On C. lifting and Semi*perfect Modules

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Abstract: *In this paper, we introduce c. lifting module as a generalization of lifting module, we prove some results on c. lifting module, also we introduce semi*perfect module as a generalization of a semiperfect module and direct summand generalized* co-finitely weakly supplemented module as a generalization of a direct summand co-finitely weakly supplemented module and we prove under certain conditions c. lifting, semi*perfect and direct summand generalized* co- finitely weakly supplemented modules are equivalent.*

Keywords: *lifting modules,co-finitely semiperfect modules, direct summand supplemented modules, generalization of lifting modules and generalization of a generalized weakly supplemented modules.*

I. Introduction

Let R be an associative ring with identity, and let M be a unital left R- module, N≤M will mean submodule of M. E(M), Z*(M) will indicate the injective hull, co singular submodule of M, respectively. Where Z*(M) = {m∈M ; Rm is small in E(Rm) }[1]. Let N and K be submodules of M. N is called a supplement of K in M if it is minimal with respect to M=N+K, equivalently M =N+K and N ∩K is small in N, for short (N ∩K ≪ N). Following [2] M is supplemented (⊕supplemented) if every submodule of M has a supplement (which is direct summand) in M. And M is called generalized * weakly supplemented, for short (G*WS), if for any submodule N of M , there is K≤M such that M =N+K and N ∩K ≤Z*(M), K is called a generalized * weak supplement of N in M [3]. A submodule N of M is called co-finitely submodule if $\frac{M}{N}$ is finitely generated. A module M is called ⊕ generalized * co-finitely supplemented, for short (⊕G*CS), if for any co-finite submodule N of M, there exist submodules L, T of M such that M = N+L with N ∩L ≤Z*(L) and M =L⊕T,[3]. It is clear that every ⊕supplemented modules are ⊕G*CS modules. Following [4], a module M is called lifting or D_1, if for every submodule N of M, there exists K, L ≤ M such that M = K ⊕ L and N ∩L ≪ L, clearly every (hollow, semisimple, uniserial) module is lifting. An R- module M is said to be semiperfect module, if every factor module of M has a projective cover [5].

In this paper we will introduce a direct summand generalized* co-finitely weakly supplemented module (⊕G*CWS), lifting module and semi*perfect module. We called an R- module M is a ⊕G*CWS module, if every co-finite submodule of M has a generalized* weak supplement in that is a direct summand of M, every ⊕CSmodule is a ⊕G*CWS, but the converse is not true in general. And M is called c. lifting, if for every co-finite submodule N of M, there exists a direct summand submodule K of M (i.e. M = K ⊕T, for some T) such that K ≤ N with N ∩T ≪ Z*(M), clearly, every (semisimple, hollow, uniserial,) module is c. lifting. Every lifting module is c. lifting but the converse is not true as se see in the Z- module Q and every c. lifting is a ⊕G*CWS. M is called semiperfect, if every finitely generated factor module of M has a projective cover [5]. M is called semi*perfect module, if every factor module of M has a generalized* projective cover, every semiperfect module is semi*perfect and every semi*perfect module is a ⊕G*CS module. If M is projective R- module then ⊕G*CWS, c. lifting and semi*perfect modules are equivalent. Also we will prove some results and properties of these modules.

II. On C. lifting modules

In this section we will recall the definition of lifting modules with some properties that we need to it later which are appeared in [4], [6]. And as a generalization of this type of modules we will introduce the C. lifting modules and prove some properties of these modules.

Recall that an R- module M is called lifting or D_1 module, if for every submodule N of M there exists a direct summand submodule K of M (i.e. M = K ⊕ L, for some L ≤ M) such that K ≤ N with N ∩L ≪ L,[4].
Equivalently, M is called lifting (D_1), if for every submodule N of M there exists a direct summand submodule K of M (i.e. M = K ⊕ L, for some L ≤ M) such that K ≤ N with N ∩L ≪ M,[6].

