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HLLY'S Theorem in Banach Lattice

With order continuous norm in term of a double sequence

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Abstract

In this paper, we introduce Helly and Helly -Bray theorems in term double sequence in the context of Riesz space with order continuous norm, and we review some of the results that are needed to prove our theorems.

We state some definitions, like as the moment double sequence and complete moment. Later we prove the corresponding generalized moment theorem and representation in term of double sequence of positive operators.

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1. Introduction

Helly's theorem had been of some importance along time above all in the probability theory in connection with a problem of moments of distributions.

Let $f, g: [\alpha, \beta] \times [\alpha', \beta'] \to R \times R$, be a monotone map.

Then the following is true:

1. f, g has countable set of discontinuity points .

2. If (f_{nm}, g_{nm}) is a double sequence of functions from $[\alpha, \beta] \times [\alpha', \beta']$ to $R \times R$, which is uniformly bounded and monotone. The there exists a sub double sequence $(f_{n_j m_k}, g_{n_j m_k})$ of (f_{nm}, g_{nm}) converging to a monotone map f, g.

3. Let (f_{nm}, g_{nm}) be a double sequence from $[\alpha, \beta] \times [\alpha', \beta']$ to $R \times R$, which is monotone and converging to a map$f, g: [\alpha, \beta] \times [\alpha', \beta'] \to R \times R$.

Then for any continuous maps $h, w: [\alpha, \beta] \times [\alpha, \beta] \to R \times R$, we have

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} h(t,u) df_{nm}(t) df_{nm}(u) = \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} h(t,u) df(t) df(u)$$

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} w(t,u) dg_{nm}(t) dg_{nm}(u) = \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} w(t,u) dg(t) dg(u)$$

We shall investigate these three properties for function f, g and f_{nm}, g_{nm} with values in Banach lattices. We shall see that they do not remain true for any Banach lattice and that we must confine ourselves to the narrower class of Banach lattices with order continuous norm. Next, we shall give two applications of these investigations a generalized moment theorem and a representation theorem.

2.Helly and Helly-Bray theorem in term of a double sequence.

We recall that Banach lattice $E \times E$ is said to have an order continuous norm if $\lim_{\alpha,\beta} \| x_{\alpha\beta} \| = 0$, $\lim_{\alpha\beta} \| y_{\alpha\beta} \| = 0$, for every nonincreasing double net $(x_{\alpha\beta}, y_{\alpha\beta})$ in $E \times E$ such that $inf\, x_{\alpha\beta} = 0$, $inf\, y_{\alpha\beta} = 0$.

Proposition 2.1: [1]

The Banach lattice E has order continuous norm if and only if each order interval in E is weakly compact. Moreover, a Banach lattice with order continuous norm is necessarily Dedekind complete.

The equivalence of (1) and (2) in the following proposition are the main result in [4]. We will use only (1)⇒(2) (that is a direct application of [5,III.2,Theorem 3]).

Proposition 2.2: Suppose $E \times E$ is a σ −Dedekind complete Banach lattice, the following conditions are equivalent:

1. $E \times E$ has order continuous norm;

2. Every non-decreasing function $f: [0,1] \times [0,1] \to E \times E$ has at most countably many points of discontinuity.

Proposition 2.3: Let f be a non-decreasing function defined on an interval I × I of $R \times R$, with values in a Banach lattice $E \times E$ with order continuous norm, the folowing conditions are equivalent:

1. $f(x, y) = inf\{f(z, e)/x < z \in I , y < e \in I \}$,

2. f is right-continuous at (x, y) for the norm topology;

3. f is right-continuous at (x, y) for the weak topology.

For the left-continuity, we have the similar characterizations, and in particular for continuity.

Proof: The proof is similar in [1]

We now state and prove a Helly's theorem in term of a double sequence in the setting of Banach lattice.

Theorem 2.4: Let $E \times E$ is a Banach lattice and $[\alpha, \beta] \times [\alpha', \beta']$ be a closed interval in $R \times R$. The following conditions are equivalent:

1. $E \times E$ has order continuous norm;

2. If $(f_{nm}, g_{nm})_{n,m\in N}$ is a double sequence of nondecreasing functions on $[\alpha, \beta] \times [\alpha', \beta']$, with values in some order interval $[a, b] \times [a', b']$ in $E \times E$, then there exists a double subsequence $(f_{n_j m_k}, g_{n_j m_k})_{\, n_j m_k \in N}$ of (f_{nm}, g_{nm}) and a non-decreasing functions $f, g: [\alpha, \beta] \times [\alpha', \beta'] \to [a, b] \times [a', b']$

such that $(f_{n_j m_k}(x, y), g_{n_j m_k}(x, y))_{\, n_j m_k \in N}$ is convergent to $f(x, y), g(x, y)$ for the weak topology $\sigma(E \times E, E' \times E')$ at each continuity point $(x, y) \in]\alpha, \beta[\times]\alpha', \beta'[$ of f, g , but also for $(x \vee y) = (\alpha \vee \alpha')$ and for $(x \vee y) = (\beta \vee \beta')$.

