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Numerical Solution For Mixed Volterra-Fredholm Integral Equations Of The Second Kind By Using Bernstein Polynomials Method

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Abstract

In this paper, we have used Bernstein polynomials method to solve mixed Volterra-Fredholm integral equations(VFIE's) of the second kind, numerically. First we introduce the proposed method, then we used it to transform the integral equations to the system of algebraic equations. Finally, the numerical examples illustrate the efficiency and accuracy of this method.

Keywords: Bernestein polynomials method, linear Volterra-Fredholm integral equations.

1. Introduction

Integral equations are encountered in various fields of science and numerous applications in elasticity, plasticity, heat and mass transfer, approximation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc. Exact (closed-form) solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science (Owaied 2010).

A variety of analytic and numerical methods have been used to solve Volterra-Fredholm integral equations, for Example, Repeated Trapezoidal method and the repeated Simpson's 1/3 method are used to solve linear Volterra-Fredholm integral equations (LVFIE's) of the first and second kinds in (Majeed & Omran 2008).(Al-Jarrah & Lin 2013) presented Scaling function interpolation method to solve Linear Volterra-Fredholm integral equations (VFIE's), were scaling functions and Wavelet functions are the key element of Wavelet methods which are a very useful tool in solving integral equations. Numerical solution for Fredholm–Volterra integral equation of the second kind by using collocation and Galerkin methods is found in (Hendi & Albugami 2010). A computational methods for Volterra-Fredholm integral equations are investigated in(Lechosław 2002).(Fakhrodin 2015)used a Chebyshev wavelet operational method for solving stochastic Volterra-Fredholm integral equations. The Approximate Solutions of Nonlinear Volterra-Fredholm Integral Equations can be found in(Ahmadi Shali. et al 2012).Numerical Solution Of Linear Volterra-Fredholm Integral Equations Using Lagrange Polynomials is employed in(Muna & Iman 2014). Application of Chebyshev polynomials for solving nonlinear Volterra-Fredholm integral equations system and convergence analysis are investigated in(Ezzati & Najafalizadeh 2012) .A New Computational Method for Volterra-Fredholm Integral Equations can be found in(Maleknejad & Hadizadeh 1999). (Mashayekhi.et al 2014) presented Solution of the Nonlinear Mixed Volterra-Fredholm Integral Equations by Hybrid of Block-Pulse Functions and Bernoulli Polynomials.

In recent years, many researchers have been successfully applying Bernstein polynomials method (BPM) to various linear and nonlinear integral equations . For example, Bernstein polynomials method is applied to find an approximate solution for Fredholm integro-Differential equation and integral equation of the second kind in(AL-Juburee 2010). (Al-A'asam 2014) used Bernstein polynomials for deriving the modified Simpson's 3/8 ,and the composite modified Simpson's 3/8 to solve one dimensional linear Volterra integral equations of the second kind. Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations can be found in(Hosseini.et al 2014) .This method is used to an approximate solution of nonlinear Fredholm integral equations in(Muna & Khawla 2011).The propose method is applied to find an approximate solution to initials values problem for high-order nonlinear Volterra-Fredholm integro differential equation of the second kind in(Intisar 2015). In this paper, use Bernstein polynomial method (BPM) to solve mixed Volterra-Fredholm integral equations (VFIE's) of the second kind:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt \qquad (1)$$

Where $a \le x \le b$, λ_1, λ_2 are scalar parameters, $f(x)$, $k_1(x,t)$ and $k_2(x,t)$ are continuous functions and $u(x)$ is the unknown function to be determine.

2. Bernstein Polynomials Method

Polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions. The Bernstein polynomials of degree *n* are defined by(Joy 2000):

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \qquad \text{for} \quad i = 0,1,2,\cdots,n \qquad (2)$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \qquad (3)$$

(n) is the degree of polynomials, (i) is the index of polynomials and (t) is the variable.
The exponents on the (t) term increase by one as (i) increases, and the exponents on the $(1-t)$ term decrease by one as (i) increases. The Bernstein polynomials of degree (n) can be defined by blending together two Bernstein polynomials of degree $(n-1)$ That is, the $k^{th}n^{th}$-degree Bernstein polynomial can be written as(Joy 2000):

$$B_k^n(t) = (1-t)B_k^{n-1}(t) + tB_{k-1}^{n-1}(t) \qquad (4)$$

Bernstein polynomials of degree (n)can be written in terms of the power basis. This can be directly calculated using the equation (2) and the binomial theorem as follows(Joy 2000):

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i}\binom{i}{k} t^i \qquad (5)$$

Where the binomial theorem is used to Expand $(1-t)^{n-k}$.

