

The Modified Double Weighted Exponential Distribution with Properties

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Abstract

The weighted distributions are widely used in many real life fields such as medicine, ecology, reliability, etc., for the development of proper statistical model. The concept of double weighted distribution was introduced by Al-khadim and Hantoosh (2013) and later has been studied by other researchers. In his article, e^{nx} has been considered as suitable weight for efficient modeling of double weight exponential distribution. The statistical properties of the modified double weighted exponential distribution (MDWED) are explored. The Kolmogorov- Smirnov test has been used to choose a better fitted probability model. The result of this test shown that (MDWED) is more suitable distribution to fit rainfall data then (DWED) proposed by Al-khadim and Hantoosh (2013).

Keywords: Weighted distribution, Exponential distribution, Moment Generating Function, Fisher information.

1. Introduction

Fisher (1934) introduced the concept of weighted distributions. Rao (1965) developed this concept in general terms by in connection with modelling statistical data where the usual practice of using standard distributions were not found to be appropriate, in which various situations that can be modelled by weighted distributions, where non-observe ability of some events or damage caused to the original observation resulting in a reduced value, or in a sampling procedure which gives unequal chances to the units in the original that means the recorded observations cannot be considered as original random sample. Patil and Rao (1978) discussed the weighted distribution and size-biased sampling with applications to wildlife populations and human families. Subhan and Boudrissa (2000) suggested length biased Weibull distribution properties and estimations. Abdel and Piegorsch (2002) provided in detail the applications and examples of weighted distribution. Jing (2010) explored the Weighted Inverse Weibull and Beta Inverse Weibull Distribution using $w(x) = x$ as weight for inverse Weibull distribution and Beta inverse Weibull distribution and derive the various statistical properties. Das and Roy (2011) presented applicability of length biased Weighted Generalized Rayleigh Distribution using the PDF of Rayleigh distribution and $w(x) = x^{2c-N} \exp(-x^2(c\sigma^2 - \frac{1}{2\sigma^2}))$ as weight and find the relation for statistical properties. Ahmed et al. (2013) offered the structural properties of size biased Gamma distributions. Al-khadim and Hantoosh (2013) introduced the double weighted distribution and discuss the statistical properties of double weighted exponential distribution with one weight as x and second weight $F(cx)$, where $F(x)$ CDF of the exponential distribution. Rashawn (2013) developed the double Weighted Rayleigh distribution and estimate its properties using x as first weight and CDF of the Rayleigh distribution as second weight.

Suppose X is non negative random variable with $f(x)$ as its probability density function (PDF), then the PDF of the weighted random variable X_w is given by:

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}, \quad x > 0 \quad (1)$$

where $w(x)$ be a non-negative weight function and $\mu_w = E(w(x)) < \infty$. When we use weighted distribution as a tool in the selection of suitable models for observed data is the choice of the weight function that fits the data (see Al-khadim and Hantoosh (2013)). Depending upon the choice of the weight function $w(x)$, we have different weighted models. For example, $w(x) = x$, the resulting distribution is called length biased. More generally, when $w(x) = x^c$; $c > 0$ then the resulting distribution is called sized biased. This type of sampling is a generalization of length biased sampling and majority of literature is centered on this weight function. Another type of models is double weighted distributions originally introduced by Al-khadim and Hantoosh (2013).

The double weighted distribution is defined as:

$$f_w(x; c) = \frac{w(x)f(x)F(cx)}{W_D}, \quad x \geq 0 \quad c > 0. \quad (2)$$

Where $W_D = \int_0^\infty w(x)f(x)F(cx) dx$, and first weight is $w(x)$ and second weight is $F(cx)$. Also, $F(cx)$ depends upon the original distribution $f(x)$.

2. Proposed Double Weighted Exponential Distribution

In this section, we propose a modified double weighted exponential distribution by considering the weight function $w(x) = e^{nx}$. The probability density function and cumulative density function of the exponential distribution are given by;

$$f(x) = \lambda e^{-\lambda x} \quad x \geq 0, \lambda > 0, \quad \text{and } F(cx) = 1 - e^{-c\lambda x} c > 0,$$

$$W_D = \int_0^\infty w(x)f(x)F(cx) dx = \int_0^\infty e^{nx} \lambda e^{-\lambda x} (1 - e^{-c\lambda x}) dx = \frac{c\lambda^2}{(n-\lambda)(n-\lambda-c)}. \quad (3)$$

2.1 The PDF of MDWED

Taking the first weight function $w(x) = e^{nx}$ and substituting the values of the Equation (3) in the definition of double weighted distribution which is defined in Equation (2) results as;

$$f_w(x; c, \lambda, n) = \frac{(n-\lambda)(n-\lambda-c)}{\lambda c} e^{(n-\lambda)x} (1 - e^{-c\lambda x}), \quad x \geq 0, c > 0, \lambda > n. \quad (4)$$

The p.d.f defined in Equation (4) is named as Modified Double Weight Exponential Distribution (MDWED) and the statistical properties of this distribution will be explored in the coming sections.

2.2 The CDF of MDWED

The cdf of the MDWED is obtained as:

$$F_w(x; c, \lambda, n) = \int_0^x f_w(t; c, n, \lambda, n) dx,$$

$$F_w(x; c, \lambda, n) = \frac{(n-\lambda)(n-\lambda-c)}{\lambda c} \int_0^x (e^{(n-\lambda)t} - e^{(n-\lambda-c)t}) dt$$

$$F_w(x; c, \lambda, n) = 1 - \frac{e^{(n-\lambda)x}[(\lambda+\lambda c-n)+(n-\lambda)e^{-c\lambda x}]}{\lambda c} \quad (5)$$

The probability density function and the cumulative density function of the proposed MDWED are made for various choices of parameters and are plotted in Figures 1 and 2.

Figure 1: The probability density function of MDWED at various parameters choices.