Moreover, if $E \times E$ has order continuous norm, then the functions f, g in (2) can be assumed to be right-continuous at every points $(x, y) \in]\alpha, \beta[\times]\alpha', \beta'[$.

Proof: 1 ⟹ 2

Let $(\alpha_{kl},\alpha'_{kl})_{k,l\in N}$ be a dense double sequence in $[\alpha,\beta]\times[\alpha',\beta']$ including (α,α') and (β,β'). Since the order interval $[a,b]\times[a',b']$ is weakly compact (Proposition 1), the double sequence (f_{nm},g_{nm}) has a double subsequence $(f_{nm}^{(1)},g_{nm}^{(1)})$ such that $(f_{nm}^{(1)}(\alpha_1,\alpha'_1),g_{nm}^{(1)}(\alpha_1,\alpha'_1))_{n,m\in N}$ is weakly convergent to some $(a_1,a'_1)\in[a,b]\times[a',b']$ (by Eberlein-Smulian theorem [2, The.10.13]). By induction , for p=2,3,… , q=2,3,…, the double sequence $(f_{nm}^{((p-1),(q-1))},g_{nm}^{((p-1),(q-1))})$ has a subdouble sequence $(f_{nm}^{(p,q)},g_{nm}^{(p,q)})$ such that $(f_{nm}^{(p,q)}(\alpha_p,\alpha'_q),g_{nm}^{(p,q)}(\alpha_p,\alpha'_q))_{n,m\in N}$ is weakly convergent to some $(a_p,a'_q)\in[a,b]\times[a',b']$. Using the well known diagonal process, we define a sub double sequence $(f_{n_jm_k},g_{n_jm_k})_{j,k\in N}$ of (f_{nm},g_{nm}) by $(f_{n_jm_k},g_{n_jm_k})=(f_{jk}^{(j,k)},g_{jk}^{(j,k)})$
It is clear that the double sequence $(f_{n_jm_k}(\alpha_p,\alpha'_q),g_{n_jm_k}(\alpha_p,\alpha'_q))_{j,k\in N}$ is weakly convergent to (a_p,a'_q) $(p=1,2,\dots,q=1,2,\dots,)$ and, the considered functions being non-decreasing, that $(\alpha_{r_1}\vee\alpha'_{r_2})\le(\alpha_{s_1}\vee\alpha'_{S_2})$ implies $(a_{r_1}\vee a'_{r_2})\le(a_{s_1}\vee a'_{S_2})$.

Recalling that $E\times E$ is Dedekind complete (Proposition 1), we now define a non-decreasing function $f,g:[\alpha,\beta]\times[\alpha',\beta']\to[a,b]\times[a',b']$ by

$$f(x,y)=inf\{(a_{r_1},b_{r_2})\,|\,x\le\alpha_{r_1},y\le b_{r_2}\}$$

$$g(x,y)=inf\{(a_{r_1},b_{r_2})\,|\,x\le\alpha_{r_1},y\le b_{r_2}\}$$

It is clear that $\lim_{j,k\to\infty}f_{n_jm_k}(x,y)=f(x,y),\lim_{j,k\to\infty}g_{n_jm_k}(x,y)=g(x,y)$

for $\sigma(E\times E,E'\times E')$, if $(x,y)=(\alpha,\alpha')$ or $(x,y)=(\beta,\beta')$. Let us show that this equality remains true for any $(x,y)\in[\alpha,\beta]\times[\alpha',\beta']$ such that f,g is continuous at(x,y). To this end , we recall first that the topological dual $E'\times E'$ of $E\times E$ is the set of all differences of two positive linear functionals on $E\times E$ [2,Corollary 12.5].Let φbe any positive linear functionals on $E\times E$. For any $(\alpha_p\vee\alpha'_q)\le(x\vee y)$, we have

$$\varphi(f_{n_jm_k}(\alpha_p,\alpha'_q))\le\varphi(f_{n_jm_k}(x,y))\,,\quad\varphi(g_{n_jm_k}(\alpha_p,\alpha'_q))\le\varphi(g_{n_jm_k}(x,y))$$

and, since $(f_{n_jm_k}(\alpha_p,\beta_q))_{j,k\in N}$ is weakly convergent to $f(\alpha_p,\alpha'_q)=(a_p,a'_q)$ and $(g_{n_jm_k}(\alpha_p,\alpha'_q))_{j,k\in N}$ is weakly convergent to $(\alpha_p,\alpha'_q)=(a_p,a'_q)$, we obtain :

$$\varphi(f(\alpha_p,\alpha'_q))\le\lim_{j,k\to\infty}inf\,\varphi(f_{n_jm_k}(x,y))$$

$$\varphi(g(\alpha_p,\alpha'_q))\le\lim_{j,k\to\infty}inf\,\varphi(g_{n_jm_k}(x,y))$$

