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On the stability of an SIS epidemic model involving treatment

Hiba Abdullah Ibrahim

Department of Mathematics, Baghdad University, College of Sciences, Baghdad, Iraq

Abstract

The objective of this paper is to study the stability of SIS epidemic model involving treatment. Two types of such eco-epidemiological models are introduced and analyzed. Boundedness of the system is established. The local and global dynamical behaviors are performed. The conditions of persistence of the models are derived.

Keywords: prey-predator model, disease, stability analysis.

1. Introduction

Over the past hundred years, mathematics has been used to understand and predict the spread of the disease. Some diseases confer immunity so recovered individuals gain immunity against the disease. SIS epidemic model is one of the most basic and most important models in describing the spread of many diseases.

Almost all mathematical models of diseases start from the some basic premise: That the population can be subdivided into a set of distinct classes, dependent upon their experience with respect to the disease.

One classifies individuals as one of susceptible, infection. Such models are termed as SI or SIS models. We consider SIS model with logistic dynamics for the susceptible population.

Some SIS models with infectious disease have been considered previously.(Xue, Wen, and Mini 2009) studied an SIS model with limited recourse for treatment. (Zhou and Liu 2003) studied an SIS model with pule vaccination. (Li and Ma 2002) considered an SIS epidemic model with vaccination and temporary immunity. (Van den driessche and Watmough 2000) considered an SIS epidemic model with a non-constant contact rate and showed that this model may have multiple stable equilibria a backward bifurcation. (Gao LQ Hethcote 1992) studied disease models with density-dependent demographics.

In this paper, we considered an SIS epidemic model involving treatment. The treatment is an important method to decrease the spread of disease such as measles and flu (Feng and Thieme 1995) ,(Hyman and Li 1998).

The treatment rate of infective is assumed to be proportional to the number of the infective.

2. The mathematical model

Let $S(t)$ and $I(t)$ be the numbers of the susceptible and infected population at time t respectively. The dynamics of an SIS epidemic model involving treatment can be represented by the following set of differential equations:

$$\left.\begin{aligned} \frac{dS}{dt} &= \sigma - dS - \frac{\beta SI}{S+I} + \gamma I + T(I) \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (d+\alpha+\gamma)I - T(I) \end{aligned}\right\} \quad (1)$$

Here the positive constant σ is recrotment. The positive constants d , α represent the natural death rate of the population and the disease death rate, respectively. The positive constants γ , β represent the nature recovery rate of infected individuals and the infection rate constant, respectively. $T(I)$ represents the treatment function which has been used by (Wang 2006)

$$T(I) = \begin{cases} rI & \text{if } \; 0 \le I \le I_0 \\ k & \text{if } \; I > I_0 \end{cases} \quad (2)$$

here $k = rI_0$, which is constant.

Clearly, R_+^2 is positively invariant for system (1). Consequently system (1) for $0 \le I \le I_0$ becomes

$$\left.\begin{aligned} \frac{dS}{dt} &= \sigma - dS - \frac{\beta SI}{S+I} + \gamma I + rI \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (d+\alpha+\gamma)I - rI \end{aligned}\right\} \quad (3)$$

While for $I > I_0$ becomes

$$\left.\begin{aligned} \frac{dS}{dt} &= \sigma - dS - \frac{\beta SI}{S+I} + \gamma I + k \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (d+\alpha+\gamma)I - k \end{aligned}\right\} \quad (4)$$

Theorem (1): All the solution of system (1), which initiate in R_+^2 are uniformly bounded.

Proof: Let $(S(t), I(t))$ be any solution of the system (1) with non-negative initial conditions.