The following theorem gives another equivalent definition to lifting module which was appeared in [6].
Theorem 2.1. [6]:- For any R- module M, the following are equivalent.
1. M is lifting.
2. Every submodule N of M can be written as N = A ⊕S, where A is a direct summand of M and S is small in M (S ≪ M).
3. For each N ≤ M, $N/_L \ll M/_L$, where L is a direct summand of M with L ≤ N.

Examples 2.2. [6]:-
1. Every hollow module is lifting.
In particular Z_{P^∞} is lifting, since it is hollow.
2. Q as Z- module is not lifting.
3. If M = Z_8⊕ Z_2 as Z- module, then M is not lifting, since if we let N = { $(\bar{0}, \bar{0})$, $(\bar{2}, \bar{1})$, $(\bar{4}, \bar{0})$, $(\bar{6}, \bar{0})$ }, then we have $(\bar{0}, \bar{0})$ is the only direct summand of M that contained in N, then (theorem 2.1) fail to satisfy on M (i.e. M is not Lifting).
4. If M = Z_2⊕ Z_4 as Z- module is lifting, since Z_2 is always direct summand of M.
In particular Z_p⊕Z_{p^2} as Z- module is lifting, where p is a prime number.

Recall that an R- module M is called ⊕WS, if every submodule of M has a weak supplement that is direct summand of M [7]. It is clearly that, every lifting module is⊕WS. Every (semisimple, local) module is lifting

As a generalization of a lifting module we will define the following.
Definition 2.3:- An R- module M is called C. lifting, if for co-finite submodule N of M there exists a direct summand submodule K of M (i.e. M = N ⊕ H, for some H ≤ M) such that K ≤ N with N ∩H ≤ Z*(M).
Remark 2.4:- Every lifting module is C. lifting. But the converse is not true. Notice that Q as Z- module is C. lifting, since the only co-finite submodule of Q is Q itself, but Q as Z- module is not lifting.
Examples 2.5:-
1. Every hollow R- module is C. lifting.
Proof: - Let M be a hollow R- module and let N be a cofinite submodule of M.
To prove that ∃ a direct summand submodule K of M such that K ≤ N with N ∩L ≤ Z*(M),for some L in M.
Since {0} is trivially direct summand of M(i.e. M = M ⊕ 0) such that 0 ≤ N with N ∩M =N ≪ M[since M is hollow], but since M ≤ E(M), then N = N ∩Z*(M) = Z*(N) ≤ Z*(M) [1], therefore M is C. lifting.
In particular Z_{P^∞} is C. lifting, since it is hollow.

2. Every local R- module is C. lifting, since (every local is hollow)
3. We know that an R- module M is called uniserial module, if it's submodules are linearly ordered by inclusion [8], Clearly every uniserial module is hollow and hence by (example 1) it is C. lifting.
4. Let p be a prime integer number and consider the Z- module M = $\frac{Z}{pZ} \oplus \frac{Z}{p^3Z}$, where $\frac{Z}{pZ}$ and $\frac{Z}{p^3Z}$ are hollow and local modules, hence $\frac{Z}{pZ}$ and $\frac{Z}{p^3Z}$ are G*CS.[6].
Now: Let L = $0 \oplus \frac{Z}{p^3Z}$ and N = Z(1+pZ, p+p^3Z), then
M = N + L and N ∩L = $0 \oplus \frac{p^2Z}{p^3Z}$, thus N ≅ $\frac{Z}{p^2Z}$, hence N is hollow and N ∩L ≪ M, therefore N G*WS. But N is not direct summand of M, therefore M is not C. lifting.
We introduce the following.
Definition 2.6:- An R- module M is called a direct summand generalized* co-finitely weakly supplemented, notationally (⊕G*CWS), if for every co-finite submodule N of M, ∃ L, K ≤ M such that M = N + K = K⊕ L and N ∩ K ≤ Z*(M).
Remark 2.7:- Every C. lifting R- module is ⊕G*CWS.
Proof:- Let N be a co-finite submodule of M, then by assumption ∃ M_1, M_2 ≤ M such that M_1 ≤ N with M = M_1⊕ M_2 and N ∩M_2 ≤ Z*(M), then M is ⊕G*CWS, since [M_2 is a direct summand of M].
We know that every module over a semisimple ring is semisimple. [9]. Hence we getthe following.
Proposition 2.8:- If R is semisimple ring, then every R- module is C. lifting.
Recall that an R- module M is called co-singular if M = Z*(M), [1].