But $\{(\alpha_p,\alpha'_q)|\alpha_p\le x\,,\alpha'_q\le y\}$ is a directed upwards set converging to (x,y) and, by continuity of f,g at (x,y) , $\varphi(f(x,y))\,,\varphi(g(x,y))$ is the limit of the double net $\left\{\varphi\left(f(\alpha_p,\alpha'_q)\right)\middle|\alpha_p\le x\,,\alpha'_q\le y\right\}$,

$\{\varphi\left(g(\alpha_p,\alpha'_q)\right)\,|\,\alpha_p\le x\,,\alpha'_q\le y\,\}$. It follows that,

$$\varphi(f(x,y))\le\lim_{j,k\to\infty}inf\,\varphi\left(f_{n_jm_k}(x,y)\right)$$

$$\varphi(g(x,y))\le\lim_{j,k\to\infty}inf\,\varphi\left(g_{n_jm_k}(x,y)\right)$$

and , similarly , by considering $(\alpha_p\vee\alpha'_q)\ge(x\vee y\,)$, we also obtain

$$\lim_{j,k\to\infty}sup\,\varphi\left(f_{n_jm_k}(x,y)\right)\le\varphi(f(x,y))$$

$$\lim_{j,k\to\infty} sup\ \varphi\left(g_{n_j m_k}(x,y)\right) \le \varphi\big(g(x,y)\big).$$

We conclude that $\varphi\big(f(x,y)\big) = \lim_{j,k\to\infty} \varphi(f_{n_j m_k}(x,y))$, $\varphi\big(g(x,y)\big) = \lim_{j,k\to\infty} \varphi(g_{n_j m_k}(x,y))$ and, finally, that $f(x,y), g(x,y)$ is the weak limit of the double sequence $(f_{n_j m_k}(x,y), g_{n_j m_k}(x,y))_{\ j,k\in N}$.

To prove that we may assume f, g right-continuous at every point $(x,y) \in [\alpha,\beta] \times [\alpha',\beta']$, it suffices to replace f, g by the functions $h, w: [\alpha,\beta] \times [\alpha',\beta'] \to [a,b] \times [a',b']$, defined by

$$\begin{cases} h(\alpha,\alpha') = f(\alpha,\alpha')\,, h(\beta,\beta') = f(\beta,\beta') \\ h(x,y) = inf\{f(z,e)\,\big|\, x \le z\ \in [\alpha,\beta]\ , \\ \qquad\qquad y \le e \in [\alpha',\beta']\} \end{cases} \qquad \text{for } (x,y) \in [\alpha,\beta] \times [\alpha',\beta']$$

$$\begin{cases} w(\alpha,\alpha') = g(\alpha,\alpha'), w(\beta,\beta') = g(\beta,\beta') \\ w(x,y) = in\,f\ g(z,e)\,\big|\, x \le z\ \in [\alpha,\beta], \\ \qquad\qquad y \le e \in [\alpha',\beta']\} \end{cases} \qquad \text{for } (x,y) \in [\alpha,\beta] \times [\alpha',\beta']$$

It is obvious that $f \le h$, $g \le w$, h, w are non-decreasing, right-continuous on $]\alpha,\beta[\times]\alpha',\beta'[$ and also that $f(x,y) = h(x,y)$, $g(x,y) = w(x,y)$ for some $(x,y) \in]\alpha,\beta[\times]\alpha',\beta'[$ if and only if f, g are right-continuous at (x,y) . We now show that f, g is continuous at $(x,y) \in]\alpha,\beta[\times]\alpha',\beta'[$ if and only if h, w is continuous at (x,y) , we have successively

$$h(x,y) = f(x,y) = sup\{f(z,e)\,\big|\, z < x, e < y\} \le sup\{h(z,e)\,\big|\, z < x,\ e <\ y\}$$

$$w(x,y) = g(x,y) = sup\big\{g(z,e)\,\big|\, z < x, e < y\big\} \le sup\big\{w(z,e)\,\big|\, z < x, e < y\big\}$$

and, consequently

$$h(x,y) = sup\big\{h(z,e)\,\big|\, z < x, e < y\big\},$$

$$w(x,y) = sup\{w(z,e)\,\big|\, z < x, e < y\}$$

Conversely, if h, w is continuous at (x,y) , we then have:

$$f(x,y) \le h(x,y) = sup_{\substack{z<x \\ e<y}}\ h(z,e) = sup_{\substack{z<x \\ e<y}}\left(inf_{\substack{z<z_1 \\ e<e_1}} f(z_1,e_1)\right)$$

$$\le sup_{\substack{z<x \\ e<y}}\ (inf_{\substack{z<z_1<x \\ e<e_1<y}} f(z_1,e_1) \le f(x,y)$$

$$g(x,y) \le w(x,y) = sup_{\substack{z<x \\ e<y}}\ w(z,e) = sup_{\substack{z<x \\ e<y}}\left(inf_{\substack{z<z_1 \\ e<e_1}}\ g(z_1,e_1)\right)$$

$$\le sup_{\substack{z<x \\ e<y}}\ (inf_{\substack{z<z_1<x \\ e<e_1<y}}\ g\ (z_1,e_1) \le g(x,y)$$

