

## Special Cases for Numerical Radius and Spectral Radius Inequalities

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### Abstract

In this paper, the aim of this study is to find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space, Finally, some new results about this subject are obtained.

*Keywords:* Spectral norm, Numerical radius, Spectral radius.

### 1.Introduction

Let  $B(H)$  denote the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $H$  with inner product  $\langle \cdot, \cdot \rangle$ . For  $A \in B(H)$ , let  $\omega(A)$  and  $\|A\|$  denote the numerical radius and the usual operator norm of  $A$ , respectively. It is well known that  $\omega(\cdot)$  defines a norm on  $B(H)$ , and that for every  $A \in B(H)$ ,  $w(A) \leq \|A\|$

The concepts of numerical range and numerical radius play an important role in various fields of Con-temporary Mathematics, including Operator Theory, Operator Trigonometry, Numerical Analysis and other see [1], [7],.....

#### Theorem 1.1[7]

Let  $X_i \in B(H)$  ( $i = 1, 2, \dots, n$ ). Then

$$\omega^r \left( \sum_{i=1}^n X_i \right) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n \left( |X_i|^{2r\alpha} + |X_i^*|^{2r(1-\alpha)} \right) \right\| \quad \forall \alpha \in (0,1), r \geq 1 \dots (1)$$

#### Theorem 1.2 [1]

Let  $A$  and  $B$  be self-adjoint operators in  $B(H)$ , and  $r \geq 2$ . Then

$$\omega^r(A + B) \leq 2^{r-2} \left( \| |A + B|^r + |A - B|^r \| \right). \quad (2)$$

**Theorem 1.3[6]**

If  $A, B \in B(H)$ , then

$$r(AB) \leq \frac{1}{4} \left( \|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4 \min(\|A\| \|BAB\|, \|B\| \|ABA\|)} \right) \quad (3)$$

**Theorem 1.4[7]**

Let  $A, B, C, D, S, T \in B(H)$ . Then

$$\omega^r(ATB + CSD) \leq 2^{r-2} \left( \| (A | T^* | A^*)^r + (B^* | T | B)^r + (C | S^* | C^*)^r + (D^* | S | D)^r \| \right). \quad (5)$$

In the following results , we find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space

**2.Main results**

**Theorem 2.1**

If  $A$  is an operator in  $B(H)$ , then

$$w(A) \leq \left\| |A|^2 \right\|^{\frac{1}{2}} \quad \dots\dots(5)$$

**Proof**

Let  $r=2$  in (2) ,we get the result.

**Theorem 2.2**

Let  $X_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) .Then

$$\omega\left(\sum_{i=1}^n X_i\right) \leq \frac{1}{2} \left\| \sum_{i=1}^n (|X_i| + |X_i^*|) \right\| \quad (6)$$

**Proof**

*If we put  $r=1$  in (1), then we obtain the result.*

**Corollary 2.1**

*If  $X \in B(H)$ , then*

$$\omega(X) \leq \frac{1}{2} \left\| |X| + |X^*| \right\| \quad (7)$$

**Proof**

*By taking  $X_i = X, \forall i = 1, 2, \dots, n$  in (6), so we get the result.*

**Theorem 2.3**

*If  $A$  is positive operator in  $B(H)$  and  $n$  is any scalar in  $R$ , then*

$$r(A) \leq \|A^{n+1}\|^{\frac{1}{n+1}} \quad (8)$$

**Proof**

*Take  $B = A^n$  in (3), we obtain*

$$r(A^{n+1}) \leq \frac{1}{4} \left( \|A^{n+1}\| + \|A^{n+1}\| + \sqrt{(\|A^{n+1}\| - \|A^{n+1}\|)^2 + 4 \min(\|A\| \|A^{2n+1}\|, \|A^n\| \|A^{n+2}\|)} \right)$$

$$\leq \frac{1}{4} \left( 2 \|A^{n+1}\| + \sqrt{4 \min(\|A\| \|A^{2n+1}\|, \|A^n\| \|A^{n+2}\|)} \right)$$

$$\leq \frac{1}{2} \left( \|A^{n+1}\| + \sqrt{\|A\|^{2n+2}} \right)$$

We know that  $r(A^{n+1}) = r(A)^{n+1}$ , so we get the result.

### Corollary 2.2

If  $A$  is positive operator in  $B(H)$ , then

$$r(A) \leq \|A^2\|^{\frac{1}{2}} \quad (9) \quad \text{In general for}$$

$$n \geq 1, r(A) \leq \|A^n\|^{\frac{1}{n}}$$

### Theorem 2.4

Let  $A, B, C, D, S, T \in B(H)$ . Then

$$\omega(ATB + CSD) \leq \left\| (A|T^*|A^*)^2 + (B^*|T|B)^2 + (C|S^*|C^*)^2 + (D^*|S|D)^2 \right\|^{\frac{1}{2}} \quad (10).$$

**Proof**

Let  $r=2$  in (5), we get the result.

### Corollary 2.3

Let  $A \in B(H)$ . Then

$$\omega(A^3) \leq \frac{1}{2} \left\| 4(A|A^*|A^*)^2 \right\|^{1/2} \quad (11).$$

## Proof

Let  $A=B=C=D=S=T$  in (10), we get the result.

## Open Problems

The first open problem is possible to complement the all bounds (5,7,8,9,11) by giving an upper bound estimate for the zeros of

$p(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$  of degree  $n \geq 2$ , with complex coefficients  $a_1, a_2, \dots, a_n$ , where  $a_1 \neq 0$ .

The second open problem is possible to complement the upper all bounds (5,7,8,9,11) by giving a lower bound estimate for the zeros of  $p$ . To see this, observe that the zeros of the polynomial

$q(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$  are the reciprocals of those of  $p$ . Thus, applying the upper bound (9) to the zeros of

$q$  yields the desired lower bound estimate for the zeros of  $p$ . this enables us to present a new annulus containing the zeros of  $p$ ,

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