The following theorem is a generalization of theorem 2.1

Theorem 2.9:- For any R- module M the following are equivalent.

1. M is C. lifting.
2. Any co-finite submodule N of M can be written as
N = H ⊕ T, where H is a direct summand of M and T = Z*(T).
3. For any co-finite submodule N of M, there exists a direct summand submodule K of M such that
$\frac{N}{K}$= Z*($\frac{N}{K}$).

Proof:- (1 ⟹ 2)

Let N be a co-finite submodule of M (i.e. $\frac{M}{N}$ is finitely generated), then by(1) ∃ K ≤ N such that M = K ⊕ K', for some K' ≤ M and N ∩K' ≤ Z*(M).

Now: N = N ∩M = N ∩(K ⊕ K') = K ⊕ (N ∩K').

Take H = K and T = N ∩K', therefore T = T∩ Z*(M) = Z*(T).

(2 ⟹ 3) Let N be a cofinite submodule of M, then by (2),∃ a direct summand submodule A of M and S = Z*(S) such that N = A ⊕S.

It is enough to prove that $\frac{N}{A}$ ≤ Z*($\frac{M}{A}$).

$\frac{N}{A} = \frac{A+S}{A} \leq \frac{A+Z^*(M)}{A} \leq z^*(\frac{M}{A})$, then $\frac{N}{A} = \frac{N}{A} \cap z^*(\frac{M}{A}) = z^*(\frac{N}{A})$.

(3 ⟹ 1) Let N be a cofinite submodule of M, then by (3) ∃ a direct summand submodule K of M such that K ≤ N and M = K ⊕ K' and $\frac{N}{K}$= Z*($\frac{N}{K}$).

We have to show that N ∩K' ≤ Z*(M).

Since N = N ∩M = N ∩(K ⊕ K') = K ⊕ (N ∩K'), then

N ∩K' ≅ $\frac{N}{K}$ ≤ Z*($\frac{M}{K}$)≅ Z*(K') ≤ Z*(M), thus N ∩K' ≤ Z*(M), and hence M is C. lifting.

Recall that an R- module M is called indecomposable, if M cannot be written as a direct sum of two nonzero proper submodules, [5].

Proposition 2.10:- Let M be an indecomposable C. lifting module, then every co-finite (proper) submodule of M is co-singular.

Proof:- Let L be a co--finite submodule of M, then by (theorem 2.9) L = A ⊕S, where A is a direct summand submodule of M and S is cosingular submodule of M, but since 0 is the only direct summand submodule of M, then A =0 and hence L =S ≤ Z*(M), thus L = L ∩ Z*(M) = Z*(L), so L is cosingular submodule of M.

The following proposition shows that among certain conditions the submodule of C. lifting module is again C. lifting.

Proposition 2.11:- Let M be a finitely generated C. lifting module, then every direct summand submodule of M is C. lifting.

Proof:- Let L be a direct summand submodule of M, then there exists a submodule K of M such that M = L⊕K. Let N be a co-finite submodule of L, but since M is finitely generated, then N is co-finite submodule of M, thus by (theorem 2.9) N = A⊕ S, where A is a direct summand of M with A ≤ L and S is co-singular submodule of M.

Since A is a direct summand of M, then M = A ⊕B, for some B ≤ M, hence L = L∩M = L∩(A ⊕B) = A ⊕ (L∩B), therefore A is a direct summand of L.

Now: S = N ∩K ≤ Z*(M) ∩ (N ∩K) = Z*(N ∩K) ≤ Z*(N) ≤ Z*(L) ≤ Z*(M), therefore S is co-singular, then L is C. lifting.

In the following proposition we can prove that the factor module of a C. lifting module is C. lifting.

Proposition 2.12:- Let M be a C. lifting module, then for each submodule N of M, $\frac{M}{N}$ is C. lifting.