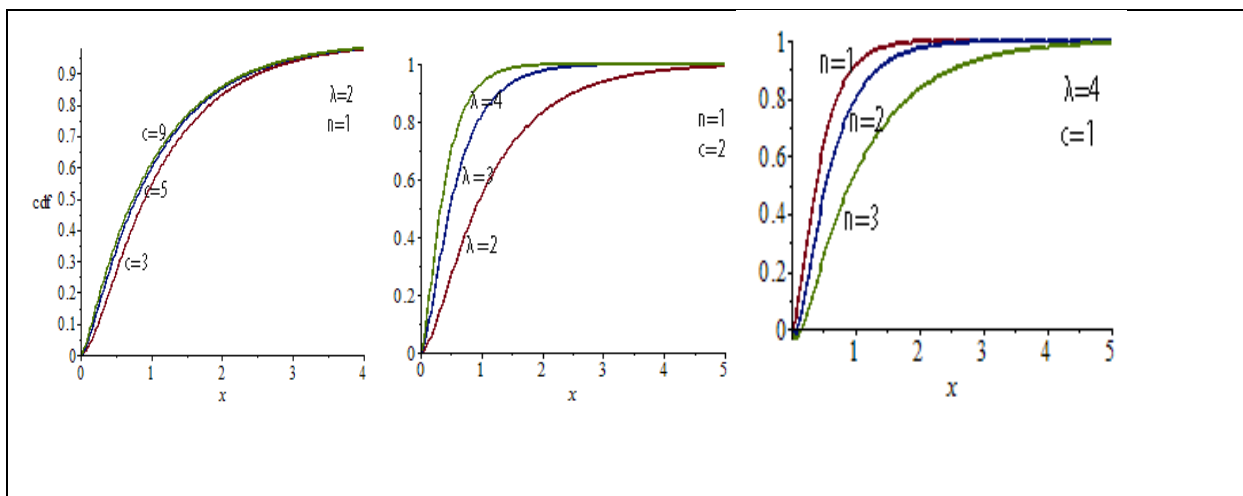
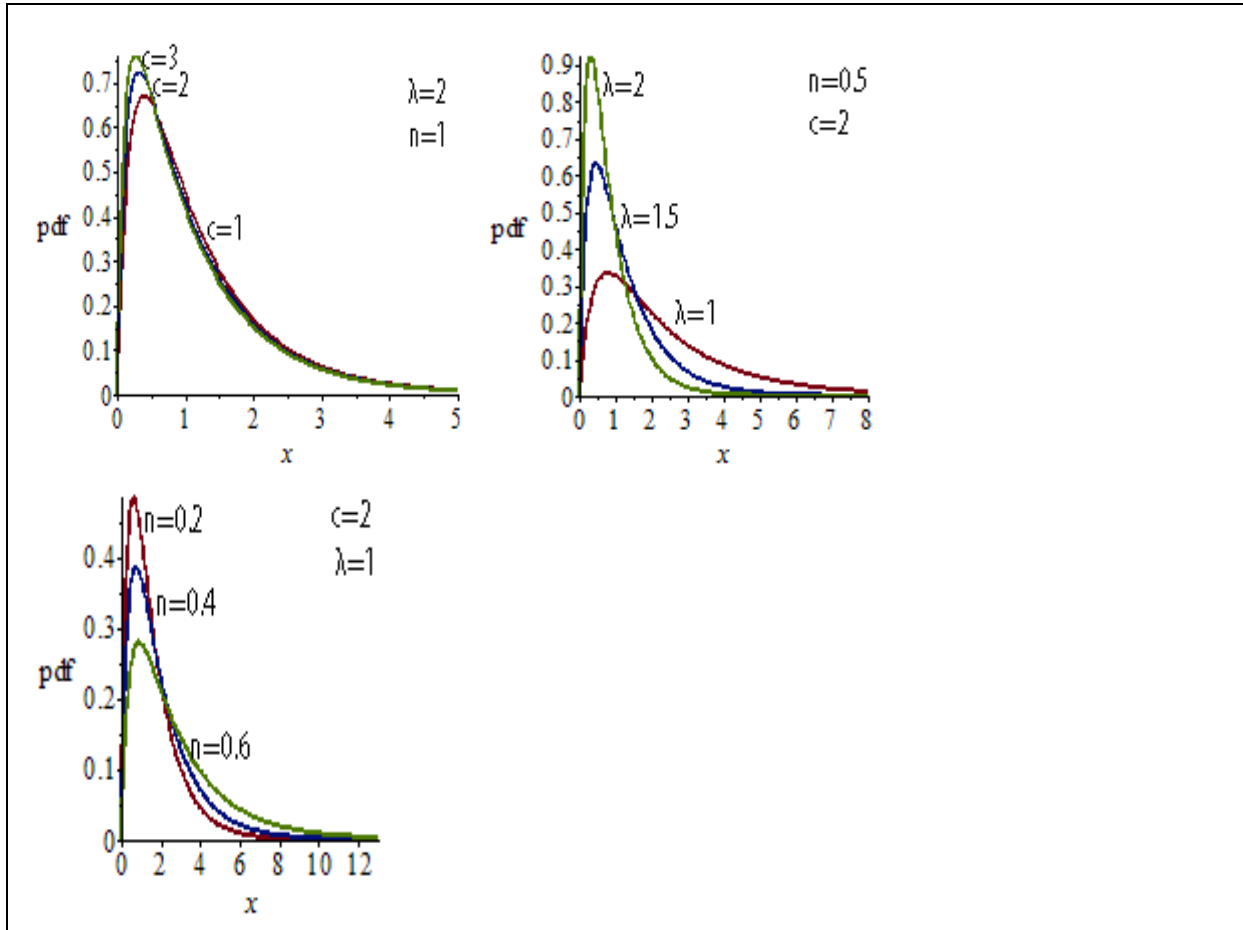


Figure 2: The CDF of the MDWED for the various choices of parameters.

From the Figure 1, it can be seen that the p.d.f of the proposed distribution increases when c and λ increases while it decreases when n increases. The figure 2 shows that the c.d.f remain lesser for lesser choices of c and λ while it is greater for the lesser values of n . All the curves given in figure 2 approaches to 1 when $x \rightarrow \infty$.

3. Particular case of distribution

If we put $n = 0$ in the p.d.f of MDWED given by Equation (4), then it becomes:

$$f_w(x; c, \lambda) = \frac{\lambda(1+c)}{c} e^{(-\lambda)x} (1 - e^{-c\lambda x}), x \geq 0, c > 0, \lambda > n \quad (6)$$

Which is new class of weighted exponential distribution given by Gupta and Kundu (2009). Hence, the Equation (6) is special case of the proposed MDWED.

