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CHARACTERIZATION AND ESTIMATION OF TRANSMUTED KUMARASWAMY DISTRIBUTION

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Abstract: In this article, a generalization of the Kumaraswamy distribution so-called transmuted Kumaraswamy distribution is proposed and studied. We will use the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distributions taking Kumaraswamy distribution as the base value distribution by introducing a new parameter that would offer more distributional flexibility. We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior.

Keywords: Kumaraswamy distribution, Reliability Function, Maximum Likelihood Estimation, Order Statistics

1. Introduction

Kumaraswamy (1980) proposed a two-parameter Kumaraswamy distribution on (0, 1), and denoted by Kum (θ,α). Its cumulative distribution function is given by

$$G(x;\theta,\alpha)=1-(1-x^{\theta})^{\alpha}\ ,x\in(0,1) \qquad (1.1)$$

and the probability density function (pdf) corresponding to (1.1) is

$$g(x;\theta,\alpha)=\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}\quad ,x\in(0,1)\ ,\theta,\alpha>0 \qquad (1.2)$$

Here θ and α are the shape parameters. Kumaraswamy (1980) and Ponnambalam, et al., (2001) have pointed out that depending on the choice of the parameter θ and α. Kum araswamy's distribution can be used to approximate many distributions, such as uniform, triangular, or almost any single model distribution and can also reproduce results of beta distribution. Kumaraswamy's distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures, hydrological data such as daily rain fall, daily stream flow, etc.

The purpose of this study is to present a new generalization of Kumaraswamy distribution called the transmuted Kumaraswamy distribution. We will derive the proposed distribution using the Quadratic rank transmutation map proposed by Shaw et al. (2009).

A random variable X is said to have transmuted distribution if its cumulative distribution function is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \quad ,|\lambda| \le 1 \tag{1.3}$$

where $G(x)$ is the cdf of the base distribution. If we put $\lambda = 0$, we get the base distribution. Afaq et al. (2014) studied transmuted inverse Rayleigh distribution and discussed its properties. Aryal and Tsokos (2009,2011) studied the transmuted extreme distributions. The authors provided the mathematical characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data sets. Faton Merovci (2013) studied the transmuted Rayleigh distribution, Ashouret et al (2013). studied the transmuted exponentiated Lomax distribution and discussed some properties of this family. In the present study we will provide the mathematical formulation of the transmuted Kumaraswamy distribution and some of its structural properties.

2. Transmuted Kumaraswamy Distribution

In this section we studied the transmuted Kumaraswamy distribution and the sub-models of this distribution. Now using (1.1) in (1.3), we have the cdf of transmuted Kumaraswamy distribution given by

$$F(x;\theta,\alpha,\lambda) = 1-(1-x^{\theta})^{\alpha}\left[1+\lambda(1-x^{\theta})^{\alpha}\right] \tag{2.1}$$

Hence the pdf of TMIR distribution with parameters $\alpha, \theta \; and \; \lambda$ is given as

$$f(x;\theta,\alpha,\lambda) = \theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right] \tag{2.2}$$

Particular cases

1. If we take $\lambda = 0$ in (2.2), we get

$$f(x;\theta,\alpha) = \theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}$$

which is pdf of Kumaraswamy distribution.

2 If we take $\alpha = 1, \theta = 1 \; and \; \lambda = 0$ in (2.2), we get

$$f(x) = 1$$

which is pdf of Uniform distribution on [0,1].

3. Reliability Analysis

In this sub-section, we present the reliability function and the hazard function for the proposed transmuted Kumaraswamy distribution. The reliability function is otherwise known as the survival or survivor function. It is the probability that a system will survive beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (cdf).

The survivor function is given by

$$R(x) = 1 - F(x)$$

$$R(x) = (1-x^{\theta})^{\alpha}\left[1+\lambda(1-x^{\theta})^{\alpha}\right]$$

The hazard function is also known as the hazard rate, failure rate, or force of mortality. The hazard rate function is given by

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\theta\alpha\, x^{\theta-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right]}{(1-x^{\theta})\left[1+\lambda(1-x^{\theta})^{\alpha}\right]}$$

4. Statistical Properties

In this section we shall discuss structural properties of transmuted Kumaraswamy (TK) distribution. Specially moments, order statistics, maximum likelihood estimation, moment generating function,