Let $W(t) = S(t) + I(t)$, then we get that

$\frac{dw}{dt} = \frac{ds}{dt} + \frac{dI}{dt}$

$dw/dt = \sigma - ds - \beta SI/(S+I) + \gamma I + T(I) + \beta SI/(S+I) - (d+\alpha+\gamma)I - T(I)$

$\frac{dw}{dt} \le \sigma - d(S+I)$. Hence

$\frac{dw}{dt} + dw \le \sigma$

Again, by applying the comparison theorem on the above differential inequality we obtain that

$\lim_{t\to\infty} Sup\, W(t) \le \frac{\sigma}{d} \Longrightarrow W(t) \le \frac{\sigma}{d}; \ \forall t.$

Hence all the solutions of system (3) that initiate in R_+^2 are confined in the region $\Omega = \left\{(S,I) \in R_+^2 : W \le \frac{\sigma}{d} + \varepsilon \text{ for any } \varepsilon > 0\right\}$.

Thus these solutions are uniformly bounded, and then the proof is complete. ■

The analysis of system (3) gives the following results:

1. The disease free equilibrium point is given by $P_1 = \left(\frac{\sigma}{d}, 0\right)$ and it's always exists.
2. The endemic equilibrium point $P_2 = (\bar{S}, \bar{I})$, exists in the *Int.* R_+^2 of $S - I$ plane under the following condition

$$\beta > A \quad (5)$$

where $A = (d + \alpha + \gamma + r)$, and

$$\bar{S} = \frac{\sigma A}{dA + (\beta - A)(d+\alpha)} \ ; \ \bar{I} = \frac{(\beta - A)\sigma}{dA + (\beta - A)(d+\alpha)} \quad (6)$$

In addition to above, P_1 is locally asymptotically stable provided that

$$A > \beta \quad (7)$$

Note that, satisfying condition (7) means that there is no positive equilibrium point in the *Int.* R_+^2 of $S - I$ plane or equivalent the infected population will face extinction while the susceptible population grows to its maximum value.

Finally, the characteristic equation of the Jacobian matrix at the endemic equilibrium point $P_2 = (\bar{S}, \bar{I})$ can be written as

$$\lambda^2 + C\lambda + E = 0 \tag{8}$$

where $C = -(\alpha + \gamma + r) + \beta$, $E = (d + \alpha)\beta^2 - 2(d + \alpha)A\beta + \alpha A^2$.

According to the Routh-Hurwitz criterion, the endemic equilibrium point $P_2 = (\bar{S}, \bar{I})$ is locally asymptotically stable point if and only if $C > 0$ and $E > 0$, which are satisfied under the following conditions

$$(d + \alpha)\beta^2 + \alpha A^2 > 2(d + \alpha)A\beta \tag{8.a}$$

Now the global stability of the endemic equilibrium point is studied in the following theorem.

Theorem (2): Suppose that the endemic equilibrium point $P_2 = (\bar{S}, \bar{I})$ of the system (3) exists, then it is a globally asymptotically stable in the $Int.R_+^2$ of the $S - I$ plane.

Proof: Let $H(S, I) = 1/SI$; $G\_1\ (S, I) = \sigma - dS - \beta SI/(S + I) + \gamma I + rI$;

$G\_2\ (S, I) = \beta SI/(S + I) - (d + \alpha + \gamma)I - rI$.

Clearly, $H(S, I) > 0$ be C^1 function in the $Int.\ R_+^2$ of the $S - I$ plane Now, since

$$HG_1(S, I) = \frac{\sigma}{SI} - \frac{d}{I} - \frac{\beta}{S+I} + \frac{(\gamma+r)}{S} ;$$

$$HG_2(S, I) = \frac{\beta}{S+I} - \frac{(d+\alpha+\gamma+r)}{S}.$$

Hence,

$$\Delta(S, I) = \frac{\partial(HG_1)}{\partial S} + \frac{\partial(HG_2)}{\partial I} = -\left(\frac{\sigma I}{(SI)^2} + \frac{(\gamma+r)}{S^2}\right) < 0$$

Note that $\Delta(S, I)$ does not change sign and is not identically zero in the $Int.\ R_+^2$ of the $S - I$ plane. Then according to Bendixson-Dulic criterion, there is no periodic solution in the $Int.\ R_+^2$ of the $S - I$ plane.