4. The Shape (Mode)

The shape of density function given in (4) can be clarified by studying this function defined over the positive real line $[0, \infty)$ and behavior of its derivative as follow:

4.1 Limit and mode of function

Note that the limits of the density function given in (4) is given by

$$\begin{aligned} \lim_{x \rightarrow 0} f_w(x; c, \lambda) &= \lim_{x \rightarrow 0} \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} e^{-(\lambda-n)x} (1 - e^{-c\lambda x}) = 0 \\ \lim_{x \rightarrow \infty} f_w(x; c, \lambda) &= \lim_{x \rightarrow \infty} \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} e^{-(\lambda-n)x} (1 - e^{-c\lambda x}) = 0 \text{ for } \lambda > n \end{aligned} \quad (7)$$

Since $\lim_{x \rightarrow \infty} e^{-(\lambda-n)x} = 0$, $\lim_{x \rightarrow 0} (1 - e^{-c\lambda x}) = 0$ and $\lim_{x \rightarrow \infty} (1 - e^{-c\lambda x}) = 1$

From the limits, we can concluded that (MDWED) has one mode say x_0 for $w(x) = e^{nx}$. In order to determined the modal value of the given function, the log likelihood function is

$$\log f_w(x; c, \lambda, n) = \log(n - \lambda) + \log(n - \lambda - \lambda c) + (n - \lambda)x + \log(1 - e^{-c\lambda x}) - \log(\lambda c) \quad (8)$$

Differentiating w.r.t x , we have

$$\frac{\partial \log f_w(x; c, \lambda, n)}{\partial x} = (n - \lambda) + \frac{\lambda c e^{-c\lambda x}}{1 - e^{-c\lambda x}}$$

The mode of the MDWED can be calculated by solving nonlinear equation w.r.t, x

$$(n - \lambda) + \frac{\lambda c e^{-c\lambda x}}{1 - e^{-c\lambda x}} = 0, \text{ i.e. } x_0 = \frac{-1}{\lambda c} \ln \frac{(\lambda - n)}{(\lambda c + \lambda - n)}, \quad (9)$$

The mode of the MDWED using the equation (9) is calculated for various parametric choices and tabulated in Table 1 below.

Table 1: The modal values of MDWED at various parameters.

λ	n	c	x_0	λ	n	c	x_0	λ	N	c	x_0
2	1	1	0.5493	5	1	1	0.1622	2	1	1	0.5493
		2	0.4024		2		0.1962	3			0.3054
		3	0.3243		3		0.2506	4			0.2118
		4	0.2747		4		0.3584	5			0.1622

5. Reliability Analysis

5.1 Reliability function

The Reliability function or survival function $R(x)$ can be derived using the cumulative distribution and given by:

$$R_{w(x;c,\lambda,n)} = 1 - F_w(x; c, \lambda)$$

$$R_{w(x;c,\lambda,n)} = \frac{e^{(n-\lambda)x[(\lambda+\lambda c-n)+(n-\lambda)e^{-c\lambda x}]} - 1}{\lambda c} \tag{10}$$

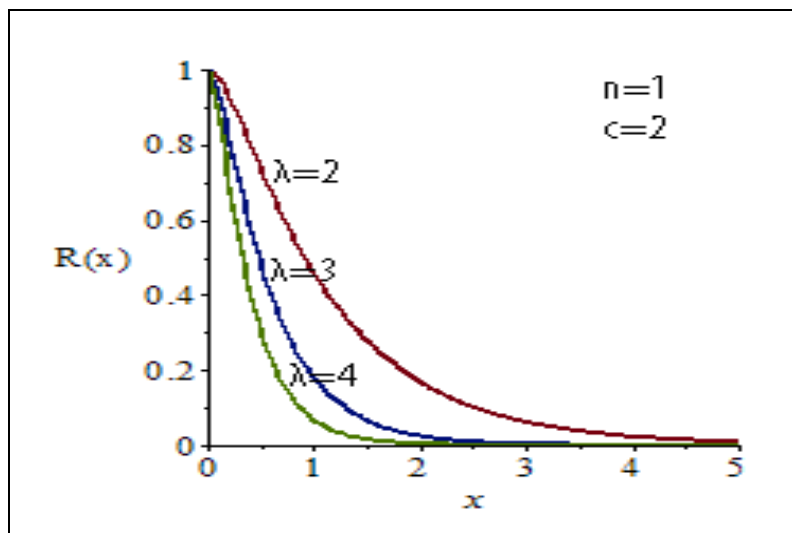


Figure 3: The reliability behavior of modified weighted exponential distribution for values of parameters $\lambda = (2, 3, 4,)$ and $n = 1, c = 2$.

It is clear that $R(x)$ decreases from 1 to zero as x increases. Also when value of λ increases $R(x)$ converges to zero as $x \rightarrow \infty$

5.2 Hazard function

The hazard function also known as the failure rate or hazard rate. Hazard function is the ratio of the probability density function to the survival function given by.

$$H_{W(x;c,\lambda,n)} = \frac{f_w(x;c,\lambda)}{R_w(x;c,\lambda)}$$

$$H_{W(x;c,\lambda,n)} = \frac{(n-\lambda)(n-\lambda-\lambda c)(1-e^{-c\lambda x})}{[(\lambda+\lambda c-n)+(n-\lambda)e^{-c\lambda x}]}$$
(11)

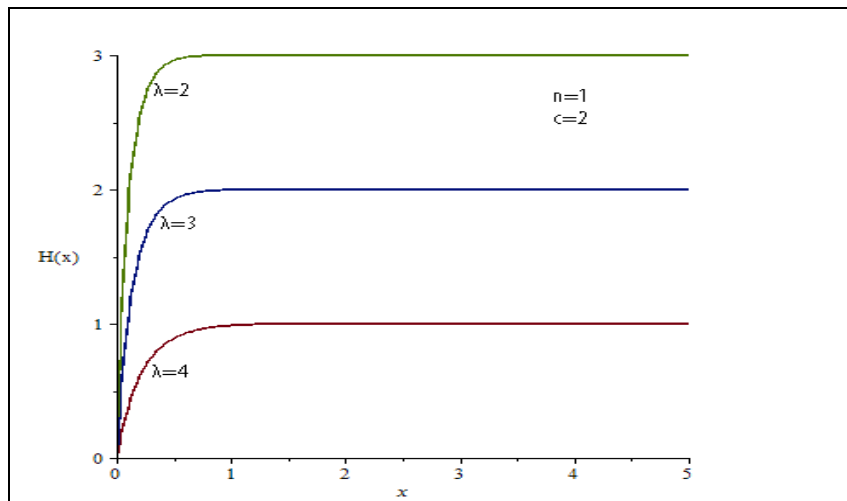


Figure 4: The behavior of hazard function for different values of λ .