4.1 Moments: The following theorem gives the rth moment of the transmuted Kumaraswamy distribution

Theorem 4.1: If X has the TK(α,θ,λ) distribution with $|\lambda| \le 1$, then the rth non-central moments are given by

$$\mu_r' = (1-\lambda)\frac{\alpha}{\theta}\beta\left(\frac{r}{\theta}+1,\alpha\right)+\frac{2\lambda\alpha}{\theta}\beta\left(\frac{r}{\theta}+1,2\alpha\right)$$

Proof: $\mu_r' = \theta\alpha\int_0^1 x^{r+\theta-1}(1-x^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right]dx$

$$= (1-\lambda)\theta\alpha\int_0^1 x^{r+\theta-1}(1-x^{\theta})^{\alpha-1}dx + 2\lambda\theta\alpha\int_0^1 x^{r+\theta-1}(1-x^{\theta})^{2\alpha-1}dx$$

Substituting $x^{\theta} = t$, we get $x = t^{\frac{1}{\theta}}$, and $dx = \frac{1}{\theta}t^{\frac{1}{\theta}-1}$

$As \quad x = t \to 0, \quad t \to 0 \quad and \; As \; x \to 1, t \to 1$

$$= (1-\lambda)\alpha\int_0^1 t^{\frac{r}{\theta}+1-1}(1-t)^{\alpha-1}dt + 2\lambda\alpha\int_0^1 t^{\frac{r}{\theta}+1-1}(1-t)^{2\alpha-1}dt$$

$$\Rightarrow \mu_r' = (1-\lambda)\alpha\,\beta\left(1+\frac{r}{\theta},\alpha\right) + 2\lambda\alpha\,\beta\left(1+\frac{r}{\theta},2\alpha\right) \qquad (4.1)$$

Put r =1 in above equation we get mean of transmuted Kumaraswamy distribution

$$\mu_1' = (1-\lambda)\alpha\,\beta\left(1+\frac{1}{\theta},\alpha\right) + 2\lambda\alpha\,\beta\left(1+\frac{1}{\theta},2\alpha\right)$$

$$\mu_1' = (1-\lambda)\alpha\frac{\Gamma\left(1+\frac{1}{\theta}\right)\Gamma\alpha}{\Gamma\left(1+\frac{1}{\theta}+\alpha\right)} + 2\lambda\alpha\frac{\Gamma\left(1+\frac{1}{\theta}\right)\Gamma(2\alpha)}{\Gamma\left(1+\frac{1}{\theta}+2\alpha\right)}$$

If we put $\lambda = 0$ in equation (4.1) we get rth moment of Kumaraswamy distribution which is given below

$$\mu_r' = \alpha\,\beta\left(1+\frac{r}{\theta},\alpha\right)$$

and its mean is

$$\mu_1' = \alpha\,\beta\left(1+\frac{1}{\theta},\alpha\right) \qquad \Rightarrow \mu_1' = \alpha\frac{\Gamma\left(1+\frac{1}{\theta}\right)\Gamma\alpha}{\Gamma\left(1+\frac{1}{\theta}+\alpha\right)}$$

4.2 Moment generating function

In this sub section we derived the moment generating function of TK(α,θ,λ) distribution.

Theorem 4.2: If *X* has the TK(α,θ,λ) distribution with $|\lambda| \le 1$, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left[(1-\lambda)\alpha\,\beta\left(1+\frac{j}{\theta},\alpha\right) + 2\lambda\alpha\,\beta\left(1+\frac{j}{\theta},2\alpha\right)\right]$$

Proof: We begin with the well known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^1 e^{tx}\, f(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^1 \left[1 + tx + \frac{(tx)^2}{2!} + ...\right] f(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^1 \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x;\alpha,\theta,\lambda)\, dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j'$$

$$\Rightarrow M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left[(1-\lambda)\alpha\, \beta\left(1+\frac{j}{\theta},\alpha\right) + 2\lambda\alpha\, \beta\left(1+\frac{j}{\theta},2\alpha\right)\right]$$

5. Maximum Likelihood Estimation

We estimate the parameters of the TK distribution using the method of maximum likelihood estimation (MLE) as follows;

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from TK distribution. Then the likelihood function is given by

$$L(x\,|\,\alpha,\theta,\lambda) = \prod_{i=1}^{n}\left[\theta\alpha\, x_i^{\theta-1}(1-x_i^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x_i^{\theta})^{\alpha}\right]\right] \qquad (5.1)$$