Now, since all the solutions of the system (3) are uniformly bounded and P_2 is a unique equilibrium point in the $Int.\ R_+^2$ of the $S - I$ plane. Hence, by using the Poincare-Bendixson theorem P_2 is a globally asymptotically stable. ■

Now, to study the persistence of the system (3), the dynamical behavior of the solution of system (3) near the boundary equilibrium point is studied above. The persistence condition of the system (3) is established in the following theorem.

Theorem (3): Assume if the following condition holds.

$$\beta > A \tag{9}$$

Then system (3) is persist.

Proof: The boundedness of the solution of the system (3) is proved in theorem (1). Consider the function $v(S, I) = S^{\alpha_1} I^{\alpha_2}$, where $\alpha_i (i = 1,2)$ are the undetermined positive constants. Obviously $v(S, I)$ is a C^1 postive definite function defined in R_+^2 , and $v(S, I) \to 0$ if $S \to 0$ or $I \to 0$.

Consequently since

$$\frac{dv}{dt} = S^{\alpha_1} \alpha_2 I^{\alpha_2 - 1} \frac{dI}{dt} + I^{\alpha_2} \alpha_1 S^{\alpha_1 - 1} \frac{dS}{dt}$$

$$\Psi(S,I) = \frac{v^{\cdot}}{v} = \frac{S^{\alpha_1}\,\alpha_2 I^{\alpha_2-1}\frac{dI}{dt} + I^{\alpha_2}\,\alpha_1 S^{\alpha_1-1}\frac{dS}{dt}}{S^{\alpha_1} I^{\alpha_2}}$$

$$= \alpha_2\left[\frac{\beta S}{S+I} - A\right] + \alpha_1\left[\frac{\sigma}{S} - d - \frac{\beta I}{S+I} + \frac{(\gamma+r)I}{S}\right]$$

Now, since the only possible omega limit sets of the system (3) on the boundary of SI −plane is the disease equilibrium point P_1. Thus according to the Gard technique the proof is follows and the system is uniformly persists if we can proof that $\Psi(S,I) > 0$ at this point. So , since

$$\Psi(S,I)|_{P_1} = \alpha_2[\beta - A]$$

Obviously, $\Psi(P_1) > 0$ for all values α_2, provided that condition (9) holds.

The existence of the equilibrium points and stability analysis of the system (4) are discussed. It is observed that, the following equilibrium points exist for the system (4):

1- The disease free equilibrium point $F_1 = \left(\frac{\sigma+k}{d}, 0\right)$ always exists.

2- The endemic equilibrium point $F_2 = (S^*, I^*)$ where

$s^* = \frac{RI^{*2} + kI^*}{(\beta-R)I^* - k}$, while $I^*$is the positive root of the second equation,

$$aI^{*2} + bI^* - k\sigma = 0 \qquad (10)$$

Where $R = d + \alpha + \gamma$, $a = \big((\beta - R)(\gamma - R) - dR\big), b = \big(\sigma(\beta - R) - dk - k(\gamma - R)\big)$.

Exists uniquely in $Int.R_+^2$ provided that

$$\beta\gamma + R^2 > (\beta + \gamma + d)R \qquad (11)$$

Clearly the positive root of eq.(10) can be written as

$$I^* = \frac{-b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac} \qquad (12)$$

In the following theorems the bounded and stability analysis of system (4) is discussed.

Now to analyze the local stability of system (4) around each of these equilibrium points, the Jacobian matrix $J(F_i)\,; i = 1,2$ of system (4) at disease free and endemic equilibrium points are computed and then the eigenvalues are determined. The following results are obtained.