The value of $H(x)$ decreases when λ increases as it is clear from figure 3.

5.3 Reverse hazard function

The reverse hazard function is ratio of PDF and CDF of the distribution

$$\varphi_{W(x;c,\lambda,n)} = \frac{f_w(x; c, \lambda)}{F_w(x; c, \lambda)}$$

$$\varphi_{W(x;c,\lambda,n)} = \frac{(n-\lambda)(n-\lambda-\lambda c)e^{(n-\lambda)x}(1-e^{-c\lambda x})}{\lambda c - e^{(n-\lambda)x}[(\lambda+\lambda c-n)+(n-\lambda)e^{-c\lambda x}]}$$
(12)

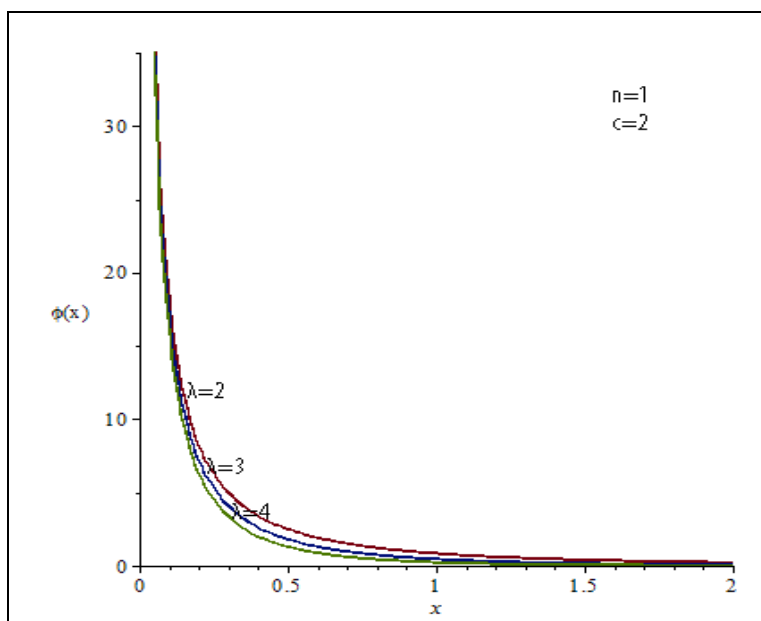


Figure 5: Represent the behavior of reverse hazard function for different choice λ which indicate that values of $\varphi_{W(x)}$ decreases as x increases.

6. Moments of MDWED

The k^{th} moment of the MDWED is given by:

$$E_{f_w}(x^k) = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \int_0^\infty x^k e^{-(n-\lambda)x} (1 - e^{-c\lambda x}) dx \quad k = 1, 2, 3, \dots$$

$$E_{f_w}(x^k) = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \int_0^\infty [x^k e^{-(\lambda-n)x} - x^k e^{-(\lambda+c\lambda-n)x}] dx$$

$$E_{f_w}(x^k) = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \left[\frac{\Gamma(k+1)}{(\lambda-n)^{k+1}} - \frac{\Gamma(k+1)}{(\lambda+\lambda c-n)^{k+1}} \right]$$

$$E_{f_w}(x^k) = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \Gamma(k+1) \left[\frac{(\lambda+\lambda c-n)^{k+1} - (\lambda-n)^{k+1}}{(\lambda-n)^{k+1}(\lambda+\lambda c-n)^{k+1}} \right]$$

$$E_{f_w}(x^k) = \Gamma(k+1) \left[\frac{(\lambda+\lambda c-n)^{k+1} - (\lambda-n)^{k+1}}{\lambda c (\lambda-n)^k (\lambda+\lambda c-n)^k} \right] \quad k=1, 2, 3, \dots$$

Let $\epsilon_c = \lambda + \lambda c - n$, and $\epsilon = \lambda - n$

$$E_{f_w}(x^k) = \Gamma(k+1) \left[\frac{(\epsilon_c)^{k+1} - (\epsilon)^{k+1}}{\lambda c (\epsilon)^k (\epsilon_c)^k} \right] \quad k=1, 2, 3, \dots \quad (13)$$

From the above equation we can find mean, variance, coefficient of variation, skewness and kurtosis as follow;

6.1 The Mean

To find mean put $k = 1$ in

$$\mu = \Gamma(2) \left[\frac{(\epsilon_c)^2 - (\epsilon)^2}{(\lambda c)(\epsilon)(\epsilon_c)} \right] \quad (14)$$

$$\mu = \left[\frac{(\epsilon_c)^2 - (\epsilon)^2}{(\lambda c)(\epsilon)(\epsilon_c)} \right] \text{ as } \Gamma(2) = 1$$

6.2 The Variance

$$\text{Var.} = E(x^2) - \mu^2$$

$$\sigma^2 = \Gamma(3) \left[\frac{(\epsilon_c)^3 - (\epsilon)^3}{\lambda c (\epsilon)^2 (\epsilon_c)^2} \right] - \Gamma(2)^2 \left[\frac{(\epsilon_c)^2 - (\epsilon)^2}{(\lambda c)(\epsilon)(\epsilon_c)} \right]^2$$

$$\sigma^2 = \frac{1}{(\lambda c)^2 (\epsilon)^2 (\epsilon_c)^2} [\Gamma(3) \lambda c ((\epsilon_c)^3 - (\epsilon)^3) - (\Gamma(2))^2 ((\epsilon_c)^2 - (\epsilon)^2)^2] \quad (15)$$

$$\sigma^2 = \frac{1}{(\lambda c)^2 (\epsilon)^2 (\epsilon_c)^2} [2\lambda c ((\epsilon_c)^3 - (\epsilon)^3) - ((\epsilon_c)^2 - (\epsilon)^2)^2] \text{ as } \Gamma(3) = 2, \quad \Gamma(2) = 1$$

6.3 The Standard deviation (S.D)

$$\sigma = \frac{[2\lambda c ((\epsilon_c)^3 - (\epsilon)^3) - ((\epsilon_c)^2 - (\epsilon)^2)^2]^{\frac{1}{2}}}{(\lambda c)(\epsilon)(\epsilon_c)}$$

