Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.13, 2015
www.iiste.org

Modeling and analysis of bilharzia disease

Raid Kamel Naji and Hassan Fadhil Ridha
Department of Mathematics, College of Science, University of Baghdad, Baghdad, IRAQ

Abstract: In this paper the dynamics of bilharazia disease in the humans, which represents its main host, is formulated mathematically. The proposed system is studied analytically. The local stability is investigated for all possible equilibrium points. Using suitable Lyapunov functions the basin of attraction of each point is specified. The conditions of occurring local bifurcation in the system are established. Numerical simulations are performed to study the global dynamics of the system and specify the set of control parameters. It is observed that the system has no periodic dynamics and the disease is controlled under some conditions on the parameters.
Keywords: Bilharzia; Parasite disease; Stability; Local bifurcation.

1. Introduction

It is well known that, Bilharzia or Schistosomiasis, which has other names as Snail fever or Katayama fever, is a disease caused by parasitic worms of the Schistosoma type. It may infect the urinary tract or the intestines. Signs and symptoms may include abdominal pain, diarrhea, bloody stool, or blood in the urine. Also it may cause liver damage, kidney failure, infertility, or bladder cancer. Further, it may cause poor growth and learning difficulty in children [1]. The disease is spread by contact with water contaminated with the parasites, which are released from infected freshwater snails. The disease comes from contaminated water and is especially common among children, farmers, fishermen, and people using unclean water.

Population of the environment plays a key role in spread of parasites disease (such as cholera, HIV and bilharzias). In deed the parasites need to the host to complete their life cycle and then it become a serious disease on the host life. Parasites and disease are frequently cited as important drivers of population and community dynamics [2]. Some parasites exploit their hosts in a prudent way, taking the resources that the need without causing noticeable damage. Prudent exploitation yields sustainable benefits to the parasite as long as the host remains healthy. Other parasites attack their host more quickly and vigorously. Rapid exploitation may allow the parasites to achieve higher reproductive rates, but damage to the host reduces the parasites opportunity for sustainable yield [3].

There are number of mathematical models has been developed to study the transmission dynamics of these disease. Ebert et al. [4] proposed and studied an epidemiological micro-parasite model using mass action incidence function. However, Hwang and Kuang [5] studied the same system with standard incidence function. Later on many researcher have been used the framework of Hwang and Kuang model in some host-parasites models, see for example [6-7] and the references therein. Recently, Wang and Kuang [8] studied the fluctuation and extinction dynamics in host-microparasite systems. They proposed models involving SI type of diseases caused by parasites with different infection rate function and then studied the stability through the computing the reproduction number.

Keeping the above in view, in this paper the model of Wang and Huang with mass action incidence function is modified so that its suite the parasite disease caused by bilharzias, which is SIS type of disease. The local as well as global stability analysis of the model is investigated analytically as well as numerically. The local bifurcations that may occur in the system are also investigated analytically as well as numerically.

2. The model construction

Bilharzia disease is a parasitic disease caused by parasitic worms affects many people, which represents the main host for it, in developing countries. Such as many parasitic diseases, bilharzia is not a directly deadly disease, but it leads to the speed of the body of the patient consumption, which leads then to death. It is an SIS type of disease that means the person which has the disease may recover and return back to incidence with disease when the humans down to the water canals and drains that contaminated by parasite worms. Therefore in order to formulate a mathematical model that describes the dynamics of bilharzia disease in the humans the following hypotheses are adopted.

1. Bilharzia disease divides the host population (humans) in to two compartments namely susceptible population, which denotes to its population size at time t by $x(t)$, and infected population, which denoted to its size at time t by $y(t)$.

2. It is assumed that the host population grows logistically with intrinsic growth rate $r > 0$ and carrying capacity $\frac{1}{L} > 0$. Since the disease causes weakness in the body of the patient, then it is assumed that the relative fecundity of the infected person is given by $0 < \alpha < 1$.
3. The natural death rate of the host population is given by $d_1 > 0$, while the disease death rate is assumed to be $e > 0$. The disease transmitted to the susceptible persons due to contact of them with the parasite represented by schistosomes type's worms with contact rate $\gamma > 0$. However the infected person will be recovered and return back to be susceptible compartment with recover rate $\beta > 0$.
4. Since the parasite that denoted to its population size at time t by $z(t)$ is released from the infected person due to urinate and defecate in waterways or on the shores of canals. It is assumed that the parasite population is grows exponentially with growth rate $\mu > 0$ and facing natural death with death rate $d_2 > 0$.

Consequently the dynamics of the interaction between the humans and bilharzia disease can be described in the following set of nonlinear differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= r(x+\alpha y)(1-L(x+y)) - d_1 x + \beta y - \gamma xz = f_1(x,y,z) \\
\frac{dy}{dt} &= \gamma xz - (d_1+e)y - \beta y = f_2(x,y,z) \qquad (1) \\
\frac{dz}{dt} &= \mu y - d_2 z = f_3(x,y,z)
\end{aligned}$$

with $x(0) \geq 0, y(0) \geq 0$ and $z(0) \geq 0$. Clearly, the right hand side functions are continuous and have continuous partial derivatives, therefore they are Lipschitzain. Thus system (1) has a unique solution of the domain $R_+^3 = \{(x,y,z) \in R^3 : x \geq 0, y \geq 0, z \geq 0\}$. More over in the following the boundedness of the solution of system (1) is established.

Theorem (1): All solutions of system (1) which initiate in R_+^3 are uniformly bounded.

Proof: Let $(x(t), y(t), z(t))$ be any solution of the system (3.1) with non-negative initial condition $(x(0), y(0), z(0))$. According to the system (1), we have

$$\frac{d(x+y)}{dt} \leq r(x+y)[1-L(x+y)]$$

which implies that

$$x(t) + y(t) \leq \frac{1}{L}, \forall t > 0$$

Consider the function $M(t) = x(t) + y(t) + z(t)$, by taking the derivative of $M(t)$ with respect to time along the solution of system (1) gives

$$\frac{dM}{dt} \leq \frac{1}{L} - \sigma M; \ 0 < \sigma = \min\{d_1 - \mu, d_2\}$$

Thus, $\frac{dM}{dt} + \sigma M \leq \frac{1}{L}$

Now, it is easy to verify that the solution of the above linear differential inequality can be written

$$M(t) \leq \frac{1}{\sigma L} + \left(M_0 - \frac{1}{\sigma L}\right) e^{-\sigma t}$$

here $M_0 = (x(0), y(0), z(0))$. Therefore, $M(t) \leq \frac{1}{\sigma L}, \forall t > 0$, hence all the solution of system (1) are uniformly bounded and therefore we finished the proof. ■

3. Existence and stability analysis of equilibrium points

System (1) has three non negative equilibrium points. The existence and the stability analysis of each of them are summarized in the following. The trivial equilibrium point $E_0 = (0,0,0)$ always exists. The disease free equilibrium point $E_1 = (\tilde{x},0,0)$, where

$$\tilde{x} = \frac{r - d_1}{rL} \tag{3a}$$

exists under the condition.

$$r > d_1 \tag{3b}$$

The endemic equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z})$ can be written in as

$$\hat{x} = \frac{d_2(d_1 + e + \beta)}{\mu\gamma}, \hat{z} = \frac{\mu}{d_2}\hat{y} \tag{3c}$$

while $\hat{y}$ represents a positive root of the following quadratic equation

$$A_1 y^2 + A_2 y + A_3 = 0 \tag{3d}$$

here

$$A_1 = rL\alpha;\ A_2 = \left[rL(1+\alpha) + \frac{\gamma\mu}{d_2} \right]\hat{x} - r\alpha - \beta;\ A_3 = \left[rL\hat{x} + d_1 - r \right]\hat{x}.$$

Clearly E_2 exists uniquely in the interior of R_+^3 if the following condition is satisfied

$$rL\hat{x} + d_1 < r \tag{3e}$$

In the following the local stability conditions of each equilibrium point of system (1) are established.