Proof:- Let $\frac{K}{N}$ be a co-finite submodule of $\frac{M}{N}$, then K is co-finite submodule of M and by assumption there exists a direct summand submodule L of M such that L ≤ K and M = L ⊕ L', for some L' ≤ M with K ∩L' ≤ Z*(M).

Now: $\frac{M}{N} = \frac{L'+N}{N} + \frac{K}{N}$ with $\frac{K}{N} \cap \frac{L'+N}{N} = \frac{(K\cap L')+N}{N} \leq$ Z*($\frac{M}{N}$) and

$\frac{L+N}{N} \cap \frac{L'+N}{N} = \frac{(L\cap L')+N}{N} = \frac{N}{N}$, hence $\frac{M}{N} = \frac{L+N}{N} \oplus \frac{L'+N}{N}$, therefore $\frac{M}{N}$ is C. lifting.

Corollary 2.13:- Any homomorphic image of a C. lifting module is again C. lifting.

Proof:- Since the homomorphic image is isomorphic to the quotient module.

Lemma 2.14:- Let M = M_1⊕ M_2 be a C. lifting module, then M_1 and M_2 are C. lifting.

Proof:- trivially by (proposition 2.12), since each of M_1 and M_2 are direct summand.

Recall that a submodule N of an R- module M is called fully invariant, if for any f ∈End (M), f (N) ≤ N, if every submodule of M is fully invariant, then M is called duo module [5].
Proposition 2. 15:- Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are C. lifting, then M is C. lifting.
Proof:- Let N be a co-finite submodule of M, then $N = N \cap M = N \cap M_1 \oplus N \cap M_2$, hence $N \cap M_1$ is co-finite submodule of M_1 and $N \cap M_2$ is co-finite submodule of M_2, therefore $\exists K_1, H_1 \le M_1$ such that $K_1 \le N \cap M_1$ and
$M_1 = (N \cap M_1) + K_1 = K_1 \oplus H_1$ with $(N \cap M_1) \cap H_1 \le Z^*(M_1)$. Also $\exists K_2, H_2 \le M_2$ such that $K_2 \le N \cap M_2$ and
$M_2 = (N \cap M_2) + K_2 = K_2 \oplus H_2$ with $(N \cap M_2) \cap H_2 \le Z^*(M_2)$, then $M = M_1 + M_2 = (K_1 + K_2) + (H_1 + H_2)$ and $N = (N \cap M_1) + (N \cap M_2) = ((N \cap M_1) + (N \cap M_2)) \cap (H_1 + H_2) = ((N \cap M_1) \cap H_1) + ((N \cap M_2) \cap H_2) \le Z^*(M_1) + Z^*(M_2) = Z^*(M)$.
Now: $M = M_1 \oplus M_2 = (K_1 \oplus H_1) \oplus (K_2 \oplus H_2) = (K_1 \oplus K_2) \oplus (H_1 \oplus H_2)$, hence $K_1 \oplus K_2$ is a direct summand of M.
Corollary 2.16:- Let $M = M_1 \oplus M_2$ be a duo module, then M is C. lifting iff M_1 and M_2 are C. lifting..

Recall that an R- module M is called π- projective module, if for any two submodules N and K of M with M =N + K, there exists f ∈End(M) such that Im f ≤ N and Im (I-f) ≤ K, where End(M) denotes the endomorphism of M.
The following theorem appeared in[6. Theorem 3.3.4] which gives some properties of π- projective module.
Theorem 2.17:- Let M be a π- projective module, the we have.
1. Every direct summand of M is π- projective.
2. If U and V are mutual supplements in M, then $U \cap V = 0$ and hence $M = U \oplus V$.
3. If M = U + V and U is a direct summand of M, then there exists V' ≤ V such that $M = U \oplus V'$.
According to these properties we can prove the following.