3. A Matrix Representation For Bernstein Polynomials

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions (AL-Juburee 2010).

$$B(t) = c_0 B_0^n(t) + c_1 B_1^n(t) + c_2 B_2^n(t) + \cdots + c_n B_n^n(t) \qquad (6)$$

The dot product of two vectors

$$B(t) = \begin{bmatrix} B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad (7)$$

which can be converted to the following form:

$$B(t)=\begin{bmatrix}1 & t & t^2\cdots & t^n\end{bmatrix}\begin{bmatrix}b_{00} & 0 & 0 & \cdots & 0\\ b_{10} & b_{11} & 0 & \cdots & 0\\ b_{20} & b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix} \quad (8)$$

where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials, we note that the matrix in this case is lower triangular.

4. Solution For Mixed Volterra-Fredholm Integral Equations Of The Second Kind

In this section Bernstein polynomials method is proposed to find the solution for Volterra-Fredholm integral equations of the second kind.
Consider the mixed Volterra-Fredholm integral equations of the second kind in equation (1):

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt \quad (9)$$

Let $u(x) = B_i^n(t)$ then,

$$u(x)=\begin{bmatrix}B_0^n(x) & B_1^n(x) & B_2^n(x) & \cdots & B_n^n(x)\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix} \quad (10)$$

Substituting (10) into equation (9) we get:

$$\begin{bmatrix}B_0^n(x) & B_1^n(x) & B_2^n(x) & \cdots & B_n^n(x)\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix} = f(x)+$$

$$\lambda_1\int_a^x k_1(x,t)\begin{bmatrix}B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t)\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}dt+\lambda_2\int_a^b k_2(x,t)\begin{bmatrix}B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t)\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}dt \quad (11)$$

Applying equation (8) into equation(11)we have:

$$\begin{bmatrix} B_0^n(x) & B_1^n(x) & B_2^n(x) & \cdots & B_n^n(x) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = f(x) +$$

$$\lambda_1 \int_a^x k_1(x,t) \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt +$$

$$\lambda_2 \int_a^b k_2(x,t) \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt \qquad (12)$$

Now to find all integration in equation (12). Then in order to determine $c_0, c_1, \cdots, c_n$ we need *n* equations. Now chose $x_i, i = 1, 2, 3, \cdots, n$ in the interval $[a, b]$, which gives *n* equations. Solve the *n* equations by Gauss elimination to find the values of $c_0, c_1, \cdots, c_n$.The following algorithm summarizes the steps for finding the solution for the second kind of mixed Volterra-Fredholm integral equations.

5- Algorithm (BPM)

Input: $(f(x), k(x,t), y(x), a, b, x, \lambda_1, \lambda_2)$

Output: polynomials of degree *n*

Step1: Choice *n* the degree of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \qquad for \;\; i = 0, 1, 2, \cdots, n$$

Step2: Put the Bernstein polynomials in linear mixed Volterra-Fredholm integral equations of second kind.

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t) \begin{bmatrix} B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt +$$

$$\lambda_2 \int_a^b k_2(x,t) \begin{bmatrix} B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$$

Step3:

Compute Volterra integral $\lambda_1 \int_a^x k_1(x,t) \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$

Compute Fredholm integral $\lambda_2 \int_a^b k_2(x,t) \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$

Step4:

Compute $c_0, c_1, \cdots, c_n$, where $x_i, i = 1, 2, 3, \cdots, n$, $x_i \in [a, b]$

End

6- Numerical Experiments

In this section we apply **BPM** to solving linear mixed Volterra-Fredholm integral equations of second kind. Also we presented four examples, the first example was solved by Lagrange Polynomials method(Mustafa & Ghanim 2014), example two was solved by Scaling Function Interpolation Method(Al-Jarrah & Lin 2013).and the last two examples are solved by modified adman's decomposition method and series solution method respectively(Wazwaz 2011). The computations associated with these examples were performed using Matlab ver.2013a

Exampe1 Consider the linear mixed Volterra-Fredholm integral equation of the second kind (Mustafa & Ghanim 2014).