Theorem (2): The trivial equilibrium point E_0 of the system (1) is locally asymptotically if the following condition is satisfied

$$r < d_1 \tag{4a}$$

Proof: The Jacobian matrix of system (1) at E_0 can be written as:

$$J(E_0) = \begin{pmatrix} r - d_1 & r\alpha + \beta & 0 \\ 0 & -(d_1 + e + \beta) & 0 \\ 0 & \mu & -d_2 \end{pmatrix} = (b_{ij}) \tag{4b}$$

Clearly, $J(E_0)$ has the following eigenvalues:

$$\lambda_x = r - d_1; \lambda_y = -(d_1 + e + \beta) < 0;\ \lambda_z = -d_2 < 0$$

Therefore, E_0 is locally asymptotically stable if and only if the eigenvalue $\lambda_x < 0$, which is satisfied provided that condition (4a) holds and hence the proof is complete. ■

Theorem (3): The disease free equilibrium point $E_1 = (\tilde{x},0,0)$ of system (1) is locally asymptotically stable if the following sufficient condition is satisfied

$$(d_1 + e + \beta)d_2 > \frac{\gamma(r - d_1)}{rL}\mu \tag{5a}$$

Proof: The Jacobian matrix of system (1) at E_1 is given by

$$J(E_1) = \begin{pmatrix} -r + d_1 & r\alpha + \beta - (1+\alpha)(r - d_1) & \frac{-\gamma(r-d_1)}{rL} \\ 0 & -(d_1 + e + \beta) & \frac{\gamma(r-d_1)}{rL} \\ 0 & \mu & -d_2 \end{pmatrix} = (c_{ij}) \tag{5b}$$

Then the characteristic equation of the Jacobian matrix $J(E_1)$ is given by:

$$(-r+d_1-\lambda)\left[\lambda^2+\beta_1\lambda+\beta_2\right]=0 \qquad (5c)$$

here $\beta_1=\left[d_1+e+\beta+d_2\right]>0;\ \beta_2=d_1d_2+ed_2+\beta d_2-\dfrac{\gamma(r-d_1)}{rL}\mu.$

Consequently equation (5c) has the following roots, which represent the eigenvalues of $J(E_1)$:

$$\lambda_x=-r+d_1;\ \lambda_y,\lambda_z=\frac{-\beta_1}{2}\pm\frac{1}{2}\sqrt{\beta_1^2-4\beta_2}$$

where $\lambda_x,\lambda_y,\lambda_z$ describe the dynamics in the $x-, y-, z-$ direction respectively. Clearly λ_y and λ_z have negative real parts provided that condition (5a) holds, while $\lambda_x<0$ under existence condition (3b). Hence E_1 is locally asymptotically stable in the R_+^3. However, it is a saddle point otherwise. ■

Theorem (4): The endemic equilibrium point $E_2=(\hat{x},\hat{y},\hat{z})$ of system (1) is locally asymptotically stable if the following conditions hold

$$r<d_1+2rL\hat{x}+rL(1+\alpha)\hat{y}+\gamma\hat{z} \qquad (6a)$$

$$r\alpha+\beta<rL(1+\alpha)\hat{x}+2rL\alpha\hat{y} \qquad (6b)$$

$$\left[rL(1+\alpha)\hat{x}+2rL\alpha\hat{y}-(r\alpha+\beta)\right]>d_2 \qquad (6c)$$

Proof: The Jacobian matrix of system (1) at the endemic equilibrium point can be written

$$J(E_2)=(d_{ij})_{3\times3} \qquad (6d)$$

here $d_{11}=r-d_1-2rL\hat{x}-rL(1+\alpha)\hat{y}-\gamma\hat{z}$, $d_{12}=r\alpha+\beta-rL(1+\alpha)\hat{x}-2rL\alpha\hat{y}$, $d_{13}=-\gamma\hat{x}$, $d_{21}=\gamma\hat{z}$, $d_{22}=-(d_1+e+\beta)$, $d_{23}=\gamma\hat{x}$, $d_{31}=0$, $d_{32}=\mu$, $d_{33}=-d_2$.

Hence, the characteristic equation of $J(E_2)$ is given by

$$\lambda^3+\hat{A}_1\lambda^2+\hat{A}_2\lambda+\hat{A}_3=0 \qquad (6e)$$

here $\hat{A}_1=-(d_{11}+d_{22}+d_{33})$,

$$\hat{A}_2=d_{11}d_{33}+d_{11}d_{22}-d_{12}d_{21}$$

$$\hat{A}_3=d_{33}(d_{12}d_{21}-d_{11}d_{22})+d_{32}(d_{11}d_{23}-d_{13}d_{21})$$
$$=d_2\gamma\hat{z}\left((d_1+e+\beta)-\left[r\alpha+\beta-rL(1+\alpha)\hat{x}-2rL\alpha\hat{y}\right]\right).$$

with

$$\hat{\Delta}=\hat{A}_1\hat{A}_2-\hat{A}_3=-(d_{11}+d_{22})d_{11}d_{22}+\hat{A}_1d_{11}d_{33}-d_{11}d_{23}d_{32}$$
$$+d_{21}(d_{11}d_{12}+d_{22}d_{12}+d_{13}d_{32})$$

Now according to Routh-Hurwitz criterion E_2 will be locally asymptotically stable provided that $\hat{A}_1>0$, $\hat{A}_3>0$ and $\hat{\Delta}=\hat{A}_1\hat{A}_2-\hat{A}_3>0$. Clearly, $\hat{A}_1$ and $\hat{A}_3$ are positive provided that the conditions (6a) and (6b) are satisfied respectively. While $\hat{\Delta}$ is positive provided that conditions (6a)-(6c) hold. Hence the proof is complete. ■

Now the stability analysis of the above equilibrium point is investigated using the suitable Lyapunov functions. The objective is to specify the basin of attraction for each of them.