The eigenvalues of the Jacobian matrix $J(F_1)$ are given by

$$\lambda_1 + \lambda_2 = \beta - (2d + \alpha + \gamma)$$

$$\lambda_1 . \lambda_2 = -d\beta + dR$$

Obviously, F_1 is locally asymptotically stable if and only if the following condition holds:

$$R > \beta \qquad (13)$$

The eigenvalues of the Jacobian matrix $J(F_2)$ are given by

$$\lambda_3 + \lambda_4 = -(2d + \alpha + \gamma) - \frac{\beta(I-S)}{(S+I)}$$

$$\lambda_3 . \lambda_4 = \frac{\beta d(I^2 - S^2) + d(d+\alpha+\gamma)(S+I)^2 + \alpha\beta I^2}{(S+I)^2}$$

F_2 is locally asymptotically stable if and only if the following condition holds:

$$(2d + \alpha + \gamma)(S + I) + \beta I > \beta S \quad (14.a)$$

$$\beta d I^2 + d(d + \alpha + \gamma)(S + I)^2 + \alpha\beta I^2 > \beta d S^2 \quad (14.b)$$

Theorem (4): suppose that the endemic equilibrium point F_2 of system (4) exists, then it is a globally asymptotically stable in the $Int. R_+^2$ provided that

$$(\sigma + \gamma I + k)I > kS \quad (15)$$

Proof: Assume that $H(S,I) = \frac{1}{SI}$, $h_1(S,I) = \sigma - dS - \frac{\beta SI}{S+I} + \gamma I + k$,
$h_2(S,I) = \frac{\beta SI}{S+I} - (d + \alpha + \gamma)I - k$.

Clearly, $H(S,I) > 0$ be C^1 function in the $Int. R_+^2$ of SI-plane.

Now, since $\Delta(S,I) = \frac{-\sigma}{IS^2} - \frac{\gamma}{S^2} - k\left(\frac{I-S}{S^2 I^2}\right)$

$\Delta(S,I) < 0$ if and only if the following condition holds:

$$(\sigma + \gamma I + k)I > kS$$

Then according to Bendixson-Dulic criterion, there is no periodic solution in the $Int. R_+^2$ of the $S - I$ plane.

Now, since all the solutions of the subsystem (4) are uniformly bounded and F_2 is a unique positive equilibrium point in the $Int. R_+^2$ of the $S - I$ plane. Hence, by using the Poincare-Bendixson theorem, F_2 is a globally asymptotically stable. ■

3. Numerical analysis

In this section the global dynamics of system (3) is studied numerically. System (3) is solved numerically for set of parameters and different sets of initial conditions, and then the attracting set and their time series are drown. For the following set of parameters

$$\sigma = 500, d = 0.1, \beta = 2.3, \gamma = 1, r = 1, \alpha = 0.01, I_0 = 75 \quad (16)$$

The attracting set along with their time series of system (3) is drown in figure 1, starting from different sets of initial conditions. Note that from now onward, in time series figures,