Hence $f(x,y) = h(x,y)$, $g(x,y) = w(x,y)$ and f, g is right-continuous at (x,y). Moreover, by Proposition 2, there exists an increasing double sequence (z_{nm}, e_{nm}) in $[\alpha,\beta] \times [\alpha',\beta']$, converging to (x,y), such that f, g is continuous at each (z_{nm}, e_{nm}) .It follows that:

$$f(x,y) = h(x,y) = sup_{n,m}\left(inf_{\substack{z_{nm}<z_1 \\ e_{nm}<e_1}} f(z_1,e_1)\right) = sup_{n,m}\, f(z_{nm},e_{nm})$$

$$\leq sup_{\substack{z<x \\ e<y}}\, f(z,e$$

$$g(x,y) = w(x,y) = sup_{n,m}\left(inf_{\substack{z_{nm}<z_1 \\ e_{nm}<e_1}} g(z_1,e_1)\right) = sup_{n,m}\, g(z_{nm},e_{nm})$$

$$\leq sup_{\substack{z<x \\ e<y}}\, g(z,e\ )$$

We conclude that $f(x,y) = sup_{\substack{z<x \\ e<y}} f(x,y)$, $g(x,y) = sup_{\substack{z<x \\ e<y}} g(x,y)$ and, finally, f,g are also left-continuous at (x,y) .

2 ⇒ 1.

Suppose 2 is true , then for every order interval $[a,b]\times[a',b']$ in $E\times E$, for every double sequence in $[a,b]\times[a',b']$ must have a double subsequence weakly converging to some point in $[a,b]\times[a',b']$. Thus order interval is weakly compact (by Eberlein-Smulian theorem again [1, The. 10.13]). By Proposition 1 that E has order continuous norm.

◙

The following example shows that functions h,w in the previous proof is not necessarily to be right-continuous at (α,α') or left-continuous at (β,β').

Example 2.5: For n=1,2,…, define

$f_n\,, g_n\colon [0,1]\times[0,1]\to[0,1]\times[0,1]$ by

$f_n(x,y) = (0,0)$, $g_n(x,y) = (0,0)$ if $x,y \in \left[0,\frac{1}{n}\right] \wedge \left[0,\frac{1}{n}\right]$

$f_n(x,y) = \left(\frac{1}{2},\frac{1}{2}\right)$, $g_n(x,y) = \left(\frac{1}{2},\frac{1}{2}\right)$ if $x,y \in]\frac{1}{n},1-\frac{1}{n}[\wedge [\frac{1}{n},1-\frac{1}{n}[$

$f_n(x,y) = (1,1)$, $g_n(x,y) = (1,1)$ if $x,y \in \left[1-\frac{1}{n},1\right] \wedge \left[1-\frac{1}{n}\right]$

Thus, the functions h,w (which coincide with f,g) is clearly not right-continuous at $(0,0)$, nor left-continuous at $(1,1)$.

In order to consider a Helly-Bray theorem in the context of banach lattices, we need the following lemma.

Lemma2.6: Suppose $E\times E$ be a σ −Dedekind complete vector lattice, $[\alpha,\beta]\times[\alpha',\beta']$ a closed interval in $R\times R$ and f a non-decreasing functions from $[\alpha,\beta]\times[\alpha',\beta']$ into $E\times E$.

Let also $\alpha = x_0 < x_1 < \cdots < x_p = \beta$, $\alpha' = y_0 < y_0 < \cdots < y_q = \beta'$ assume that f is order continuous at $(x_1,y_1),\ldots,(x_{p-1},y_{q-1})$, and consider a functions

$h\colon [\alpha,\beta]\times[\alpha',\beta']\to R\times R$ such that

$$\begin{cases} h(x,y) = constant\ (z_j,e_i) & if \quad x\in[x_{j-1},x_j[\,,\ \ y\in[y_{i-1},y_i[\\ & \quad (1\le j\le p)\,,(1\le i\le q) \\ h(\beta,\beta') = (z_p,e_q) & \end{cases}$$

Then h is integrable with respect to f and

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h(t,u)df(t)df(u) = \sum_{j=1}^{p}\sum_{i=1}^{q}(z_j + e_i)[f(x_j, y_i) - f(x_{j-1}, y_{i-1})]$$

Proof: For simplicity, we only prove that

$$\int_{\alpha}^{x_1}\int_{\alpha'}^{y_1} h(t,u)df(t)df(u) = (z_1, e_1)[f(x_1, y_1) - f(\alpha, \alpha')]$$

Given any partition $\alpha = x'_0 < x'_1 < \cdots < x'_{r_1} = x_1$ of $[\alpha, x_1]$,