Theorem (5): Suppose that the trivial equilibrium point E_0 is locally asymptotically stable, then it is a globally asymptotically stable in R_+^3, if the following condition holds

$$r<\frac{d_1+e-\mu}{\alpha} \qquad (7)$$

Proof: Consider the following function $V_0(x,y,z)=x+y+z$. It is easy to check that $V_0(x,y,z)\in C^1\left(R_+^3,\ R\right)$ in addition, $V_0(0,0,0)=0$ while $V_0(x,y,z)>0;\ \forall(x,y,z)\in R_+^3$ with $(x,y,x)\neq(0,0,0)$. Furthermore

$$\frac{dV_0}{dt}=-rLx^2+(r-d_1)x-rL\alpha y^2+(r\alpha+\mu-d_1-e)y-rL(1+\alpha)xy-d_2z$$

So due to condition (7), it is obtained that $\frac{dV_0}{dt}$ is negative definite and hence the proof is complete.

■

Theorem (6): Assume that the free disease equilibrium point E_1 is locally asymptotically stable then it is globally asymptotically stable in the region $\Omega_1=\left\{(x,y,z)\in R^3: x>H, y\geq 0, z\geq 0\right\}$ provided that

$$(r\alpha+\beta)\tilde{x}+d_1+e+\beta>\mu \qquad (8)$$

here $H=\max\left\{\tilde{x}+\frac{r\alpha+\beta}{rL(1+\alpha)},\tilde{x}+1\right\}$.

Proof: Consider the following function

$$V_1(x,y,z)=\frac{(x-\tilde{x})^2}{2}+y+z.$$

It is easy to see that $V_1(x,y,z)\in C^1\left(R_+^3,R\right)$, in addition, $V_1(\tilde{x},0,0)=0$, while $V_1(x,y,z)>0;\ \forall(x,y,z)\in R_+^3$ and $(x,y,z)\neq(\tilde{x},0,0)$. Furthermore, by differentiating the function with respect to time and then simplifying the resulting terms we get:

$$\frac{dV_1}{dt}=-[rLx](x-\tilde{x})^2-[rL(1+\alpha)(x-\tilde{x})-(r\alpha+\beta)]xy-\alpha L(x-\tilde{x})y^2$$
$$-[(r\alpha+\beta)\tilde{x}+d_1+e+\beta-\mu]y-d_2z-\gamma[x-\tilde{x}-1]xz$$

Then according to condition (8), its easy to check that $\frac{dV_1}{dt}$ is negative definite in the region $\Omega_1\subset R_+^3$ and hence V_1 is Laypunov function with respect to E_1. So E_1 is a globally asymptotically stable.

■

Theorem (7): Assume that the endemic equilibrium point E_2 is locally asymptotically stable then it is globally asymptotically stable in the interior of region that satisfy the following conditions

$$r-d_1<rL(x+\hat{x})+rL(1+\alpha)\hat{y}+\gamma\hat{z} \qquad (9a)$$
$$q_{12}^2<q_{11}q_{22} \qquad (9b)$$
$$(\gamma x)^2<q_{11}d_2 \qquad (9c)$$
$$(q_{23})^2<q_{22}d_2 \qquad (9d)$$

here $q_{11}=rL(x+\hat{x})-(r-d_1)+rL(1+\alpha)\hat{y}+\gamma\hat{z}$, $q_{22}=d_1+e+\beta$

$q_{12}=rL(1+\alpha)x+\alpha L(y+\hat{y})-(r\alpha+\beta)-\gamma\hat{z}$, $q_{23}=\gamma x+\mu$

Proof: Consider the following function

$$V_2(x,y,z)=\frac{(x-\hat{x})^2}{2}+\frac{(y-\hat{y})^2}{2}+\frac{(z-\hat{z})^2}{2}$$

It is easy to see that $V_2(x,y,z)\in C^1\left(R_+^3,R\right)$, in addition $V_2(\hat{x},\hat{y},\hat{z})=0$, while $V_2(x,y,z)>0$; $\forall(x,y,z)\in R_+^3$ and $(x,y,z)\neq(\hat{x},\hat{y},\hat{z})$. Furthermore by differentiating the function with respect to time and then simplifying the resulting terms we get:

$$\frac{dV_2}{dt} = -\left[\frac{q_{11}}{2}(x-\hat{x})^2 + q_{12}(x-\hat{x})(y-\hat{y}) + \frac{q_{22}}{2}(y-\hat{y})^2\right]$$
$$-\left[\frac{q_{11}}{2}(x-\hat{x})^2 + \gamma x(x-\hat{x})(z-\hat{z}) + \frac{d_2}{2}(z-\hat{z})^2\right]$$
$$-\left[\frac{q_{22}}{2}(y-\hat{y})^2 - q_{23}(y-\hat{y})(z-\hat{z}) + \frac{d_2}{2}(z-\hat{z})^2\right]$$

By using the above conditions, we obtained that

$$\frac{dV_2}{dt} < -\left[\sqrt{\frac{q_{11}}{2}}(x-\hat{x}) + \sqrt{\frac{q_{22}}{2}}(y-\hat{y})\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(x-\hat{x}) + \sqrt{\frac{d_2}{2}}(z-\hat{z})\right]^2$$
$$-\left[\sqrt{\frac{q_{22}}{2}}(y-\hat{y}) - \sqrt{\frac{d_2}{2}}(z-\hat{z})\right]^2$$

Clearly, q_{11} is positive provided that condition (9a) holds. Consequently, due to conditions (9b)-(9d), $\frac{dV_2}{dt}$ is negative definite and hence V_2 is Laypunov function with respect to E_2. So E_2 is globally asymptotically stable and hence the proof is complete.■

4. Local bifurcation analysis

In this section, the occurrence of local bifurcation (such as transcritical, pitchfork and saddle-node) around each one of the system's equilibrium points is studied. Recall that the general Jacobian matrix $J(x, y, z)$ of system (1) is given by:

$$J(\mathrm{X}) = DF(\mathrm{X}) = (a_{ij})_{3\times 3};\ F = (f_1, f_2, f_3)^T,\ \mathrm{X} = (x, y, z) \qquad (10)$$

here $a_{11} = r - d_1 - 2rLx - rL(1+\alpha)y - \gamma z$, $a_{12} = r\alpha + \beta - rL(1+\alpha)x - 2rL\alpha y$, $a_{13} = -\gamma x$, $a_{21} = \gamma z$, $a_{22} = -(d_1 + e + \beta)$, $a_{23} = \gamma x$, $a_{31} = 0$, $a_{32} = \mu$, $a_{33} = -d_2$. Therefore, it is easy to verify that for any non-zero vector $V = (v_1, v_2, v_3)^T$, we get

$$D^2F(\mathrm{X}).(V,V) = \begin{pmatrix} -2\left(rLv_1^2 + rL(1+\alpha)v_1v_2 + \gamma v_1v_3 + \gamma\alpha L v_2^2\right) \\ 2\gamma v_1 v_2 \\ 0 \end{pmatrix} \qquad (11)$$

Theorem (8): The system (1) at equilibrium point E_0 with the parameter r passes through the value $r^* = d_1$, has transcritical bifurcation but neither saddle node bifurcation nor pitchfork bifurcation can occur.