6.4 The coefficient of variance

$$C.V = \frac{[2\lambda c ((\epsilon_c)^3 - (\epsilon)^3) - ((\epsilon_c)^2 - (\epsilon)^2)^2]^{\frac{1}{2}}}{(\epsilon_c)^2 - (\epsilon)^2}$$

7.5 The Coefficient of skewness

$$C.S = \frac{(\lambda c)^2 \Gamma(4) [(\epsilon_c)^4 - (\epsilon)^4] - 3 \Gamma(2) \Gamma(3) \lambda c [(\epsilon_c)^3 - (\epsilon)^3] [(\epsilon_c)^2 - (\epsilon)^2] + 2 \Gamma(2) ((\epsilon_c)^2 - (\epsilon)^2)^3}{[\Gamma(3) \lambda c ((\epsilon_c)^3 - (\epsilon)^3) - (\Gamma(2))^2 ((\epsilon_c)^2 - (\epsilon)^2)^2]^{\frac{3}{2}}}$$

$$C.S = \frac{6(\lambda c)^2 [(\epsilon_c)^4 - (\epsilon)^4] - 6 \lambda c [(\epsilon_c)^3 - (\epsilon)^3] [(\epsilon_c)^2 - (\epsilon)^2] + 2 ((\epsilon_c)^2 - (\epsilon)^2)^3}{[\Gamma(3) \lambda c ((\epsilon_c)^3 - (\epsilon)^3) - (\Gamma(2))^2 ((\epsilon_c)^2 - (\epsilon)^2)^2]^{\frac{3}{2}}}, \Gamma(4) = 6, \tag{18}$$

6.6 Coefficient of kurtosis

Let $\epsilon_c = \lambda + \lambda c - n$, and ϵ

C.K

$$= \frac{(\lambda c)^3 \Gamma(5) [\epsilon_c^5 - \epsilon^5] - 4(\lambda c)^2 \Gamma(4) \Gamma(2) [\epsilon_c^4 - \epsilon^4] [\epsilon_c^2 - \epsilon^2] + 6 \lambda c \Gamma(3) (\Gamma(2))^2 [\epsilon_c^3 - \epsilon^3] [\epsilon_c^2 - \epsilon^2]^2 - 3(\Gamma(2))^4 [\epsilon_c^2 - \epsilon^2]^3}{[\Gamma(3) \lambda c (\epsilon_c^3 - \epsilon^3) - (\Gamma(2))^2 (\epsilon_c^2 - \epsilon^2)^2]^2}$$

as $\Gamma(5) = 24, \Gamma(4) = 6, \Gamma(3) = 2, \Gamma(2) = 1$ (19)

The Table 2 Given below shows the Mod (X_w), Mean (μ), Variance (σ^2), Standard Deviation (S.D), Coefficient of variance (C.V), Coefficient of Skewness (C.S), Coefficient of kurtosis S(C.K) with some values parameters.

Table2: The values of mean, mod, variance, S.D, C.V, C.S and C.K at various choices of parameters.									
λ	N	C	X_w	μ	σ^2	S.D	C.V	C.S	C.K
2	1	1	0.5493	1.3333	1.1111	1.0541	0.7926	1.7709	7.9200
		2	0.4024	1.2000	1.0400	1.0198	0.8498	1.9008	8.5562
		3	0.3243	1.1429	1.0204	1.0101	0.8838	1.9460	8.7680
		4	0.2747	1.1111	1.0123	1.0061	0.9055	1.9662	8.8554
5	1	1	0.1622	0.3611	0.0748	0.2735	0.7574	1.6601	7.3486
		2	0.1962	0.4583	0.1267	0.3560	0.7768	1.7284	7.7029
		3	0.2506	0.6429	0.2704	0.5200	0.8088	1.8194	8.1627
		4	0.3584	1.1667	1.0278	1.0138	0.8689	1.9284	8.6844
2	1	1	0.5493	1.3333	1.1111	0.0541	0.0406	1.7709	7.9200
		3	0.3054	0.7000	0.2900	0.5385	0.7693	1.7033	7.5731
		4	0.2118	0.4762	0.1315	0.3626	0.7614	1.6753	7.4269
		5	0.1622	0.3611	0.0748	0.2735	0.7574	1.6601	7.3486

7. Moment generating function of (MDWED)

$$M_{x(t)} = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \int_0^\infty e^{nx} e^{tx} e^{-\lambda x} (1-e^{-c\lambda x}) dx$$

$$M_{x(t)} = \frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c} \int_0^\infty e^{(n+t-\lambda)x} (1-e^{-c\lambda x}) dx$$

$$M_{x(t)=\frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c}} \int_0^\infty [e^{-(n+t-\lambda)x} - e^{-(n+t-\lambda-c\lambda)x}] dx$$

$$M_{x(t)=\frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c}} \int_0^\infty [e^{-(\lambda-n-t)x} - e^{-(\lambda+c\lambda-n-t)x}] dx$$

$$M_{x(t)=\frac{(n-\lambda)(n-\lambda-\lambda c)}{\lambda c}} \left[\frac{1}{(\lambda-n-t)} - \frac{1}{(\lambda+c\lambda-n-t)} \right] \tag{20}$$

8. Fisher information of (MDWED)

The fisher information of the above modified weighted exponential distribution can be fined as follow:

$$\log f_w(x; c, \lambda, n) = \log(n - \lambda) + \log(n - \lambda - \lambda c) + (n - \lambda)x + \log(1 - e^{-c\lambda x}) - \log(\lambda c) \tag{21}$$

Differentiating w.r.t c

$$\frac{\partial \log f_w(x; c, \lambda, n)}{\partial c} = \frac{(-\lambda)}{(n-\lambda-\lambda c)} + \frac{x\lambda e^{-c\lambda x}}{(1-e^{-c\lambda x})} - \frac{1}{c} \tag{22}$$

$$\frac{\partial^2}{\partial c^2} (\log f_w(x; c, \lambda, n)) = \frac{-\lambda^2}{(n-\lambda-\lambda c)^2} - \frac{x^2 \lambda^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} + \frac{1}{c^2}$$

$$E\left[\frac{\partial^2}{\partial c^2} (\log f_w(x; c, \lambda, n))\right] = E\left(\frac{-\lambda^2}{(n-\lambda-\lambda c)^2}\right) - E\left(\frac{x^2 \lambda^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2}\right) + E\left(\frac{1}{c^2}\right)$$

$$E\left[\frac{\partial^2}{\partial c^2} (\log f_w(x; c, \lambda, n))\right] = \frac{-\lambda^2}{(n-\lambda-\lambda c)^2} - \int_0^\infty f_w(x; c, \lambda, n) \frac{x^2 \lambda^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} dx + \frac{1}{c^2} \tag{23}$$

$$E\left[\frac{\partial^2}{\partial c^2} (\log f_w(x; c, \lambda, n))\right] = \frac{-\lambda^2}{(n-\lambda-\lambda c)^2} - \frac{(n-\lambda-\lambda c)(n-\lambda)}{\lambda^2 c^4} \sum_{j=0}^\infty \sum_{i=0}^\infty \frac{(-1)^{i+j} \Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}\right) \Gamma(j)}{\Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}-1\right) (i!) (j!)} \times \frac{2!}{(i+1)!} + \frac{1}{c^2} \tag{24}$$

