the new scheme on the selected problems shows the effectiveness of the scheme. Even though it may not solve every problem, it is a signal that this method can be employed and can compare favourably with other methods . It can also be better than the finite element method that is widely use for this class of problems.

The new scheme compared favourably with the existing standard schemes like that of Fatunla (1976), Ibijola (1997) and Ogunrinde (2010) .From the graphs we can infer that the results shows a measure of convergence towards the theoretical solution . The analysis of the schemes properties and stability analysis will be available in another article.

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