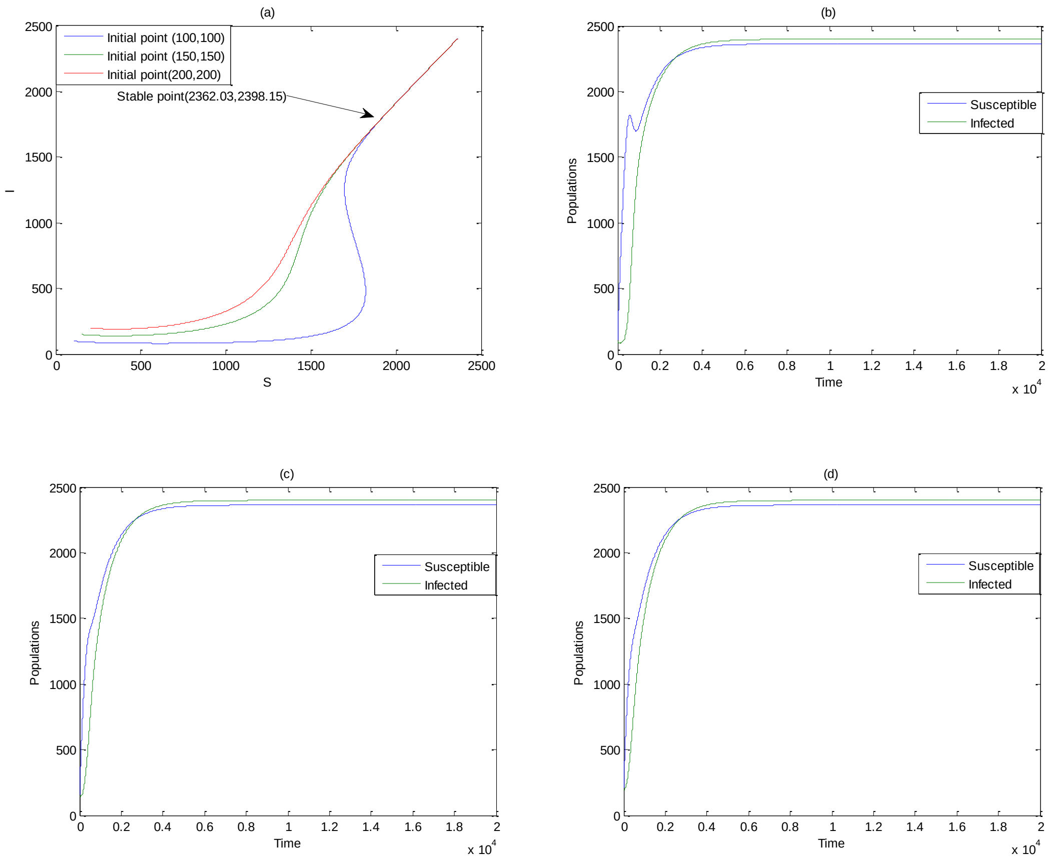

Fig. (1): The phase plot of system (3) in case 1. (a) The solution of system (1) approaches asymptotically to stable positive point initiated at different initial points. (b) Time series of the attractor in (a) initiated at (100,100). (c) Time series of the attractor in (a) initiated at (150,150). (d) Time series of the attractor in (a) initiated at (200,200).

Further analysis for the role of changing the infection rate constant β on the dynamics of system (3) is performed system (3) approaches asymptotically to disease free equilibrium point with the rest of parameters as given in Eq. (16), $P_1 = (5000,0)$, in the *Int.* R_+^2 for $\beta = 2$ as shown in figure 2.

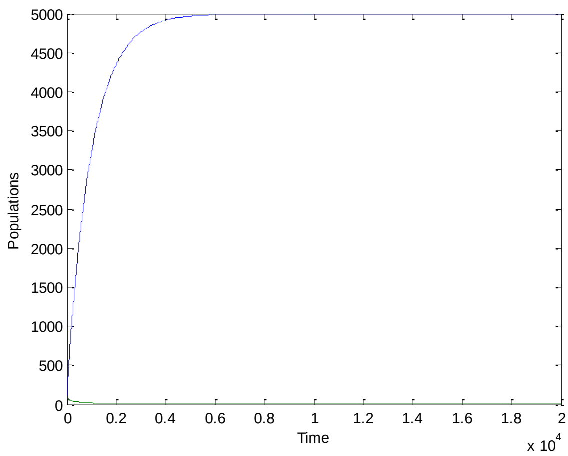

Fig2. Time series of the solution of system (3)

4. Conclusions and Discussion

In this paper, we proposed and analyzed an epidemiological model that described the dynamical behavior of an epidemic model, where the infectious disease transmitted directly from contact between them. The model included for non-linear autonomous differential equations that describe the dynamics of two different populations namely susceptible individuals (S) infected individuals for disease (I). The boundedness of system (3) has been discussed. The conditions for existence, stability for each equilibrium points are obtained. Further, it is observed that the disease free equilibrium point P_1 exists. The second disease free equilibrium point P_2 exists. Finally, to understand the effect of infection rate on the global system (3) and confirm our above analytical results, the system (3) has been solved numerically for different sets of initial points for set of parameters given by Eq. (16), and the following observations are made:

1. The system (3) do not has periodic dynamic, instead it they approach either to the all equilibrium point.
2. For the set of hypothetical parameters values given Eq. (16), system (3) approaches asymptotically to a globally asymptotically stable P_1 and P_2.

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