Differentiating (20) w.r.t λ

$$\frac{\partial \log f_w(x; c, \lambda, n)}{\partial \lambda} = \frac{(-1)}{(n-\lambda)} - \frac{(1+c)}{(n-\lambda-\lambda c)} - x + \frac{cx e^{-c\lambda x}}{(1-e^{-c\lambda x})} - \frac{1}{\lambda} \tag{25}$$

$$\frac{\partial^2}{\partial \lambda^2} (\log f_w(x; c, \lambda, n)) = \frac{-1}{(n-\lambda)^2} - \frac{(1+c)^2}{(n-\lambda-\lambda c)^2} - \frac{c^2 x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} + \frac{1}{\lambda^2} \tag{26}$$

$$E\left[\frac{\partial^2}{\partial \lambda^2} (\log f_w(x; c, \lambda, n))\right] = E\left(\frac{-1}{(n-\lambda)^2}\right) - E\left(\frac{(1+c)^2}{(n-\lambda-\lambda c)^2}\right) - E\left(\frac{c^2 x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2}\right) + E\left(\frac{1}{\lambda^2}\right)$$

$$E\left[\frac{\partial^2}{\partial \lambda^2} (\log f_w(x; c, \lambda, n))\right] = \frac{-1}{(n-\lambda)^2} - \frac{(1+c)^2}{(n-\lambda-\lambda c)^2} - E\left(\frac{c^2 x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2}\right) + \frac{1}{\lambda^2}$$

$$E\left[\frac{\partial^2}{\partial \lambda^2} (\log f_w(x; c, \lambda, n))\right] = \frac{-1}{(n-\lambda)^2} - \frac{(1+c)^2}{(n-\lambda-\lambda c)^2} - \frac{(n-\lambda-\lambda c)(n-\lambda)}{\lambda^4 c^2} \sum_{j=0}^\infty \sum_{i=0}^\infty \frac{(-1)^{i+j} \Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}\right) \Gamma(j)}{\Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}-1\right) (i!) (j!)} \times \frac{2!}{(i+1)!} + \frac{1}{\lambda^2} \tag{27}$$

Differentiating (20) w.r.t n

$$\frac{\partial \log f_w(x; c, \lambda, n)}{\partial n} = \frac{1}{(n-\lambda)} + \frac{1}{(n-\lambda-\lambda c)} + x \tag{28}$$

$$\frac{\partial^2}{\partial n^2} (\log f_w(x; c, \lambda, n)) = -\frac{1}{(n-\lambda)^2} - \frac{1}{(n-\lambda-\lambda c)^2} \tag{29}$$

$$E \left[\frac{\partial^2}{\partial n^2} (\log f_w(x; c, \lambda, n)) \right] = E \left[-\frac{1}{(n-\lambda)^2} - \frac{1}{(n-\lambda-\lambda c)^2} \right]$$

$$E \left[\frac{\partial^2}{\partial n^2} (\log(x; c, \lambda, n)) \right] = -\frac{1}{(n-\lambda)^2} - \frac{1}{(n-\lambda-\lambda c)^2} \quad (30)$$

Differentiating (21) w.r.t λ

$$\frac{\partial^2}{\partial \lambda \partial c} (\log f_w(x; c, \lambda, n)) = \frac{(-n)}{(n-\lambda-\lambda c)^2} - \frac{c\lambda x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} + \frac{x e^{-c\lambda x}}{(1-e^{-c\lambda x})}$$

$$E \left[\frac{\partial^2}{\partial \lambda \partial c} (\log f_w(x; c, \lambda, n)) \right] = E \left[\frac{(-n)}{(n-\lambda-\lambda c)^2} \right] - E \left[\frac{c\lambda x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} \right] + E \left[\frac{x e^{-c\lambda x}}{(1-e^{-c\lambda x})} \right]$$

$$E \left[\frac{\partial^2}{\partial \lambda \partial c} (\log f_w(x; c, \lambda, n)) \right] = \frac{(-n)}{(n-\lambda-\lambda c)^2} - E \left[\frac{c\lambda x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} \right] + E \left[\frac{x e^{-c\lambda x}}{(1-e^{-c\lambda x})} \right]$$

$$E \left[\frac{\partial^2}{\partial \lambda \partial c} (\log f_w(x; c, \lambda, n)) \right] = \frac{(-n)}{(n-\lambda-\lambda c)^2} - \frac{(n-\lambda-\lambda c)(n-\lambda)}{\lambda^3 c^3} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j} \Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}\right) \Gamma(j)}{\Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}-1\right) (i!) (j!)} \times \frac{2!}{(i+1)!} + \frac{n-\lambda}{n-\lambda-\lambda c} \quad (31)$$

Where $E \left[\frac{x e^{-c\lambda x}}{(1-e^{-c\lambda x})} \right] = \frac{n-\lambda}{n-\lambda-\lambda c}$

Differentiating (21) w.r.t n

$$\frac{\partial^2}{\partial n \partial c} (\log f_w(x; c, \lambda, n)) = \frac{\lambda}{(n-\lambda-\lambda c)^2} = \frac{\partial^2}{\partial c \partial n} (\log f_w(x; c))$$

$$E \left[\frac{\partial^2}{\partial n \partial c} (\log(x; c, \lambda, n)) \right] = \frac{\lambda}{(n-\lambda-\lambda c)^2} = E \left[\frac{\partial^2}{\partial c \partial n} (\log f_w(x; c)) \right] \quad (32)$$