Proof : According to the Jacobian matrix at E_0 that given by Eq.(4b), system (1) has zero eigenvalue (say $\lambda_x = 0$) at $r^* = d_1$, so that Jacobian matrix $J(E_0)$ with $r^* = d_1$ becomes

$$J_0^* = J(E_0, r^*) = (b_{ij}^*)_{3\times 3}$$

where $b_{ij}^* = b_{ij};\ \forall i, j = 1,2,3$ with $b_{11}^* = 0, b_{12}^* = d_1\alpha + \beta$.

Now, let $K = (k_1, k_2, k_3)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_x = 0$ of the matrix J_0^*. Thus $(J_0^* - \lambda_x I)K = 0$, which gives that k_1 be any nonzero real number while $k_2 = k_3 = 0$

Let $\Psi=(\psi_1,\psi_2,\psi_3)^T$ be the eigenvector associated with the eigenvalue $\lambda_x=0$ of the matrix J_0^{*T}. Then we have $\left(J_0^{*T}-\lambda_x I\right)\Psi=0$. By solving this equation for Ψ we obtain $\Psi=\left(\psi_1,\frac{r^*\alpha+\beta}{d_1+e+\beta}\psi_1,0\right)^T$, where ψ_1 be any nonzero real number. Now, consider

$$\frac{\partial F}{\partial r}=F_r(\mathrm{X},r)=\left(\frac{\partial f_1}{\partial r},\frac{\partial f_2}{\partial r},\frac{\partial f_3}{\partial r}\right)^T=\left((x+\alpha y)\left[1-L(x+y)\right],0,0\right)^T$$

So, $F_r\left(E_0,r^*\right)=(0,0,0)^T$ and then $\Psi^T F_r\left(E_0,r^*\right)=0$

Thus, according to Sotomayor's theorem [? ?], the saddle-nod bifurcation can't occur, while the first condition of transcritical bifurcation is satisfied. Now, since

$$DF_r(\mathrm{X},r)=\begin{pmatrix}1-2Lx-L(1+\alpha)y & \alpha-L(1+\alpha)x-2L\alpha y & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

where $DF_r(\mathrm{X},r)$ represents the derivative of $F_r(\mathrm{X},r)$ with respect to the vector X. Moreover, $\Psi^T\left[DF_r\left(E_0,r^*\right)K\right]=k_1\psi_1\neq 0$. Now, by substituting K instead of V and E_0 instead of X in Eq. (11) we get that

$$D^2F\left(E_0,r^*\right)(K,K)=\left(-2r^*Lk_1^2,0,0\right)^T$$

Further, it is observed

$$\Psi^T\left[D^2F\left(E_0,r^*\right)(K,K)\right]=-2r^*L\psi_1k_1^2\neq 0$$

Thus, according to Sotomayor's theorem, system (1) has transcritical bifurcation at E_0 with the parameter $r^*=d_1$, while pitchfork can't occur. Hence the proof is complete. ■

Theorem (9): The system (1) at equilibrium point E_1 with the parameter d_2 passes through the value $d_2^*=\frac{\gamma(r-d_1)\mu}{rL(d_1+e+\beta)}$, has a transcritical bifurcation, but neither saddle-nod bifurcation, nor pitchfork bifurcation can occur.

Proof: According to the Jacobian matrix $J(E_1)$ given by Eq. (5b), system (1) has zero eigenvalue (say $\lambda_z=0$) at $d_2=d_2^*$, so the Jacobian matrix $J(E_1)$ with $d_2=d_2^*$ becomes:

$$J_1^*=J_1\left(E_1,d_2^*\right)=(c_{ij}^*)$$

where $c_{ij}^*=c_{ij};\ \forall i,j=1,2,3$ with $c_{33}^*=d_2^*$.

Now, let $M=(m_1,m_2,m_3)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_z=0)$. Thus $\left(J_1^*-\lambda_z I\right)M=0$, which gives

$$m_1=\frac{\gamma[\alpha d_1-(r+e)]}{rL(d_1+e+\beta)}m_3,\ m_2=\frac{\gamma(r-d_1)}{rL(d_1+e+\beta)}m_3$$

while m_3 be any nonzero real number. Let $N=(n_1,n_2,n_3)^T$ be the eigenvector associated with the eigenvalue $\lambda_z=0$ of the matrix J_1^{*T}. Then we have $\left(J_1^{*T}-\lambda_z I\right)N=0$. By solving this equation for N, we obtain that

$$N=\left(0,\frac{\mu n_3}{d_1+e+\beta},n_3\right)^T$$

where n_3 be any nonzero real number. Now, consider

$$\frac{\partial F}{\partial d_2} = F_{d_2}(\mathrm{X}, d_2) = \left(\frac{\partial f_1}{\partial d_2}, \frac{\partial f_2}{\partial d_2}, \frac{\partial f_3}{\partial d_2} \right)^T = (0,0,-z)^T$$

So, $F_{d_2}\left(E_1, d_2^*\right) = (0,0,0)^T$, and hence $N^T F_{d_2}\left(E_1, d_2^*\right) = 0$, thus according to Sotomayor's theorem saddle-node bifurcation can't occur, while the first condition of transcritical bifurcation is satisfied. Now, since

$$DF_{d_2}(\mathrm{X}, d_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $DF_{d_2}(\mathrm{X}, d_2)$ represents the derivative of $F_{d_2}(\mathrm{X}, d_2)$ with respect to vector X. Furthermore we have $N^T \left[DF_{d_2}\left(E_1, d_2^*\right) M \right] = -n_3 m_3 \neq 0$.

Now, by substituting M instead of V and E_1 instead of X in Eq. (11) we get

$D^2 F\left(E_1, d_2^*\right)(M, M) = (u_1, u_2, 0)^T$

where

$$u_1 = -2m_3^2 \left(rL\left(\frac{\gamma[\alpha d_1 - (r+e)]}{rL(d_1+e+\beta)} \right)^2 + rL(1+\alpha) \frac{\gamma^2[\alpha d_1 - (r+e)](r-d_1)}{[rL(d_1+e+\beta)]^2} + \frac{\gamma^2[\alpha d_1 - (r+e)]}{rL(d_1+e+\beta)} + \gamma L\alpha \left(\frac{\gamma(r-d_1)}{rL(d_1+e+\beta)} \right)^2 \right)$$

$$u_2 = 2 \frac{\gamma^3[\alpha d_1 - (r+e)](r-d_1)}{[rL(d_1+e+\beta)]^2} m_3^2$$

So, $N^T \left[D^2 F\left(E_1, d_2^*\right)(M, M) \right] = 2 \frac{\gamma^3[\alpha d_1 - (r+e)](r-d_1)}{[rL(d_1+e+\beta)]^2} \left(\frac{\mu}{d_1+e+\beta} \right) m_3^2 n_3$

Since $d_1 < r$ due to existence condition of E_1 and $0 < \alpha < 1$ then we obtain that $N^T \left[D^2 F\left(E_1, d_2^*\right)(M, M) \right] \neq 0$. Thus, according to Sotomayor's theorem, system (1) has transcritical bifurcation at E_1 with the parameter $d_2 = d_2^*$ while the pitchfork bifurcation does not occur.