Proposition 2.18:- Let M be a π- projective module, then M is C. lifting iff M is ⊕G*CWS.
Proof:- (⟹) Let N be a co-finite submodule of M, then by assumption, there exists a direct summand K of M such that K ≤ N and $M = K \oplus L$ for some L ≤ M with $N \cap L \le Z^*(M)$.
Now: since K is a direct summand and M is π- projective, then by (theorem 2.17(3)), $\exists L_1 \le L$ such that $M = K \oplus L_1$, thus $M = N + L_1$ and $N \cap L_1 \le N \cap L \le Z^*(M)$, but since L_1 is also direct summand of M, therefore M is ⊕G*CWS.
(⟸) Let N be a co-finite submodule of M, then by assumption ∃ L ≤ M, where L is a direct summand of M with L ≤ N such that $M = N + L = K \oplus L$, fore some K ≤ M and since M is π- projective, then by (theorem 2.18(3)), $\exists N_1 \le N$ such that $M = N_1 \oplus L$, where N_1 is a direct summand of M and L is co-singular, then M is C. lifting.
Corollary 2.19:- Let $M = M_1 \oplus M_2$ be a π- projective module, then M is C. lifting iff M_1 and M_2 are C. lifting
Proof:- (⟸) Let M_1 and M_2 be C. lifting modules, then by (proposition 2.19) M_1 and M_2 are ⊕G*CWS, hence by [3] M is ⊕G*CWS, then by (proposition 2.19) M is C. lifting.
(⟹) trivially by (Lemma 2.15).

Recall that an R- module M is called quasi- projective, if M is M- projective [6].
It is known that every quasi- projective is π- projective [see 6, proposition 3.3.2], hence have the following.
Corollary 2.20:- Let $M = M_1 \oplus M_2$ be a quasi- projective module, then M is C. lifting iff M_1 and M_2 are C. lifting.

III. Semi*perfect modules

In this section we will study a semi*perfect module as a generalization of semiperfect module that appeared in[5], and also we will introduce the Generalized* cover(projective cover) with some properties and examples. f

It is know that an epimorphism f: P → M is called cover of M if kernel f is small in P, and in addition if P is projective module on M, then f is called projective cover.[5].
Definition 3.1:- Let M and N be two R- modules and an epimorphism f: N → M, then we say that f is Generalized* cover of M if kerf ≤ Z*(M), where kerf is the kernel of f. In case N is projective module on M, then f is called Generalized* projective cover of M.
Lemma 3.1:- Let M, K and N be R- modules and let
f: K → M and g: M → N be two Generalized* cover for M and N respectively, such that f(Z*(K)) = Z*(M), then $g_\circ f$ is Generalized* cover for N.
Proof:- If f and g are cover then so is $g_\circ f$ [11].
Suppose that both of f and g are Generalized* cover for M and N respectively.
We have to show that $\ker (g_\circ f) \le Z^*(K)$.

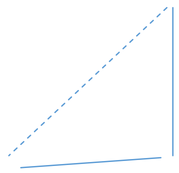
Let m ∈ker (g $_o$f), then g $_o$f(m) = 0, hence f(m) ∈ker g but since g is Generalized* cover for N, then we have f(m) ∈ker g ≤ Z*(M).
Now. Since ker f ≤ Z*(k), then ∃ x ∈ Z*(k) such that f(m) = f(x) hence f(m- x) = 0, therefore m- x ∈ ker f ≤ Z*(k), thus m ∈ Z*(k), then ker (g $_o$f) ≤ Z*(K).
Proposition 3.2:- Any finite direct sum of Generalized* cover is Generalized* cover.
Proof:- let f_i : P_i → M_i be a Generalized* cover of M_i, ∀ i= 1, 2, …, n.
We want to prove that $\oplus_{i=1}^{n} f_i$: $\oplus_{i=1}^{n} P_i \rightarrow \oplus_{i=1}^{n} M_i$ is a Generalized* cover of M_i?
By assumption, since ker (f_i) ≤ Z*(P_i) ∀ i= 1, 2, …, n, then we have ker ($\oplus_{i=1}^{n} f_i$) = $\oplus_{i=1}^{n}$ker (f_i), thus $\oplus_{i=1}^{n}$ker (f_i) ≤ $\oplus_{i=1}^{n}$ Z*(P_i) , therefore $\oplus_{i=1}^{n} f_i$ is Generalized* cover of M_i.
Lemma 3.3:- Let M be an R- module and N be a submodule of M with natural epimorphism and let P be any R- module with g: P → $\frac{M}{N}$ and k: P → M such that k(Z*(P)) = Z*(M), where g and k have a composition with f, then g is Generalized* cover epimorphism if and only if Im k is Generalized* supplemented of N with ker f ≤ Z*(P), where Im k is the image of k.
Proof:- (⟹) We have to show that N ∩ Im k ≤ Z*(Im k).
Let x ∈ N ∩ Im k, then we have
x = k(y) for some y ∈P, x ∈N, to show that.
g(y) = f(k(y)) = f(x) = 0, by [first isomorphism theorem we have N = ker f, x ∈N], then y x ∈ker g and k(y) ∈ k(ker g).
Now. If x x ∈ k(ker g), then x= k(y), for some y ∈ ker g but f(k(y)) = g(y), therefore f(k(y)) = g(y) = 0[y ∈ ker g] hence x ∈ker f and x ∈ Im k, but ker f = N thus
x ∈ N ∩Im k = k(ker g) ≤ Z*(Im k) = Z*(k(P)), therefore Im k is a Generalized* supplement of N, where Im k = k(P). But since g is epimorphism, then ker k ≤ ker g ≤ Z*(P).
(⟸) trivially by (lemma 3.1)
Recall that an R- module M is called semiperfect module, if every factor module of M has a projective cover, [2].