Differentiating (24) w.r.t n

$$\frac{\partial^2}{\partial n \partial \lambda} (\log f_w(x; c, \lambda, n)) = \frac{1}{(n-\lambda)^2} + \frac{(1+c)}{(n-\lambda-\lambda c)^2} = \frac{\partial^2}{\partial \lambda \partial n} (\log f_w(x; c))$$

$$E \left[\frac{\partial^2}{\partial n \partial \lambda} (\log f_w(x; c, \lambda, n)) \right] = \frac{1}{(n-\lambda)^2} + \frac{(1+c)}{(n-\lambda-\lambda c)^2} = E \left[\frac{\partial^2}{\partial \lambda \partial n} (\log f_w(x; c)) \right] \quad (33)$$

Differentiating (24) w.r.t c

$$\frac{\partial^2}{\partial c \partial \lambda} (\log f_w(x; c, \lambda, n)) = \frac{(-n)}{(n-\lambda-\lambda c)^2} - \frac{c\lambda x^2 e^{-c\lambda x}}{(1-e^{-c\lambda x})^2} + \frac{x e^{-c\lambda x}}{(1-e^{-c\lambda x})}$$

$$E \left[\frac{\partial^2}{\partial n \partial \lambda} (\log f_w(x; c, \lambda, n)) \right] = \frac{(-n)}{(n-\lambda-\lambda c)^2} - \frac{(n-\lambda-\lambda c)(n-\lambda)}{\lambda^3 c^3} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j} \Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}\right) \Gamma(j)}{\Gamma\left(\frac{\lambda+\lambda c-n}{\lambda c}-1\right) (i!) (j!)} \times \frac{2!}{(i+1)!} + \frac{n-\lambda}{n-\lambda-\lambda c} \quad (34)$$

Now the fisher information matrix for (MDWED) is given by:

$$I_w(x; c, \lambda, n) = \begin{bmatrix} -E \left[\frac{\partial^2}{\partial c^2} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial \lambda \partial c} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial n \partial c} (\log(x; c, \lambda, n)) \right] \\ -E \left[\frac{\partial^2}{\partial c \partial \lambda} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial \lambda^2} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial n \partial \lambda} (\log(x; c, \lambda, n)) \right] \\ -E \left[\frac{\partial^2}{\partial n \partial c} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial \lambda \partial n} (\log(x; c, \lambda, n)) \right] & -E \left[\frac{\partial^2}{\partial n^2} (\log(x; c, \lambda, n)) \right] \end{bmatrix} \quad (35)$$

9. Estimation of parameters of the MDWED

This section presented the Method of moment and maximum likelihood to estimate the parameter of the proposed MDWED.

9.1 Method of Moment estimator

The Method of Moment estimators are not efficient as Maximum Likelihood. They are often use due to simple computation. Since there are three parameters are used so we find the first 3 moment estimator. If X follows (MDWED) with parameter n, c and λ then k^{th} moment of X is given by:

$$E_{f_w}(x^k) = \Gamma(k + 1) \left[\frac{(\lambda + \lambda c - n)^{k+1} - (\lambda - n)^{k+1}}{\lambda c (\lambda - n)^k (\lambda + \lambda c - n)^k} \right]$$

For $k=1$ and $X_1, X_2, X_3 \dots \dots \dots X_n$ be an independent simple from the (MDWED). We obtain the 1st simple as:

$$\frac{1}{n} \sum_{j=0}^n X_j = \Gamma(2) \left[\frac{(\lambda + \lambda c - n)^2 - (\lambda - n)^2}{\lambda c (\lambda - n) (\lambda + \lambda c - n)} \right] \tag{36}$$

$$\bar{X} = \left[\frac{(\lambda + \lambda c - n)^2 - (\lambda - n)^2}{\lambda c (\lambda - n) (\lambda + \lambda c - n)} \right] \text{ where } \Gamma(2) = 1$$

$$\frac{1}{n} \sum_{j=0}^n X_j^2 = 2 \left[\frac{(\lambda + \lambda c - n)^3 - (\lambda - n)^3}{\lambda c (\lambda - n)^2 (\lambda + \lambda c - n)^2} \right] \text{ where } \Gamma(3) = 2 \tag{37}$$

$$\frac{1}{n} \sum_{j=0}^n X_j^3 = 6 \left[\frac{(\lambda + \lambda c - n)^4 - (\lambda - n)^4}{\lambda c (\lambda - n)^3 (\lambda + \lambda c - n)^3} \right] \tag{38}$$

Solving above equations for n, c and λ we can estimate the parameters.

We can find n, c and λ from above equations.

9.2 Maximum likelihood estimator

This is the best identified, most widely used, and most important method of estimation. First we write a likelihood function $L(\theta; x)$, and then find the value ε of θ which maximize $L(\theta; x)$ the log-likelihood function based on the random sample $x_1, x_2, x_3, \dots \dots \dots x_m$ is given by:

$$L(x; \lambda, n, c) = m \log(n - \lambda) + m \log(n - \lambda - \lambda c) + (n - \lambda) \sum x_i + \sum_{i=1}^m \log(1 - e^{-c \lambda x_i}) - m \log(\lambda c) \tag{39}$$

Taking partial derivative w.r.t λ, c and n then we have

$$\frac{\partial L(x; \lambda, n, c)}{\partial \lambda} = \frac{-m}{(n-\lambda)} - \frac{m(1+c)}{(n-\lambda-\lambda c)} - \sum_{i=1}^m x_i + \sum_{i=1}^m \frac{c x_i (e^{-c \lambda x_i})}{(1 - e^{-c \lambda x_i})} - \frac{m}{\lambda} \tag{40}$$

$$\frac{\partial L(x; \lambda, n, c)}{\partial c} = \frac{(-m \lambda)}{(n-\lambda-\lambda c)} + \sum_{i=1}^m \frac{\lambda x_i (e^{-c \lambda x_i})}{(1 - e^{-c \lambda x_i})} - \frac{m}{c} \tag{41}$$

$$\frac{\partial L(x; \lambda, n, c)}{\partial n} = \frac{m}{(n-\lambda)} + \frac{m}{(n-\lambda-\lambda c)} + \sum_{i=1}^m x_i \tag{42}$$

Equating above equation to zero we have

$$\frac{-m}{(n-\lambda)} - \frac{m(1+c)}{(n-\lambda-\lambda c)} - \sum_{i=1}^m x_i + \sum_{i=1}^m \frac{c x_i (e^{-c \lambda x_i})}{(1 - e^{-c \lambda x_i})} - \frac{m}{\lambda} = 0 \tag{43}$$

$$\frac{(-m \lambda)}{(n-\lambda-\lambda c)} + \sum_{i=1}^m \frac{\lambda x_i (e^{-c \lambda x_i})}{(1 - e^{-c \lambda x_i})} - \frac{m}{c} = 0 \tag{44}$$

$$\frac{m}{(n-\lambda)} + \frac{m}{(n-\lambda-\lambda c)} + \sum_{i=1}^m x_i = 0 \tag{45}$$

To find the value of n, λ and c we have to solve (42), (43) and (44) using numerical technique methods. We use newton Raphson method to (see Adi (1966)) to obtain the solution of nonlinear equation given above.