■

Theorem (10): The system (1), at equilibrium point E_2 with the parameter d_1 passes through the value $d_1^* = r\alpha - rL(1+\alpha)\hat{x} - 2rL\alpha\hat{y} - e$, undergoes saddle node bifurcation but neither transcritical bifurcation nor pitchfork bifurcation can occur provided that the following conditions are satisfied.

$$r < d_1 + 2rL\hat{x} + rL(1+\alpha)\hat{y} \quad \text{(12a)}$$

$$r\alpha > rL(1+\alpha)\hat{x} + 2rL\alpha\hat{y} + e \quad \text{(12b)}$$

Proof: Consider the characteristics equation of the Jacobian matrix $J(E_2)$ that is given in Eq. (6e). Clearly this equation has zero root if and only if $\hat{A}_3 = 0$, and hence E_2 becomes a nonhyperbolic equilibrium point. Now since

$$\hat{A}_3 = d_2 \gamma \hat{z}\left((d_1 + e + \beta) - \left[r\alpha + \beta - rL(1+\alpha)\hat{x} - 2rL\alpha\hat{y} \right] \right)$$

Thus $\hat{A}_3 = 0$ at $d_1^* = r\alpha - rL(1+\alpha)\hat{x} - 2rL\alpha\hat{y} - e$, which is positive under condition (12b).

The Jacobian matrix of system (1) at E_2 with parameter $d_1 = d_1^*$ becomes $J_2^* = J_2\left(E_2, d_1^*\right) = \left(d_{ij}^*\right)_{3\times 3}$ where $d_{ij}^* = d_{ij}$ for all $i, j = 1,2,3$ with $d_{11}^* = r - d_1^* - 2rL\hat{x} - rL(1+\alpha)\hat{y} - \gamma\hat{z}$ and $d_{22}^* = -\left(r\alpha - rL(1+\alpha)\hat{x} - 2rL\alpha\hat{y} + \beta\right) = -d_{12}^*$, which are negative under the conditions (12a)-(12b) respectively.

Let $P = \left(p_1, p_2, p_3\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda = 0$. Thus $\left(J_2^* - \lambda I\right)P = 0$, which gives

$p_1 = 0$, $p_2 = qp_3$ and $p_3 \in R;\ p_3 \neq 0$

where $q = \frac{d_{13}^* d_{21}^* - d_{11}^* d_{23}^*}{d_{11}^* d_{22}^* - d_{12}^* d_{21}^*} > 0$ due to conditions (12a)-(12b).

Let $W = \left(w_1, w_2, w_3\right)^T$ be the eigenvector associated with the eigenvalue $\lambda = 0$ of the matrix J_2^{*T}. Then $\left(J_2^{*T} - \lambda I\right)W = 0$ gives

$w_1 = h_1 w_2$, $w_2 = h_2 w_2$ and $w_2 \in R;\ w_2 \neq 0$

here $h_1 = -\frac{d_{21}^*}{d_{11}^*} > 0$ and $h_2 = \frac{d_{13}^* d_{21}^* - d_{23}^* d_{11}^*}{d_{11}^* d_{33}^*} > 0$.

Now, consider

$$\frac{\partial F}{\partial d_1} = F_{d_1}(\mathrm{X}, d_1) = \left(\frac{\partial f_1}{\partial d_1}, \frac{\partial f_2}{\partial d_1}, \frac{\partial f_3}{\partial d_1}\right)^T = (-x, -y, 0)^T$$

So, $F_{d_1}\left(E_2, d_1^*\right) = \left(-\hat{x}, -\hat{y}, 0\right)^T$ and hence $W^T F_{d_1}\left(E_2, d_1^*\right) = -\left(h_1\hat{x} + \hat{y}\right)w_2 \neq 0$.

So, the pitchfork bifurcation and transcritical bifurcation can't occur. While the first condition of the saddle-node bifurcation is satisfied. Also, we have

$$W^T\left[D^2 F\left(E_2, d_1^*\right)(P, P)\right] = -2\gamma\alpha L q^2 h_1 p_3^2 w_2 \neq 0$$

Hence, the system (1) has saddle-node bifurcation at E_2 with parameter $d_1 = d_1^*$. ■

5. Numerical simulation

In order to confirm our obtained analytical results and understand the effects of each parameter on the bilharzia epidemic system, this section deals with the global dynamics of system (1). Consequently, system (1) is solved numerically for different sets of initial conditions and different sets of parameters. It is observed that, for the following set of biologically reasonable hypothetical parameters, system (1) approaches asymptotically to the endemic equilibrium point as shown in Fig. (1) below.

$$r = 1,\ L = 0.05,\ \alpha = 0.25,\ d_1 = 0.05,\ \beta = 0.5,$$
$$\gamma = 0.05,\ e = 0.2,\ \mu = 0.5,\ d_2 = 0.2 \qquad (13)$$