By taking logarithm of (5.1), we find the log likelihood function

$$l = n\log\alpha + n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i + (\alpha-1)\sum_{i=1}^{n}\log(1-x_i^{\theta}) + \sum_{i=1}^{n}\log\left[1-\lambda+2\lambda(1-x_i^{\theta})^{\alpha}\right] \qquad (5.2)$$

To obtain the MLE's of $\alpha, \theta \, and \, \lambda$, we differentiating loglikelihood with respect to $\alpha, \theta \, and \, \lambda$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1-x_i^{\theta}) + \sum_{i=1}^{n}\frac{2\lambda(1-x_i^{\theta})^{\alpha}\log(1-x_i^{\theta})}{1-\lambda+2\lambda(1-x_i^{\theta})^{\alpha}} \qquad (5.3)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i - (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\log x_i}{(1-x^{\theta})} - \sum_{i=1}^{n}\frac{2\lambda\alpha(1-x^{\theta})^{\alpha-1}x_i^{\theta}\log x_i}{1-\lambda+2\lambda(1-x_i^{\theta})^{\alpha}} \qquad (5.4)$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n}\frac{2(1-x_i^{\theta})^{\alpha}-1}{1-\lambda+2\lambda(1-x_i^{\theta})^{\alpha}} \qquad (5.5)$$

The MLE of $\alpha, \theta \, and \, \lambda$ and is obtained by solving this nonlinear system of equations. Setting these expressions to zero and solving them simultaneously yields the maximum likelihood estimates of these three parameters.

6. Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)}, X_{(2)},..., X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2,..., X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r}$$

For r = 1, 2, . . . ,n.

we have from (1.1) and (1.2) the pdf of the rth order inverse Rayleigh random variable $X_{(r)}$ is given by

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{n\alpha-r\alpha+\alpha-1}(1-(1-x^{\theta})^{\alpha})^{r-1}$$

Therefore, the pdf of the nth order inverse Rayleigh statistic $X_{(n)}$ is given by

$$g_{X_{(n)}}(x) = n\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}(1-(1-x^{\theta})^{\alpha})^{n-1} \quad (6.1)$$

and the pdf of the first order inverse Rayleigh statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x) = n\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{n\alpha-1} \quad (6.2)$$

Note that in particular case of n=2, (6.1) yields

$$g_{X_{(2)}}(x) = 2\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}(1-(1-x^{\theta})^{\alpha}) \quad (6.3)$$

and (6.2) yields

$$g_{X_{(1)}}(x) = 2\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{2\alpha-1} \quad (6.4)$$

Observe that (6.3) and (6.4) are special cases of (2.1) for $\lambda = -1$ *and* $\lambda = 1$ respectively. It has been observed that a transmuted Kumaraswamy distribution with $\lambda = 1$ is the distribution of $\min(X_1, X_2)$ and a transmuted Kumaraswamy distribution with $\lambda = -1$ is the $\max(X_1, X_2)$ where X_1 and X_2 are independent and identically distributed Kumaraswamy random variables. Now we provide the distribution of the order statistics for a transmuted Kumaraswamy random variable. The pdf of the rth order statistic for a transmuted Kumaraswamy distribution is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)(n-r)} \theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right]$$
$$\left[1-(1-x^{\theta})^{\alpha}\left(1+\lambda(1-x^{\theta})^{\alpha}\right)\right]^{r-1}\left[(1-x^{\theta})^{\alpha}\left(1+\lambda(1-x^{\theta})^{\alpha}\right)\right]^{n-r}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(r) = n\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right]\left[1-(1-x^{\theta})^{\alpha}\left(1+\lambda(1-x^{\theta})^{\alpha}\right)\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = n\theta\alpha\, x^{\theta-1}(1-x^{\theta})^{\alpha-1}\left[1-\lambda+2\lambda(1-x^{\theta})^{\alpha}\right]\left[(1-x^{\theta})^{\alpha}\, 1+\lambda(1-x^{\theta})^{\alpha}\right]^{n-1}$$

Note that $\lambda = 0$ yields the order statistics of the inverse Rayleigh distribution.

7. Conclusion

We defined a three-parameter Transmuted Kumaraswamy distribution as a generalization of the two-parameter Kumaraswamy distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the Kumaraswamy distribution as the base distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function and reliability behavior of the transmuted Kumaraswamy distribution shows that the subject distribution can be used to model reliability data. We expect that this study will serve as a reference and help to advance future research in the subject area.

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