10. Application

In this section, an illustrative example is presented to demonstrate the application of the MDWED in the real life. The rain fall data given by Khadim and Hantoosh (2013) is used to fit the MDWED. Following the Khadim and Hantoosh (2013), the MDWED is fitted on rainfall data using Newton Raphson method. Starting with $\lambda = 0.02, n = 0.01$ and $c = 0.9$ as initial guess, Newton Raphson after 11 iteration gives values of parameters correct up to 4 decimals places as $\hat{\lambda} = 0.0087, \hat{n} = -0.0137$ and $\hat{c} = 0.2253$. For the validity of our Newton Raphson code, the DWED is also fitted on the given data. The values of parameters are obtained $\hat{c} = 0.00051$ and $\hat{\lambda} = 0.03513$, which are similar to the values obtained in Khadim and Hantoosh (2013) if round off up to 4 decimal points.

We can also apply formal goodness of fit test in order to verify which distribution fits to the given data. We apply the Kolmogorov-Smirnov test (KS test) for the goodness of fit purpose. **Table 4.** Lists the MLE estimates of the parameters λ, n and c and the values of the test statistics which is KS test. The p values of KS for (MDWED) and (DWED) are 0.450 and 0.285 respectively. The results in **table 4** shows that (MDWED) fits the data as well as the (DWED).

Table 3. Parameter estimates and K-S statistics for total annual rainfall (mm) in province of Babylon.		
Distributions	DWED	MDWED
Parameters estimates	$\hat{\lambda} = 0.0351$ $\hat{c} = 0.0005$	$\hat{\lambda} = 0.0087$ $\hat{n} = -0.0137$ $\hat{c} = 0.2253$
K.S statistics	0.1876	0.1019
P values	0.285	0.450

My second judgement is based on the probability plot. Here the empirical distribution functions values are taken along x-axis and theoretical distribution values are taken along y-axis. For the (MDWED) we have computed based on (4) for the y-axis against the empirical CDF $(i - 0.5)/n$ where $i = 1, 2, 3, \dots, n$ and (x_i) are the values in the sample of data, in order from smallest to largest. The probability plot corresponding to the (DWED) and (MDWED) are given in figure 6

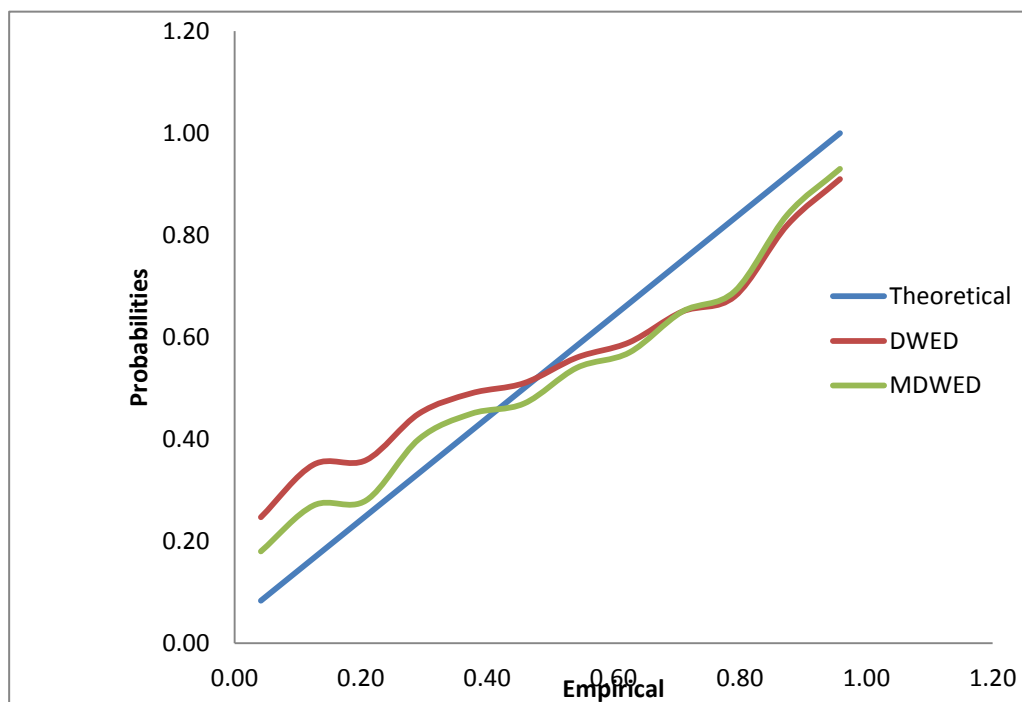


Figure-6: The probability Plot of MDWED and DWED for rainfall data.

A statistical distribution which is close to the line of theoretical probabilities gives better fit. The MDWED is better fit to rainfall data as compare to the DWED as it is obvious from figure 6. Therefore, on the basis of these two comparisons it is concluded that the MDWED provides better fit than DWED for modelling the rainfall data.

11. Discussions and Conclusions

The (MDWED) is the consequence of (DWED) hosted by Al-Khadim and Hantoosh (2013). In this practice the comparison of (DWED) and (MDWED) is made by well-known statistical test K.S. Test claims that statistical distribution better adequate for grater p values. The p values of (MDWED) and (DWED) are estimated in **Table 3**. The p value of (MDWED) is grater then p value of (DWED) which claim that (MDWED) gives better adequate for rainfall data and e^{nx} is better weight choice as compared to x . Also we found that New Class Weighted Exponential Distribution given by Gupta and Kundu (2009) is a special case of the MDWED hence (MDWED) is most broad-spectrum and useful distribution. We hope the MDWED may attract extensive applications in lifetime data analysis and other fields. The future research may consider in parameter estimation using Bayesian or other approaches.

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