$\alpha' = y'_0 < y'_1 < \cdots < y'_{r_2} = y_1$ of $[\alpha', y_1]$ and points $t_{i_1} \in [x'_{i_1-1}, x'_{i_1}]$, $(1 \le i_1 \le r_1)$, $u_{i_2} \in [y'_{i_2-1}, y'_{i_2}]$, $(1 \le i_2 \le r_2)$, we have :

$$\left|\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2} h(t_{i_1}, u_{i_2})[f(x'_{i_1}, y'_{i_2}) - f(x'_{i_1-1}, y'_{i_2-1})] - (z_1 + e_1)[f(x_1, y_1) - f(\alpha, \alpha')]\right|$$
$$= \left|h(t_{r_1}, u_{r_2}) - (z_1 + e_1)\right| [f(x_1, y_1) - f(x'_{r_1-1}, y'_{r_2-1})]$$
$$\le \left|(z_2 + e_2) - (z_1 + e_1)\right| [f(x_1, y_1) - f(x'_{r_1-1}, y'_{r_2-1})]$$

We now choose an increasing double sequence $(\alpha_{nm}, \beta_{nm})$ in $[\alpha, x_1[\times [\alpha', y_1[$, converging to (x_1, y_1) . Since f order continuous at (x_1, y_1) , we have

$$f(x_1, y_1) = \sup\{f(x,y) \,|\, \alpha \le x < x_1 , \alpha' \le y < y_1\} = sup_{n,m}(\alpha_{nm}, \beta_{nm})$$

Then $(\rho_{nm}, \rho'_{nm}) = \left|(z_2 + e_2) - (z_1 + e_1)\right| [f(x_1, y_1) - f(\alpha_{nm}, \beta_{nm})]$ is a nonincreasing double sequence in E such that

$$\inf_{n,m} \rho_{nm} \vee \inf_{n,m} \rho'_{nm} = 0\,.$$

On the other hand, letting $(\delta_{nm} \vee \delta'_{nm}) = ((x_1) - \rho_{nm}) > 0$, we see that

$$\max_{\substack{1\le i_1\le r_1\\ 1\le i_2\le r_2}} ((x'_{i_1}, y'_{i_2}) - (x'_{i_1-1}, y'_{i_2-1})) \le (\delta_{nm}, \delta'_{nm}) \Rightarrow \left|(z_2, e_2) - (z_1, e_1)\right| [f(x_1, y_1) - f(x'_{r_1-1}, y'_{r_2-1})] \le (\rho_{nm}, \rho'_{nm})$$

The proof is complete. ◙

We now state and prove a Helly-Bray theorem in term of a double sequence:

Theorem 2.7: Consider a closed interval $[\alpha,\beta]\times[\alpha',\beta']$ in $R\times R$, a Banach lattice $E\times E$ with order continuous norm and an order interval $[a,b]\times[a',b']$ in $E\times E$. Let (f_{nm},g_{nm}) be a double sequence of nondecreasing functions from $[\alpha,\beta]\times[\alpha',\beta']$ into $[a,b]\times[a',b']$ and assume there exists a nondecreasing functions $f,g:[\alpha,\beta]\times[\alpha',\beta']\to[a,b]\times[a',b']$ such that

$$\lim_{n,m\to\infty} f_{nm}(x,y)=f(x,y)\ ,\ \lim_{n,m\to\infty} g_{nm}(x,y)=g(x,y)$$

For the weak topology $\sigma(E\times E,E'\times E')$ at each continuity point $(x,y)\in[\alpha,\beta]\times[\alpha',\beta']$ of f,g , but also for $(x,y)=(\alpha,\alpha')$ and $(x,y)=(\beta,\beta')$ .Then, for each continuous function $h,w:[\alpha,\beta]\times[\alpha',\beta']\to R\times R$, we have

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h(t,u)df_{nm}(t)df_{nm}(u)=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta'} h(t,u)df(t)df(u),$$

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w(t,u)dg_{nm}(t)dg_{nm}(u)=\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w(t,u)dg(t)dg(u)$$

For $\sigma(E\times E,E'\times E')$.

Proof: By Proposition 2, we have know that f,g has at most countable many points of discontinuity. For $p=q=1,2,,\dots,,$

let $\alpha=x_0^{(p)}<x_1^{(p)}<\cdots<x_{k_p}^{(p)}=\beta,\ \alpha'=y_0^{(q)}<y_1^{(q)}<\cdots<x_{k_q}^{(q)}=\beta'$ be points of continuity of f,g , excepted perhaps α and β , such that $|h(x,y)-h(z,e)|\le\frac{1}{p+q}$, $|w(x,y)-w(z,e)|\le\frac{1}{p+q}$

if $x,z\in\left[x_{j-1}^{(p)},x_j^{(p)}\right]$, $y,e\in\left[y_{i-1}^{(q)},y_i^{(q)}\right]$ $(1\le j\le k_p),(1\le i\le k_q)$.