$$u(x) = f(x) + \int_0^x (x - t)u(t)dt + \int_0^2 xt\, u(t)$$

Where $f(x) = 2\cos(x) - x\cos(2) - 2x\sin(2) + x - 1$

With the exact solution $u(x) = cos(x)$, $x \in [0,2]$

Table 1. Numerical results for Example 1 by using BPM.

t	*Yexact*	*Bernstein polynomials method*	
		yapp,n=3	*Error ,n=3*
0	1.0000	1.0000	0
0.2000	0.9801	0.9841	0.0000
0.4000	0.9211	0.9234	0.0000
0.6000	0.8253	0.8242	0.0000
0.8000	0.6967	0.6930	0.0000
1.0000	0.5403	0.5360	0.0000
1.2000	0.3624	0.3595	0.0000
1.4000	0.1700	0.1699	0.0000
1.6000	-0.0292	-0.0266	0.0000
1.8000	-0.2272	-0.2235	0.0000
2.0000	-0.4161	-0.4146	0.0000

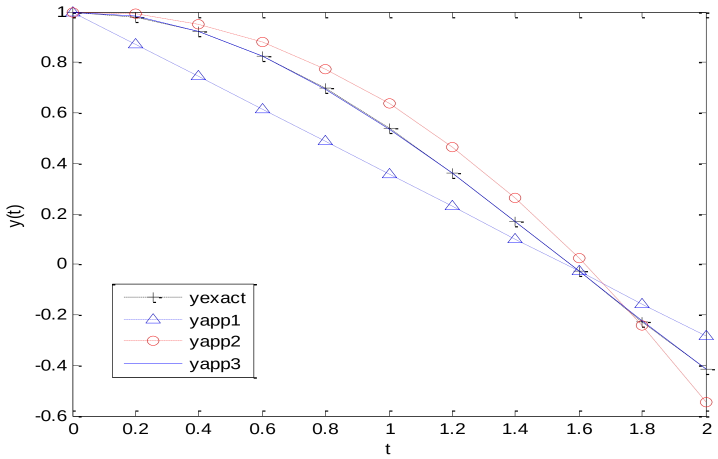

Figure 1. Approximate solutions and exact solution for Example 1 by using BPM.

Example 2 Consider the linear mixed Volterra-Fredholm integral equation of the second kind (Al-Jarrah & Lin 2013)

$$u(x) = f(x) + \int_0^x xt\, u(t)dt + \int_0^1 xt\, u(t)dt$$

Where $f(x) = \frac{2}{3}x - \frac{1}{3}x^4$, with the exact solution $u(x) = x$, $x \in [0,1]$

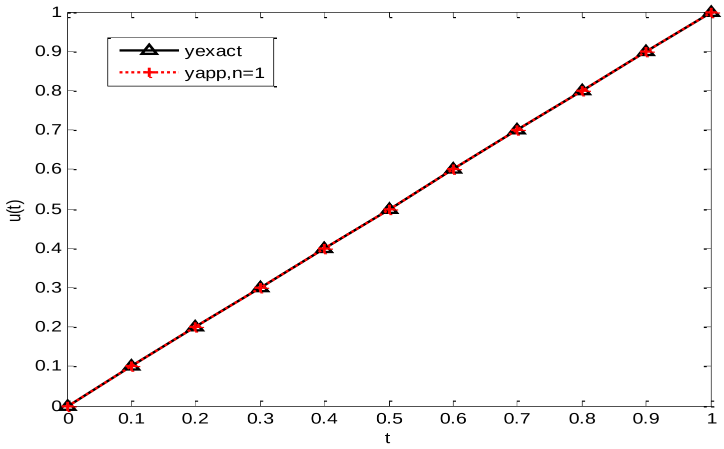

Figure 2. Approximate solution and exact solution for Example 2 by using BPM.

Example 3 Consider the linear mixed Volterra-Fredholm integral equation of the second kind (Wazwaz 2011).

$$u(x) = 3x + 4x^2 - x^3 - x^4 - 2 + \int_0^x tu(t)dt + \int_{-1}^1 tu(t)dt \ ,$$

And the exact solution is $u(x) = 3x + 4x^2$.

