

# Stability of convergence theorems of the Noor iteration method for an enumerable class of continuous hemi contractive mapping in Banach spaces

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**Abstract:** The purpose of this is to study the Noor iteration process for the sequence  $\{x_n\}$  converges to a common fix point for enumerable class of continuous hemi contractive mapping in Banach spaces.

**Key words:** Stability, Noor iterations, Hemiccontractive mapping, Convergence theorem Continuous pseudocontractive mapping.

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**Introduction:** Let  $E$  be a real Banach space and let  $J$  denote the normalized duality mapping from  $E$  to  $E^*$  and defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}; \text{ for all } x \in E,$$

Where  $E^*$  denotes the dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  denotes the generalization duality pair.

It is well known that if  $E^*$  is strictly convex then  $J$  is single-valued. In the sequel, we shall denote the single-valued duality mapping by  $j$ . Let  $K$  be a nonempty closed convex subset of Banach space  $E$  and  $T: K \rightarrow K$  be a self-mapping of  $K$ .

**Definition 1.1 [1]** (i) A mapping  $T$  with domain  $D(T)$  and range  $R(T)$  in a Banach space is called pseudocontractive mapping, if for all  $x, y \in D(T)$ , there exists  $j(x - y) \in J(x - y)$  such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \quad (1)$$

(ii) A mapping  $T$  with domain  $D(T)$  and range  $R(T)$  in  $E$  is called a hemicontractive mapping if

$F(T) \neq \emptyset$  and for all  $x \in D(T)$   $x^* \in F(T)$  such that,

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2$$

(iii) A mapping  $T: K \rightarrow K$  is called  $L$ -Lipschitzian there exists  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ For all } x, y \in K$$

**Definition 1.2 [3]** If  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  are sequences of real numbers in  $[0,1]$ . For arbitrary  $x_0 \in E$ , Let  $\{x_n\}_{n=0}^{\infty}$  be the Noor iteration and defined by,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tq_n$$

$$q_n = (1 - \beta_n)x_n + \beta_n Tr_n$$

$$r_n = (1 - \beta_n)x_n + \beta_n Tr_n$$

Where  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$  and  $\{r_n\}_{n=0}^{\infty}$  are sequences of real numbers in  $[0, 1]$ .

**Lemma 1.3 [2]** Let  $E$  be a real uniformly convex Banach space,  $K$  is nonempty closed convex subset of  $E$  and  $T$  a continuous pseudocontractive mapping of  $K$ , then  $I - T$  is demiclosed at zero, that is, for all sequences  $\{x_n\} \subset K$  with  $x_n \rightarrow p$  and  $x_n - Tx_n \rightarrow 0$  it follows that  $p = Tp$

**Lemma 1.4 [4,5]** Let  $\delta$  be a number satisfying  $0 \leq \delta < 1$  and  $\{\epsilon_n\}$  a positive sequence satisfying  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Then, for any positive sequence  $\{u_n\}$  satisfying:

$$u_{n+1} \leq \delta u_n + \epsilon_n, \text{ It follows that } \lim_{n \rightarrow \infty} u_n = 0$$

## 2. Main Results

**Theorem 2.** Let  $\{T_n\}_{n=1}^{\infty}$  be defined as above and  $F := \bigcap_{i=1}^{\infty} F(T_n) \neq \emptyset$  and let  $(E, \|\cdot\|)$  be a Banach space,  $T: E \rightarrow E$  a self map of  $E$  with a fixed point  $p$ , satisfying the contractive condition

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 \text{ For } x_0 \in E.$$

Let  $\{x_n\}_{n=1}^{\infty}$  is converge to  $p$  and defined by the iteration (1.2) where  $\{\alpha_n\}_{n=1}^{\infty}$  is a real sequence in  $(0, 1)$  and define as  $\epsilon_n = \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n Tq_n\|$  Then

- (i)  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for  $p \in F$ ;
- (ii)  $\lim_{n \rightarrow \infty} d(x_n, F) = \{ \inf \|x_n - p\| : p \in F \}$ ;
- (iii)  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_n\}_{n=1}^{\infty}$  if and only if  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$

**Proof** Let  $p \in F$  and  $n \geq 1$  by 1.1 we choose  $j(x_n - p) \in J(x_n - p)$  such that

$$\|x_{n+1} - p\|^2 = \langle x_{n+1} - p, j(x_{n+1} - p) \rangle$$

$$\|x_{n+1} - p\| \leq \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n Tq_n\| + \|(1 - \alpha_n)x_n + \alpha_n Tq_n - p\|$$

$$= \epsilon_n + \|(1 - \alpha_n)x_n + \alpha_n Tq_n - ((1 - \alpha_n) + \alpha_n)p\|$$

$$= \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|Tq_n - p\|$$

$$\leq \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|Tq_n - p\|$$

$$\begin{aligned}
 &= \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|p - Tq_n\| \\
 &\leq \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n a \|p - q_n\| \\
 &= \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n a \|q_n - p\| \tag{1}
 \end{aligned}$$