We will introduce the following propositions as a generalization of a semiperfect module.
Theorem 3.4:- If every Generalized* projective cover of an R- module M satisfies f(Z*(P)) = Z*(M), then the following are equivalent.
1. M is Generalized* cover semiperfect.
2. M is Generalized* cover by supplements have Generalized* projective cover.
Proof:- (1 ⟹2) Let N be a co-finite submodule of M such that M = N+ L, for some L ≤ M, then by(1) f: P → $\frac{M}{N}$ be a Generalized* projective cover of $\frac{M}{N}$, where P is projective module.
Now: $\frac{M}{N} = \frac{L+N}{N} \cong \frac{L}{L \cap N}$, but since P is projective, then f can be lifted to g: P → L and since f is Generalized* cover, thus by (lemma 3.5) we have the image of g(Im g) is Generalized* supplemented of $L \cap N$ (i.e. L = Im g + ($L \cap N$) and Im g ∩($L \cap N$) ≤ Z*(Im g)) with ker g ≤ ker ($\pi_o i_o g$) = ker f≤ Z*(P), where π is the natural epimorphism and i:L → M be the inclusion map, then (2) holds.
(2 ⟹ 1) Let N be a co-finite submodule of M (i.e. $\frac{M}{N}$ is finitely generated), then by (2) ∃ L ≤ M such that M = N+ L and $L \cap N$ ≤ Z*(L).
We have to show that M is Generalized* co-finitely semiperfect module.
Let f: P → L be a Generalized* projective cover of L and let g: L → $\frac{L}{L \cap N}$ and h: $\frac{L}{L \cap N} \rightarrow \frac{L+N}{N}$, where g is the canonical epimorphism on L such that $\frac{L}{L \cap N} \cong \frac{L+N}{N} = \frac{M}{N}$ hence g is Generalized* cover of $\frac{L}{L \cap N}$ (i.e. ker g = $L \cap N$ ≤ Z*(L)), thus $h_o g_o f$: P → M is Generalized* projective cover of $\frac{M}{N}$ therefore by (lemma 3.1) M is Generalized* co-finitely semiperfect module.
Proposition 3.5:- Let M be an R- module such that M is ⊕G*CS projective module, then M is Generalized* co-finitely semiperfect module.
Proof:- Let N be a co-finite submodule of M, then by assumption ∃ L, K ≤ M such that M = N+ K = K ⊕L with $N \cap K$ ≤ Z*(K), where K is projective.
Now: Let i: K → M and π: M → $\frac{M}{N}$ be the inclusion and natural epimorphism maps respectively, then π_oi: K → $\frac{M}{N}$ is an epimorphism with ker(π_oi) = $N \cap K$ ≤ Z*(K), hence M is Generalized* co-finitely semiperfect module.
Theorem 3.6:- Let M be a projective R- module, then M is ⊕G*CS iff M is semi*perfect module.