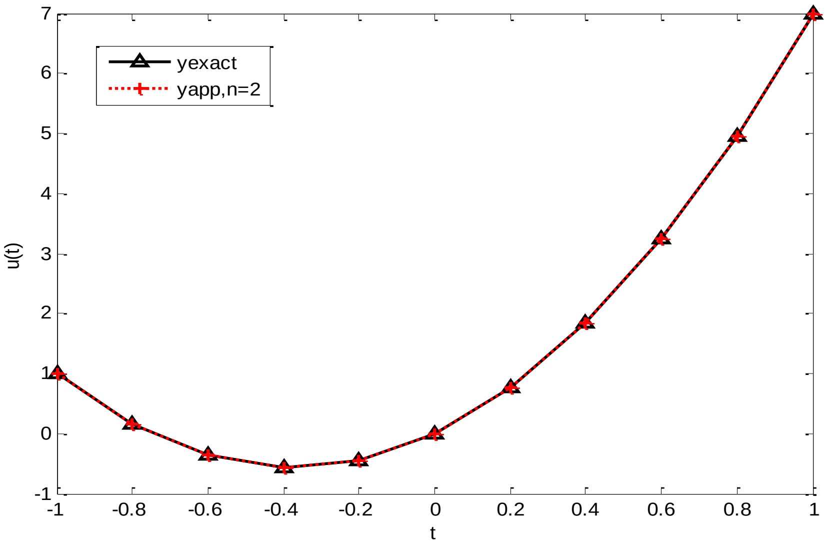

Figure 3. Approximate solutions and exact solution for Example 3 by using BPM.

Example 4 Consider the linear mixed Volterra-Fredholm integral equation of the second kind (Wazwaz 2011)

$$u(x) = e^x - 1 - x + \int_0^x u(t)dt + \int_0^1 xu(t)dt \ ,$$

and the exact solution is $u(x) = xe^x$.

Table 3. Numerical results for Example 4 by using BPM.

t	*yexact*	*Bernstein polynomials method*	
		yapp,n=4	*Error ,n=4*
0	0	0	0
0.1000	0.1105	0.1102	0.0000
0.2000	0.2443	0.2441	0.0000
0.3000	0.4050	0.4050	0.0000
0.4000	0.5967	0.5969	0.0000
0.5000	0.8244	0.8245	0.0000
0.6000	1.0933	1.0934	0.0000
0.7000	1.4096	1.4098	0.0000
0.8000	1.7804	1.7808	0.0000
0.9000	2.2136	2.2142	0.0000
1.0000	2.7183	2.7185	0.0000

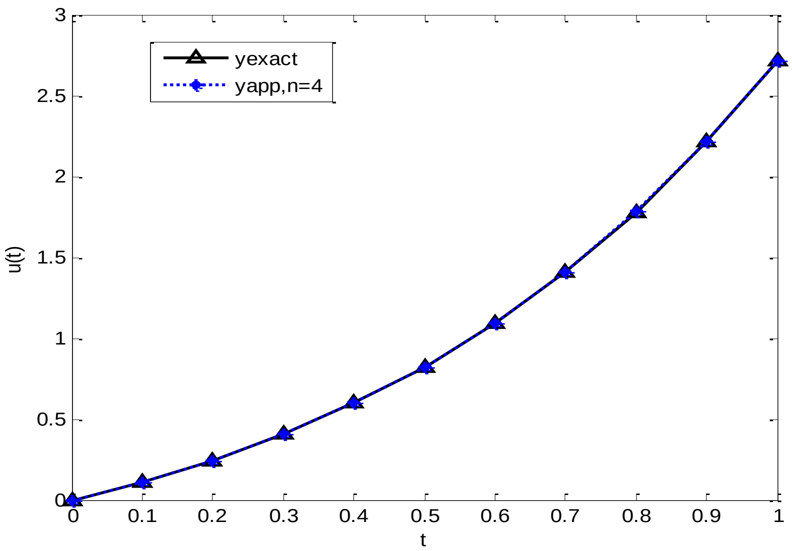

Figure 4. Approximate solutions and exact solution for Example 4 by using BPM.

7. CONCLUSIONS

In this paper, we have successfully used **BPM** for solving linear mixed Volterra-Fredholm integral equations of the second kind .It is apparently seen that **BPM** is a powerful and easy-to-use analytic tool for finding the solutions for integral equations. The integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the numerical solution, for this purpose the presented method can be proposed. From numerical examples it can be seen that the proposed method is efficient and accurate to estimate the solution of these equations. Also we noted that when the degree of Bernstein polynomials is increasing the errors decreasing to smaller values.

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