Define $h_{p,q},w_{p,q}$ on $[\alpha,\beta]\times[\alpha',\beta']$ by $h_{p,q}(x,y)=h(x_{j-1}^{(p)},y_{i-1}^{(q)})$ if $x\in[x_{j-1}^{(p)},x_j^{(p)}[,\ y\in[y_{i-1}^{(q)},y_i^{(q)}[\ ,(1\le j\le k_p),(1\le i\le k_q)$

and $h_{p,q}(\beta,\beta')=h(x_{k_p-1}^{(p)},y_{k_q-1}^{(q)})\ ,w_{p,q}(\beta,\beta)=w(x_{k_p-1}^{(p)},y_{k_q-1}^{(q)})$.

By Lemma 6 , we have

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h_{p,q}(t,u)df_{nm}(t)df_{nm}(u)=\sum_{i=1}^{k_p}\sum_{j=1}^{k_q} h\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)[f_{nm}\left(x_j^{(p)},y_i^{(q)}\right)-f_{nm}\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)]$$

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w_{p,q}(t,u)dg_{nm}(t)dg_{nm}(u)=\sum_{i=1}^{k_p}\sum_{j=1}^{k_q} w\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)[g_{nm}\left(x_j^{(p)},y_i^{(q)}\right)-g_{nm}\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)]$$

And this weakly converges in n,m to

$$\sum_{j=1}^{k_p}\sum_{i=1}^{k_q} h\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)\left[f\left(x_j^{(p)},y_i^{(q)}\right)-f\left(x_{j-1}^{(p)},y_{i-1}^{(q)}\right)\right]=\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h_{p,q}(t,u)df(t)df(u)\,.$$

$$\sum_{j=1}^{k_p}\sum_{i=1}^{k_q} w\left(x_{j-1}^{(p)}, y_{i-1}^{(q)}\right)\left[g\left(x_j^{(p)}, y_i^{(q)}\right) - g\left(x_{j-1}^{(p)}, y_{i-1}^{(q)}\right)\right] = \int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w_{p,q}(t,u)dg(t)dg(u)\,.$$

On the other hand, it follows from

$$\left|\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(h - h_{p,q})df_{nm}\right| \le \frac{1}{p+q}((b+b') - (a+a')) \quad and$$

$$\left|\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(h - h_{p,q})df\right| \le \frac{1}{p+q}((b+b') - (a+a'))$$

$$\left|\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(w - w_{p,q})dg_{nm}\right| \le \frac{1}{p+q}((b+b') - (a+a')) \quad and$$

$$\left|\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(w - w_{p,q})dg\right| \le \frac{1}{p+q}((b+b') - (a+a'))$$

that

$$\lim_{p,q\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(h - h_{p,q})df_{nm} = \lim_{p,q\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(h - h_{p,q})df = 0$$

$$\lim_{p,q\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(w - w_{p,q})dg_{nm} = \lim_{p,q\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'}(w - w_{p,q})dg = 0$$

uniformly in n, m for the norm topology. The result follows. ◙

Next, we give the following Corollary:

Corollary 2.8: With the same assumptions as in Theorem 2, we have also

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f_{nm}(t,u)dh(t)dh(u) = \int_{\alpha'}^{\beta}\int_{\alpha'}^{\beta'} f(t,u)dh(t)dh(u)$$

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g_{nm}(t,u)dw(t)dw(u) = \int_{\alpha'}^{\beta}\int_{\alpha'}^{\beta'} g(t,u)dw(t)dw(u)$$

for $\sigma(E\times E, E'\times E')$

Proof : By the formula of integration by parts [3] , we have

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f_{nm}(t,u)dh(t,u) = f_{nm}(\beta,\beta')h(\beta,\beta') - f_{nm}(\alpha,\alpha')h(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h(t,u)df_{nm}(t)df_{nm}(u) \quad \dots.(1)$$

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g_{nm}(t,u)dw(t,u) = g_{nm}(\beta,\beta')w(\beta,\beta') - g_{nm}(\alpha,\alpha)w(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w(t,u)dg_{nm}(t)dg_{nm}(u) \quad \dots(2)$$

In the same way, we can prove that

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f(t,u)dh(t,u) = f(\beta,\beta')h(\beta,\beta') - f(\alpha,\alpha')h(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h(t,u)df(t)df(u)$$

$$\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g(t,u)dw(t,u) = g(\beta,\beta')w(\beta,\beta') - g(\alpha,\alpha')w(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w(t,u)dg(t)dg(u)$$

If we take the $limit$ in (1) and (2) as $n,m \to \infty$, we have

$$\lim_{n,m\to\infty}(\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f_{nm}(t,u)dh(t,u) = f_{nm}(\beta,\beta')h(\beta,\beta') - f_{nm}(\alpha,\alpha)h(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} h(t,u)df_{nm}(t)df_{nm}(u))$$

$$\lim_{n,m\to\infty}(\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g_{nm}(t,u)dw(t,u) = g_{nm}(\beta,\beta')w(\beta,\beta') - g_{nm}(\alpha,\alpha')w(\alpha,\alpha')$$

$$-\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} w(t,u)dg_{nm}(t)dg_{nm}(u))$$

Thus

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f_{nm}(t,u)dh(t)dh(u) = \int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} f(t,u)dh(t)dh(u)$$

$$\lim_{n,m\to\infty}\int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g_{nm}(t,u)dw(t)dw(u) = \int_{\alpha}^{\beta}\int_{\alpha'}^{\beta'} g(t,u)dw(t)dw(u)$$ ◙

3. Applications

As in the classical case, we are now able to prove a moment theorem and to deduce from it a representation theorem for operators on the space of continuous function on $[0,1]$, by means of a nondecreasing function . Our proofs are easy adaptions of the classical proofs. We include these proofs for sake of completeness.