Proof:- (⟸) Let N be a co-finite submodule of M (i.e. $\frac{M}{N}$ is finitely generated, then by assumption there exists a projective cover π: P ⟶ $\frac{M}{N}$ with ker (π) = N ≤ Z*(P).
For the canonical epimorphism q: M ⟶ $\frac{M}{N}$ and since M is projective, there exists f: M ⟶ P such that π_of = q.
i.e. the diagram is commute. M

∃f

q



P π $\frac{M}{N}$

But since f is epimorphism and P is projective, then f is splits [5], and hence there exists g: P ⟶ M such that $f_o g$ = I_P [5], therefore π = $\pi_o f_o g$ = $q_o g$.
Let m ∈M, then m +N ∈ $\frac{M}{N}$ and hence ∃ p ∈P such that π(p) = m +N, since [π(p) = q_og(p) = m +N]
= g(p) + N iff m – g(p) ∈N, therefore m = g(p) + N = g(p)+ ker f, where ker f ≤ N.
Now: let x ∈g(p) ∩ker f implies that x = g(p), p ∈P and x ∈ker f, hence f(x) = 0, thus f(x) = f(g(p)) = 0 = p iff p =0, then 0 = g(p) = x and M = g(p) ⊕ker f.
To show that g(p) is Generalized*supplement of N in M, define φ: g(p) ⟶ $\frac{M}{N}$ such that φ_og = π and φ(g(p)) = g(p) + N = π(p) hence φ is an epimorphism with
ker φ = N.
ker φ = { g(p) : g(p) + N = N} iff { g(p) : g(p) ∈N, therefore ker φ = g(p) ∩N ≤ Z*(P), hence g(p) is Generalized*supplement of N in M, then M is ⊕G*CS.
(⟹) Let $\frac{M}{N}$ be a finitely generated submodule, then by assumption ∃ H,T ≤ M such that M = N+ H =H ⊕T with N ∩H ≤ Z*(H), but since M is projective, then H is a projective submodule, by[8.every direct summand of a projective module is projective].
Let i: H ⟶ M and π: M ⟶ $\frac{M}{N}$ be inclusion and the natural epimorphism maps, resp. such that π_oi: H ⟶ $\frac{M}{N}$ ⟶ 0 is onto with ker (π_oi) = { h ∈ H : π(i(h)) = h+N = N iff h ∈N} thus ker (π_oi) = N ∩H ≤ Z*(H), then M is semi*perfect module.
Proposition 3.7:- Every homomorphic image of semi*perfect module is again semi*perfect.
Proof:- Let f: M ⟶ N be any R- homomorphism, where M, N any R- module and let $\frac{f(M)}{U}$ be a finitely generated factor module of f(M).
Define an epimorphism g: M ⟶ $\frac{f(M)}{U}$ by g(m) = f(m)+ U.
Since M is semi*perfect module, then $\frac{M}{f^{-1}U}$ ≅ $\frac{f(M)}{U}$ finitely generated, therefore $\frac{f(M)}{U}$ has Generalized* projective cover h: P ⟶ $\frac{f(M)}{U}$ ⟶ 0 with ker h ≤ Z*(P), where h is epimorphism and P is a projective module, then f(M) is semi*perfect.
Corollary 3.8:- Any factor module of semi*perfect module is semi*perfect module.

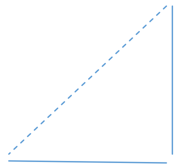
Notice that a submodule N of an R- module M is called small cover of M, if there exists an epimorphism f: N ⟶ M such that ker f ≪ N,[5].
Proposition 3.9:- Every small cover of semi*perfect module is semi*perfect.
Proof:- Let N be a small cover of M and f: N ⟶ M be a small epimorphism.
Let U be a co-finite submodule of N(i.e. $\frac{N}{U}$ is finitely generated) and a homomorphism ρ: $\frac{N}{U}$ ⟶ $\frac{M}{f(U)}$ define byρ(n+ U) = f(n) + f(U), ρ is onto with
ker ρ = { n+ U : f(n) + f(U) = f(U) iff f(n) ∈ f(U)} iff { f(n) = f(u), fore some u ∈U iff n –u ker f ≪ N} = { n = U + ker f ⟹ n ∈U}, therefore ker ρ ≪ $\frac{N}{U}$.
Notice that: $\frac{M}{f(U)}$ = $\rho(\frac{N}{U})$ ≅ $\frac{N/U}{\frac{U+\ker f}{U}}$ by the following.
Define g: $\frac{N}{U}$ ⟶ $\rho(\frac{N}{U})$ as