For the estimate of  $\|q_n - p\|$  in (1) we get

$$\begin{aligned}
 \|q_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tr_n - p\| \\
 &= \|(1 - \beta_n)x_n + \beta_n Tr_n - ((1 - \beta_n) + \beta_n)p\| \\
 &= \|(1 - \beta_n)(x_n - p) + \beta_n(Tr_n - p)\| \\
 &\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|Tr_n - p\| \\
 &= (1 - \beta_n) \|x_n - p\| + \beta_n \|p - Tr_n\| \\
 &\leq (1 - \beta_n) \|x_n - p\| + \beta_n a \|p - r_n\| \\
 &= (1 - \beta_n) \|x_n - p\| + \beta_n a \|r_n - p\| \tag{2}
 \end{aligned}$$

Substituting (2) into (1) gives

$$\|x_{n+1} - p\| \leq \epsilon_n + (1 - (1 - a)\alpha_n - \alpha_n \beta_n a) \|x_n - p\| + \alpha_n \beta_n a^2 \|r_n - p\| \tag{3}$$

For  $\|r_n - p\|$  in (3) we have,

$$\begin{aligned}
 \|r_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - p\| \\
 &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - ((1 - \gamma_n) + \gamma_n)p\| \\
 &= \|(1 - \gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\
 &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n \|Tx_n - p\| \\
 &= (1 - \gamma_n) \|x_n - p\| + \gamma_n \|p - Tx_n\| \\
 &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n a \|p - x_n\| \\
 &= (1 - \gamma_n + \gamma_n a) \|x_n - p\| \tag{4}
 \end{aligned}$$

Substituting (4) into (3) and using lemma 1.3

$$\begin{aligned}
 &= \epsilon_n + (1 - (1 - a)\alpha_n - \alpha_n \beta_n a) \|x_n - p\| + \alpha_n \beta_n a^2 (1 - \gamma_n + \gamma_n a) \|x_n - p\| \\
 &= \epsilon_n (1 - (1 - a)\alpha_n - (1 - a)\alpha_n \beta_n a - (1 - a)\alpha_n \beta_n \gamma_n a^2) \|x_n - p\| \\
 &\leq (1 - (1 - a)\alpha - (1 - a)\alpha\beta a - (1 - a)\alpha\beta\gamma a^2) \|x_{n-1} - p\| + \epsilon_n
 \end{aligned}$$

Observe that

$$0 \leq (1 - (1 - a)\alpha - (1 - a)\alpha\beta a - (1 - a)\alpha\beta\gamma a^2) < 1 \quad (5)$$

Therefore, taking the limit as  $n \rightarrow \infty$  of both sides of the inequality (5) and using lemma 1.6 we get

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0, \text{ That is } \lim_{n \rightarrow \infty} x_n = p$$

$$\text{By theorem 1.2 } \|x_n - p\| \leq \|x_{n-1} - p\|$$

Taking infimum over all  $p \in F$ , we have,

$$d(x_n, F) = \inf_{p \in F} \|x_n - p\| \leq \inf_{p \in F} \|x_{n-1} - p\| = d(x_{n-1}, F),$$

Thus  $\lim_{n \rightarrow \infty} d(x_n, F)$  exist. We finally prove (iii). suppose that  $x_n \rightarrow p \in F$  from (ii) and

$$d(x_n, F) \leq \|x_n - p\| \rightarrow 0, \text{ We have } \lim_{n \rightarrow \infty} d(x_n, F) = 0 \text{ for } n, m \in \mathbb{N} \text{ and } p \in F, \text{ it follows}$$

From (1.3) that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2 \|x_n - p\|$$

Consequently,

$$\|x_{n+m} - x_n\| \leq 2 \|x_n - p\| \rightarrow 0$$

Therefore  $\{x_n\}$  is a Cauchy sequence. Suppose  $\lim_{n \rightarrow \infty} x_n = u$  for some  $u \in E$ . then

$$d(u, F) = \lim_{n \rightarrow \infty} d(x_n, F) = 0$$

Since  $F$  is closed set,  $u \in F$

So, Noor iteration process is  $T$ -stable.

Thus, the stability of Noor iteration considerable for finding fixed point for enumerable class of continuous hemi contractive mapping in Banach spaces.

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