Let us set some definitions, which will be used for prove the theorems following.

Definition 3.1: Let E be a Banach lattice and Let $f:[\alpha,\beta]\times[\alpha,\beta]\to E$, $[\alpha,\beta]\subset R$.

f is called o-bounded variation if there exist $M\in E$ such that for any partition $(x_0,x_1,\dots,x_n),(y_0,y_1,\dots,y_m)$ of $[\alpha,\beta]\times[\alpha,\beta]$, the following inequality holds :

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left|f(x_{i+1},y_{j+1})-f(x_i,y_j)\right|\le M$$

Where $M=(M_1,M_2)$.

Definition 3.2: Let E be a Banach lattice. The double sequence (a_{kl},b_{kl}) in $E\times E$ is called moment double sequence if there exist two functions

$f,g:[0,1]\times[0,1]\to E$ with o-bounded variation , such that for any k,l , we have

$$a_{kl}=\int_0^1\int_0^1 (t^k+u^l)\,df(t)df(u)$$

$$b_{kl}=\int_0^1\int_0^1 (t^k+u^l)\,dg(t)dg(u)$$

Defintion 3.3: Let E a Banach lattice. The double sequence (a_{kl},b_{kl}) in $E\times E$ is completely monotone, if for any $,m,k,l$, we have

$$\Delta^n a_{kl} = \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j} a_{k+i,l+j} \geq 0$$

$$\Delta^m b_{kl} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j} b_{k+i,l+j} \geq 0$$

We now state and prove a Moment theorem in term of a double sequence:

Theorem 3.4: Let E be a Banach lattice with order continuous norm and (a_{kl}, b_{kl}) is a double sequence in $E \times E$. (a_{kl}, b_{kl}) is a moment double sequence of a nondecreasing functions f and g if and only if (a_{kl}, b_{kl}) is completely monotone .

Proof: From definition (3) and (4) we get that, if the double sequence (a_{kl}, b_{kl}) is the moment double sequence of nondecreasing functions f and g then it is completely monotone.

For the opposite side we have for $n = 0,1,2, \ldots$ define two functions

$f_n, g_n : [0,1] \times [0,1] \to E$ by

$$f_n(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}a_{ij}$$

for $(x,y) \in]\frac{m-1}{n},\frac{m}{n}[\wedge]\frac{m-1}{n},\frac{m}{n}[$

$$g_n(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}b_{ij}$$

$$f_n\left(\frac{m}{n},\frac{m}{n}\right) = \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}a_{ij}$$

$$f_n(0,0) = (0,0) \ , f_n(1,1) = a_{(0,0)}$$

$$g_n\left(\frac{m}{n},\frac{m}{n}\right) = \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}b_{ij}$$

$$g_n(0,0) = (0,0) \ , g_n(1,1) = b_{(0,0)}$$

The functions f_n , g_n is nondecreasing and has values in the order interval $[0, a_0] \times [0, b_0]$ of $E \times E$.

If we define the operator Λ on the space of polynomials by

$$\Lambda(\sum_{i=0}^{n}\sum_{j=0}^{n}(c_i x^i + d_j y^j)) = \sum_{i=0}^{n}\sum_{j=0}^{n}(c_i a_{ij} + d_j a_{ij})$$

$$\Lambda(\sum_{i=0}^{n}\sum_{j=0}^{n}(c_i x^i + d_j y^j)) = \sum_{i=0}^{n}\sum_{j=0}^{n}(c_i b_{ij} + d_j b_{ij})$$

It is clear that the Bernstein polynomials

$$B_{(k,l),n}(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}\left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}(x^{i}+y^{j})(1-(x+y))^{n-i-j}$$

verify

$$\Lambda\left(B_{n,(k,l)}\right)=\int_0^1\int_0^1 (t^k+u^l)df_n(t)df_n(u)$$

$$\left(B_{n,(k,l)}\right)=\int_0^1\int_0^1 (t^k+u^l)dg_n(t)dg_n(u)$$

Using Theorem 1,we can choose a sub double sequence $(f_{n_jm_k},g_{n_{k_1}m_{k_2}})_{j,k\in N}$ of (f_{nm},g_{nm}) and a nondecreasing functions f , g such that $(f_{n_jm_k}(x,y),g_{n_jm_k}(x,y))_{j,k\in N}$ weakly converges to $f(x,y),g(x,y)$