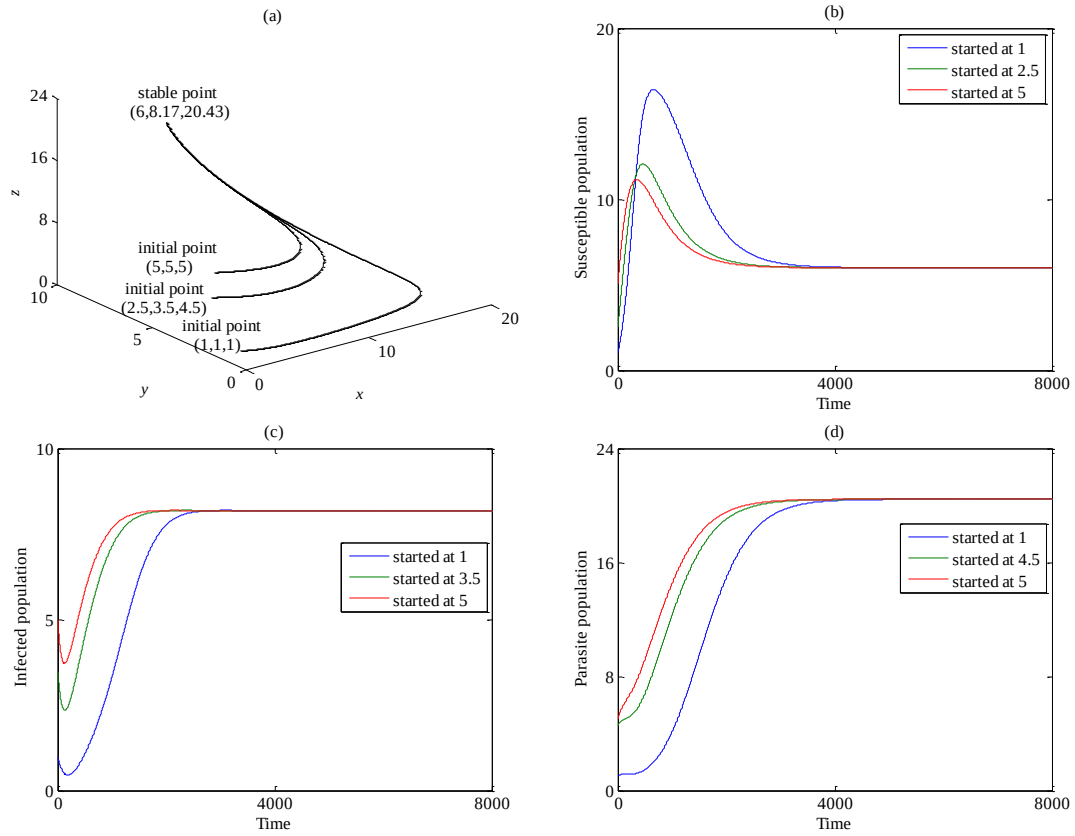

Fig. (1): The trajectories of system (1), for the data (13) started at different initial points, approaches to $E_2 = (6,8.17,20.43)$ asymptotically. (a) Phase portrait. (b) Time series of phase portrait with respect to x. (c) Time series of phase portrait with respect to y. (d) Time series of phase portrait with respect to z.

Obviously, Fig. (1) shows that, system (1) approaches asymptotically to the globally stable endemic equilibrium point E_2 from different sets of initial conditions. This is indicates to the existence of globally asymptotically stable of system (1) in the interior of positive octant, which represents the persistence of all the species too.
Now to investigate the effect of the parameters values of system (1) on the dynamical behavior of system (1), the system is solved numerically for the set of parameters values (13) with varying one parameter each time. It is observed that, for the data given in Eq. (13) with any initial point used in Fig. (1), varying the parameter value α doesn't has any effect on the dynamical behavior of system (1) and the system still approaches to a endemic equilibrium point. However, for the data given by (13) with the varying the parasite growth rate in the range $\mu \le 0.11$, the trajectories of system (1) approach asymptotically to the disease free equilibrium point E_1 as shown in the following typical figure, Fig. (2), for $\mu = 0.1$. Otherwise the solution of system (1) still approaches to the endemic equilibrium point for all other values of μ .

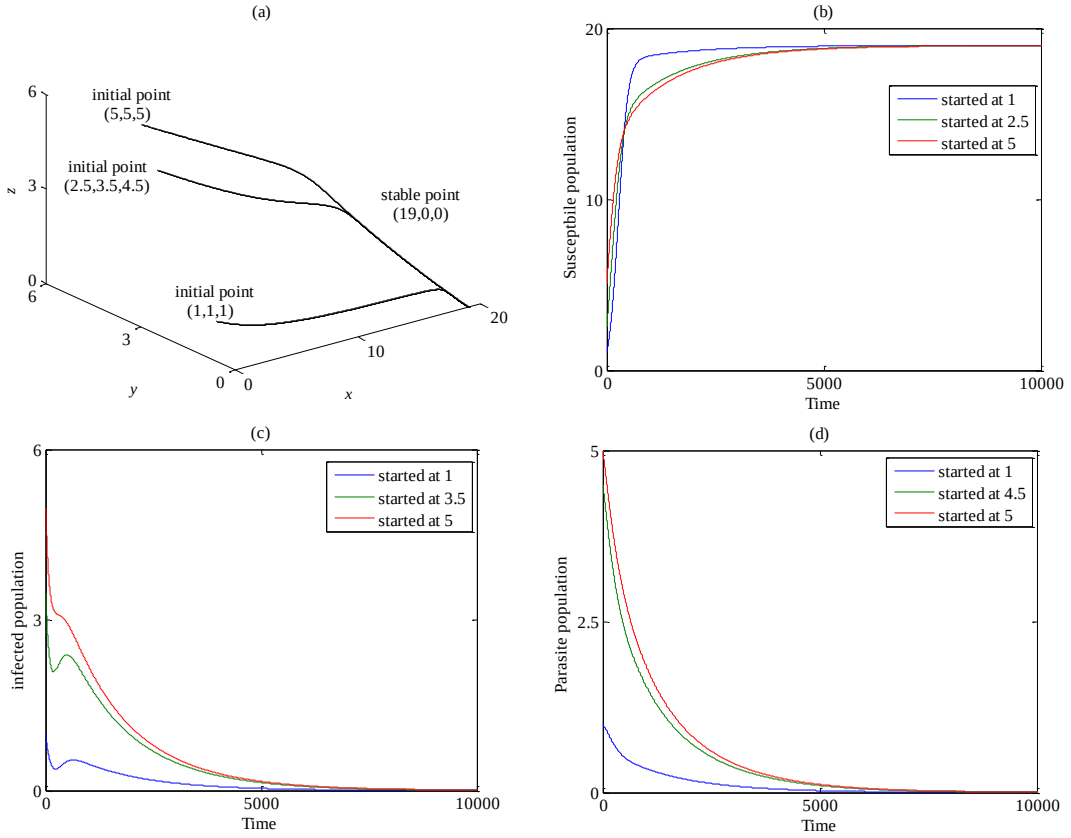

Fig. (2): The trajectories of system (1), for the data (13) with $\mu = 0.1$ started at different initial points, approaches to $E_1 = (19,0,0)$ asymptotically. (a) Phase portrait. (b) Time series of phase portrait with respect to x. (c) Time series of phase portrait with respect to y. (d) Time series of phase portrait with respect to z.

Clearly Fig. (2) shows the approaching of the solution of system (1) asymptotically to the free disease equilibrium point from different initial values as the parasite growth rate reduced to $\mu = 0.1$.

Now for the data given by Eq. (13) with $L > 0.23$, the solution of system (1) approaches asymptotically to disease free equilibrium point as shown in the following typical figure, Fig. (3), for $L = 0.25$. However the system still approaches to the endemic equilibrium point for other values of *L*.