g(n+ U) = f(n) + f(U), g is onto, therefore $\frac{N/_U}{kerg} = \rho(\frac{N}{U})$, where ker g = { n+U : g(n+ U) = f(n) + f(U) = f(U)} iff { n+ U : n –U ∈ker f } = { n+ U : n = U + ker f}, but since
$\frac{N}{U} = \frac{U+\ker f}{U}$ hence ker g = $\frac{U+\ker f}{U}$, so $\rho(\frac{N}{U}) \cong \frac{N/_U}{\frac{U+\ker f}{U}}$. But since $\frac{N}{U}$ is finitely generated, then $\frac{M}{f(U)}$ is finitely generated and since M is semi*perfect, then there exists
q: P → $\frac{M}{f(U)}$ → 0 with ker q = f(U) ≤ Z*(P) and since P is projective, then there exists h: P → $\frac{N}{U}$, h is an epimorphism such that $\rho_\circ h$ = q as shown in the following diagram.

P

∃h

q



$\frac{N}{U}\rho\frac{M}{f(U)}$

With ker h ≤ ker q = f(U) ≤ Z*(P), then N is semi*perfect.
Corollary 2.10:- If N ≪ M, then $\frac{M}{N}$ is semi*perfect module iff M is semi*perfect.
Proof:- (⟹) Let π: M → $\frac{M}{N}$ be the natural epimorphism map with ker π = N ≪ M, hence M is a small cover of $\frac{M}{N}$, then by (proposition 3.9) M is semi*perfect module.
(⟸) trivial by (corollary 3.8).

Proposition 3.11: - Let f: P → M be a projective cover for M, then the following are equivalent.
1. M is semi*perfect module.
2. P is semi*perfect module.
3. P is ⊕G*CS module.
Proof: -
(1 ⟹ 2) by (proposition 3.9).
(2 ⟹ 3) by (theorem 3.6).
(3 ⟹ 1) Since P is projective, then by (theorem 3.6) P is semi*perfect, but since f: P → M → 0 is epimorphism hence we have f(P)= M, then by (proposition 3.7) M is semi*perfect module.

Theorem 3.12:- Let P = $\oplus_{i\in I}P_i$ be a direct sum of projective modules P_i, then P is semi*perfect iff every summand P_i is semi*perfect, ∀ i ∈I.
Proof:- (⟹)Let P = $\oplus_{i\in I}P_i$, then P is projective, by [8. Any direct sum of projective modules is projective], hence $P_i \cong \frac{P}{\oplus_{j\in I, j\neq i}P_j}$, then by (corollary 3.8) P_i is semi*perfect module, ∀ i ∈I.
(⟸) Let P_i is projective semi*perfect module, then by (theorem 3.6) P_i is ⊕G*CS, ∀ i ∈I. Then by [3. Any direct sum of ⊕G*CS modules is ⊕G*CS] we have
P = $\oplus_{i\in I}P_i$ is ⊕G*CS, therefore by (theorem 3.6) P is semi*perfect module.

Corollary 3.13:- Let M be a projective module, then M is ⊕G*CS iff every direct summand of M is ⊕G*CS.
Proof:- (⟹) Let P be a direct summand submodule of M (i.e. M = P ⊕K, fore some K ≤ M), then P is projective, by[8. Every direct summand of a projective module is projective], but by assumption M is ⊕G*CS projective module, hence by (theorem 3.6) M is semi*perfect and by (theorem 3.11) P is semi*perfect and also by (theorem 3.13) we have P is ⊕G*CS.
(⟸) trivially, since (M = M ⊕0).

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