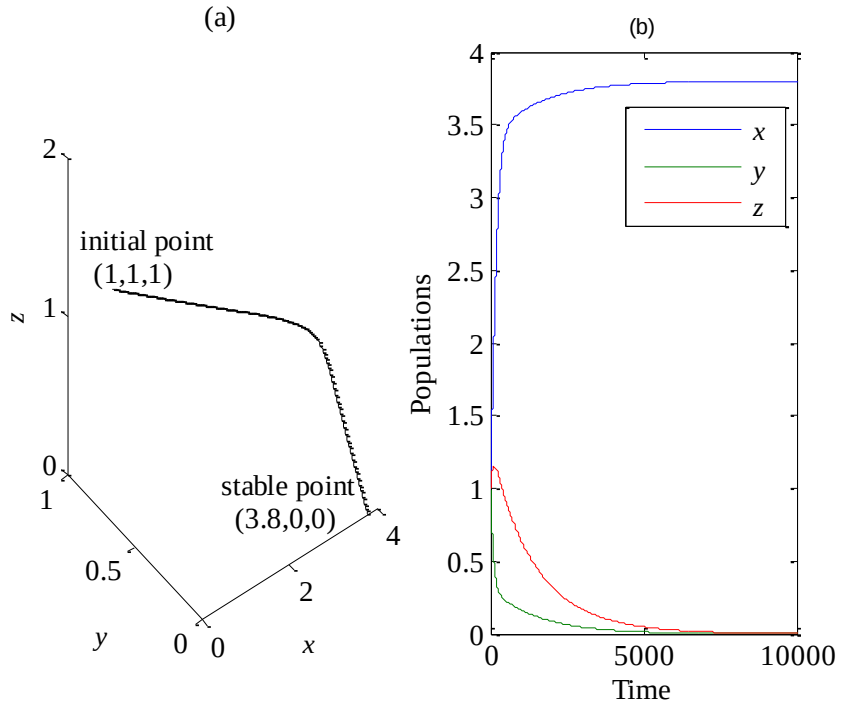

Fig. (3): (a) The trajectory of system (1), for the data (13) with $L=0.25$, approaches to $E_1=(3.8,0,0)$ asymptotically. (b) Time series of the trajectory in (a).

Again Fig.(3) shows clearly the approaching of the solution of system (1) with decreasing of carrying capacity of the host or equivalently increase the value of *L*. Now by varying the natural death rate of the host population in the range $0.61 < d_1$, keeping the rest of parameters values as in Eq. (13), system (1) approaches asymptotically to the disease free equilibrium point $E_1=(\hat{x},0,0)$ as shown in the typical figure, Fig. (4) for $d_1=0.8$. However as the intrinsic growth rate of the host population satisfies the condition $r < d_1$ for the data used in Fig. (4), its observed that the solution of system (1) approaches asymptotically to the trivial equilibrium point $E_0=(0,0,0)$ as shown in the typical figure, Fig. (5) for the date (13) with $d_1=0.8$ and $r=0.7$.

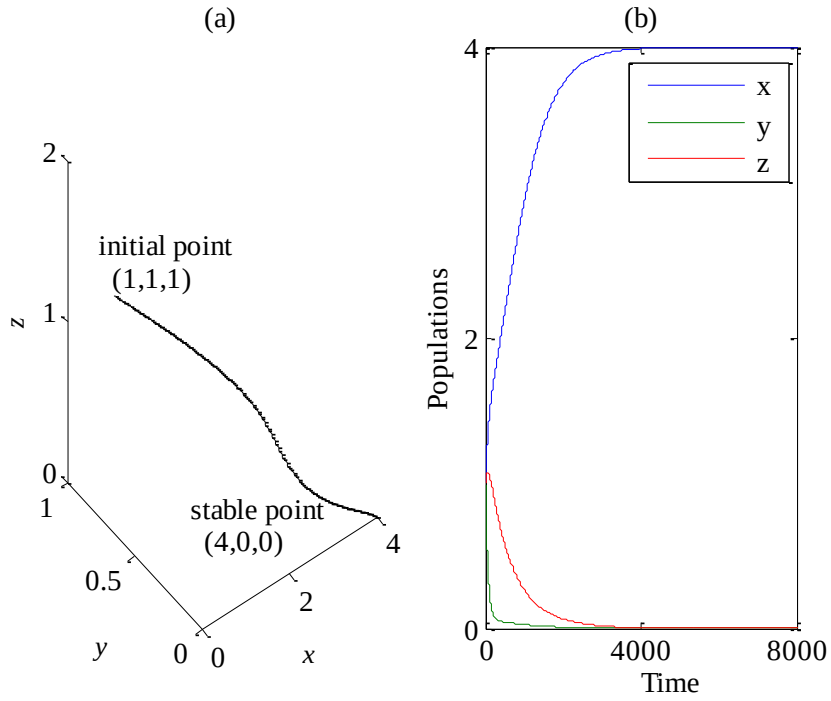

Fig. (4): (a) The trajectory of system (1), for the data (13) with $d_1 = 0.8$, approaches to $E_1 = (4,0,0)$ asymptotically. (b) Time series of the trajectory in (a).

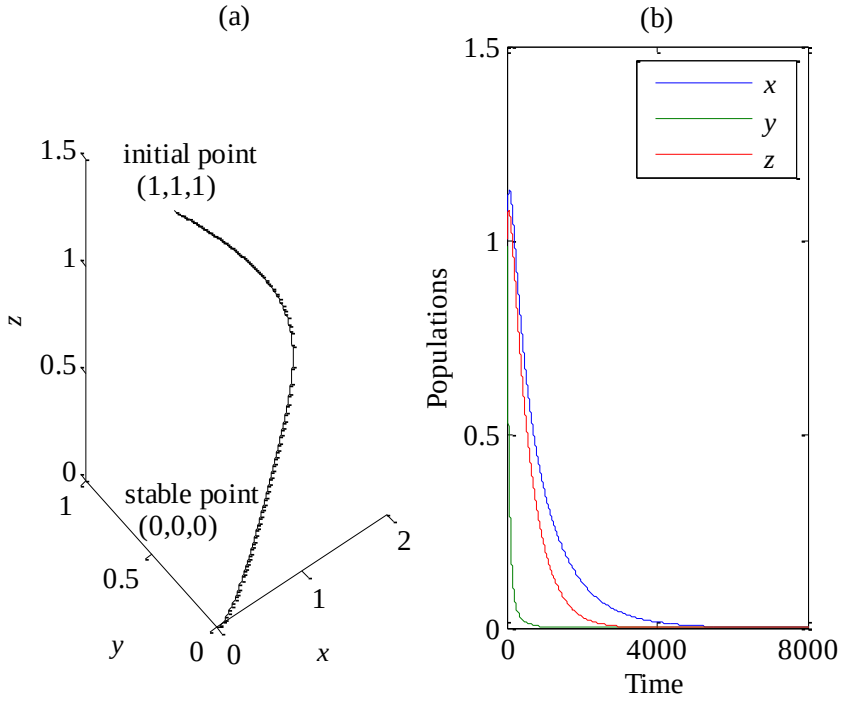

Fig. (5): (a) The trajectory of system (1), for the data (13) with $d_1 = 0.8$ and $r = 0.7$, approaches to $E_0 = (0,0,0)$ asymptotically. (b) Time series of the trajectory in (a).

Clearly, Fig. (4) explains the approaches of the solution of system (1) to the disease free equilibrium point as the natural death rate of the host increases. While, Fig. (5) shows the approaches of system (1) to the vanishing equilibrium point as the intrinsic growth rate of the host becoming less than its death rate.

Now varying the parasite natural death rate in effects the stability of the endemic equilibrium point too so that for the data in Eq. (13) with $d_2 > 0.66$ the solution of system (1) approach asymptotically to the disease free equilibrium point as shown in the typical figure, Fig. (6) below for $d_2 = 0.75$.

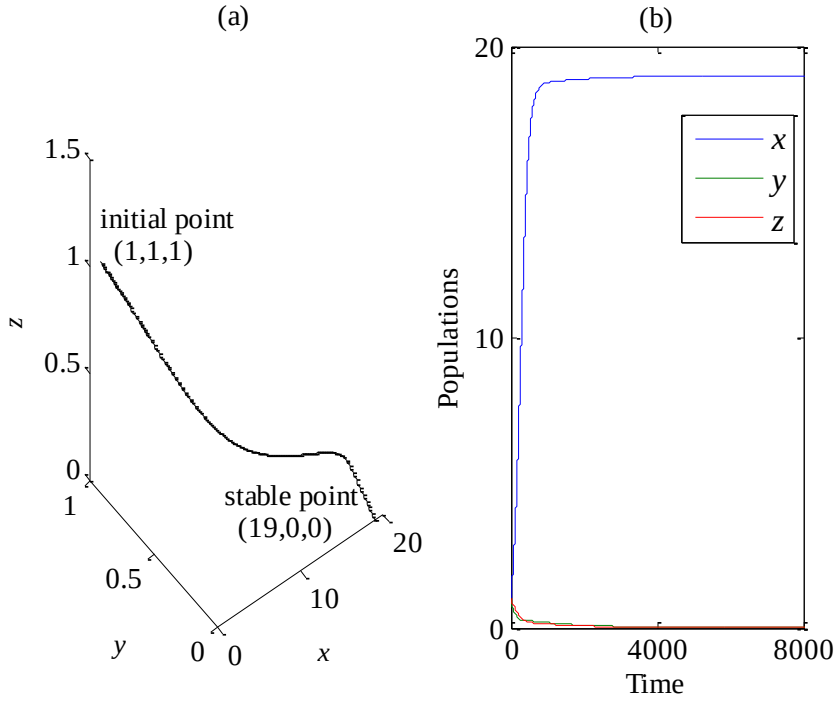

Fig. (6): (a) The trajectory of system (1), for the data (13) with $d_2 = 0.75$, approaches to $E_1 = (19,0,0)$ asymptotically. (b) Time series of the trajectory in (a).

Clearly Fig. (6) indicates to the qualitative change in stability of system (1) as the natural death rate of parasite cruses a specific value. Finally, it is observed that the parameters values β and e have similar effects on the dynamical behavior of system (1) as that of the parameter d_1. While the parameter γ has effect similar likes the effect of μ on the dynamics of system (1).

6. Conclusions and discussion

In this paper the effects of parasitic disease, represented by bilharazia, on their main host, which represented by humans, are formulated mathematically and studied analytically as well as numerically. The objective of this study is to understand the effects of all factors, which helping the spread of this type of disease and hence get the capability of control the disease.

The existence and uniqueness of solution of the proposed model are discussed. The boundedness of the solution is also studied. All possible equilibrium points with their local and global stability are investigated. The qualitative dynamical behavior as a function of varying the parameters values is studied analytically as well as numerically. Finally, for the biologically feasible set of hypothetical data as given in Eq. (13), the system (1) is solved numerically and the obtained results are explained in some typical figures and we will summarize as follows.

1. System (1) has no periodic dynamics, instead of that the solution approaching asymptotically to one of their three possible equilibrium points depending on there set of parameters values.
2. For the data given in Eq. (13), the trajectories of system (1) approached asymptotically to the global stable endemic equilibrium point E_2 in the $Int.R_+^3$, which indicates to the persistence of all the species.
3. It is observed that varying the relative fecundity of the infected person parameter value $\alpha \in (0,1)$ keeping other parameters fixed as Eq. (13) do not has any effect on the dynamical behavior of system (1) and the system still persists in the form of a globally asymptotically stable endemic equilibrium point.
4. Decreasing the parasite growth rate μ below the value 0.12 in Eq. (13) caused destabilizing to the endemic equilibrium point and the trajectories of system (1) approached asymptotically to the free disease equilibrium

point, which means losing the persistence of system (1). Otherwise the system still has a globally asymptotically stable endemic equilibrium point.

5. Increasing the inverse of host carrying capacity L above 0.23 in Eq. (13) caused destabilizing to the endemic equilibrium point and the trajectories of system (1) approached asymptotically to the free disease equilibrium point. Otherwise the system still has a globally asymptotically stable endemic equilibrium point.
6. Similarly increasing the natural death rate of host d_1 above 0.61 in Eq. (13) caused destabilizing to the endemic equilibrium point and the trajectories of system (1) approached asymptotically to the free disease equilibrium point, which indicates to occurring of the bifurcation as shown analytically. Otherwise the system still has a globally asymptotically stable endemic equilibrium point. Since the bifurcation parameter d_1^* depends on different other parameters the system undergoes bifurcations as shown in above points (4) and (5) as varying those parameters.
7. As the intrinsic growth rate r_1 of the host population becomes below its death rate d_1, then system (1) completely collapse and the trajectory approached asymptotically to the vanishing equilibrium point, which indicates to occurring of the bifurcation. Otherwise the system still approaches to the endemic point.
8. Further increasing the parasite natural death rate d_2 above 0.66 in Eq. (13) causes bifurcation in the system and the trajectory transferred from the endemic point to the disease free equilibrium point asymptotically an hence the system will losses the persistence. . Otherwise the system still approaches to the endemic point.
9. Finally, its observed that the disease death rate e and the recover rate β have similar effects on the dynamics of system (1) as that of host natural death rate d_1. While the effects of contact rate γ on the dynamics of system (1) are similar as those happened with varying parasite growth rate.

According to the above discussion, it's observed that system (1) is sensitive to varying in many of its parameters and hence there is higher possibility to control this type of disease and keep the humans safe of it.

References

1. Ross A, Bartley P, Sleigh A, et al. Schistosomiasis. N Engl J Med 2002;346:1212.
2. Anderson, R.M. & May, R.M.. Infectious Diseases of Humans. Dynamics and Control. Oxford University Press, New York. (1992)
3. Frank, A.S. Models of parasite virulence. The quarterly review of biology. 71 (1): 37-78 (1996).
4. Ebert D,. Ecology, Epidemiology, and Evolution of Parasitism in Daphnia [Internet]. Bethesda (MD): National Library of Medicine (US), National Center for Biotechnology Information. (2005).
5. Hwang T.W. and Kuang Y. Deterministic extinction effect of parasites on host populations. J. Math. Biol. 46: 17–30 (2003)
6. Eikenberry S., Hews S., Nagy J.D. And Kuang Y. the dynamics of a delay model of hepatitis b virus infection with logistic hepatocyte growth. Math. Biosci. Eng. 6 (2): 283-299 (2009).
7. Hews S.H. Eikenberry S., Nagy J.D. And Kuang Y. Rich dynamics of a hepatitis B viral infection model with logistic hepatocyte growth. J. Math. Biol. 60:573–590 (2010).
8. Wang K. and Kuang Y., Fluctuation and extinction dynamics in host-microparasite systems, Communications on pure and applied analysis,10 (5): 1537-1548 (2011)