at each continuity point $(x,y)\in]0,1[\wedge]0,1[$ of f,g for $x,y=0$ and $x,y=1$. By Theorem 2, for every k,l we have:

$$\lim_{j,k\to\infty}\int_0^1\int_0^1 (t^k+u^l)df_{n_jm_k}(t)df_{n_jm_k}(u)=\int_0^1\int_0^1 (t^k+u^l)df(t)df(u),$$

$$\lim_{j,k\to\infty}\int_0^1\int_0^1 (t^k+u^l)dg_{n_jm_k}(t)dg_{n_jm_k}(u)=\int_0^1\int_0^1 (t^k+u^l)dg(t)dg(u),$$

for $\sigma(E,E')$.

We now show that $\lim_{n\to\infty}\Lambda(B_{(k,l),n})=a_{kl}$, $\lim_{n\to\infty}\left(B_{(k,l),n}\right)=b_{kl}$ for the norm topology of $E\times E$, and the conclusion will follow . By classical algebraic computation , it is easy to show that $a_{(0,0)}=\Lambda\left(B_{(0,0),n}\right)$, $b_{(0,0)}=\Lambda(B_{(0,0),n})$

and that

$$a_{kl}=\sum_{i=k}^{n}\sum_{j=l}^{n}\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}a_{ij}$$

$$b_{kl}=\sum_{i=k}^{n}\sum_{j=l}^{n}\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)}\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}b_{ij}.$$

Consequentely , we obtain :

$$a_{kl}-\Lambda\left(B_{(k,l),n}\right)=\sum_{i=k}^{n}\sum_{j=l}^{n}\left(\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)}-\left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\right)\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}a_{kl}$$
$$-\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\binom{n}{i}\binom{n}{j}(\frac{i}{n})^{k}(\frac{j}{n})^{l}\Delta^{n-i-j}a_{ij}$$

$$b_{kl} - \Lambda\big(B_{(k,l),n}\big) = \sum_{i=k}^{n}\sum_{j=l}^{n}\left(\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)} - \left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\right)\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}b_{kl}$$
$$-\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\binom{n}{i}\binom{n}{j}(\frac{i}{n})^{k}(\frac{j}{n})^{l}\Delta^{n-i-j}b_{ij}$$

Let $(z,e) = \left(\frac{i}{n},\frac{j}{n}\right)$, and observe that

$$\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)} - \left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l} = \prod_{\substack{i=0\\ j=0}}^{\substack{k-1\\ l-1}}\frac{(nz-i)+(ne-j)}{(n-i)+(n-j)} - (z^{k}+e^{l})$$

It follows that , given $\epsilon > 0$, there exists n_0 such that

$$\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)} - \left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l} \le \epsilon$$

for $n \ge n_0$

and

$$\left|\sum_{i=k}^{n}\sum_{j=l}^{n}\left(\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)} - \left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\right)\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}a_{kl}\right| \le \epsilon$$

$$\left|\sum_{i=k}^{n}\sum_{j=l}^{n}\left(\frac{(i+j)\big((i+j)-1\big)\ldots.\big((i+j)-k-l+1\big)}{n(n-1)\ldots(n-k-l+1)} - \left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\right)\binom{n}{i}\binom{n}{j}\Delta^{n-i-j}b_{kl}\right| \le \epsilon$$

for $n \ge n_0$.

It is also clear that

$$\left|\sum_{i=k}^{n}\sum_{j=l}^{n}\binom{n}{i}\binom{n}{j}\left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\Delta^{n-i-j}a_{kl}\right| \le \left(\frac{k}{n}\right)^{k}\left(\frac{l}{n}\right)^{l}a_{(0,0)}$$

$$\left|\sum_{i=k}^{n}\sum_{j=l}^{n}\binom{n}{i}\binom{n}{j}\left(\frac{i}{n}\right)^{k}\left(\frac{j}{n}\right)^{l}\Delta^{n-i-j}b_{kl}\right| \le \left(\frac{k}{n}\right)^{k}\left(\frac{l}{n}\right)^{l}b_{(0,0)}$$

Now, it easy to conclude that if n is large enough we have

$$\left|a_{kl} - \Lambda\big(B_{(k,l),n}\big)\right| \le 2\epsilon a_{(0,0)} \vee \left|b_{kl} - \Lambda\big(B_{(k,l),n}\big)\right| \le 2\epsilon b_{(0,0)},$$

which proves the theorem. ◙

We use only an ((Helly theorem)) and not a representation theorem of [6] in prove theorem above, Making it simpler than those [7,8], such that theorem is only a special case of the ((moment theorem)) of [4,5] , And improve the ((moment theorem)) in [9] .we conclude this section by showing that , in the special setting of banach lattice with order continuous norm , is a corollary of the above theorem.

We now state and prove a Representation theorem in term of a double sequence:

Theorem 3.5: Every positive linear operator L on $C([\alpha,\beta]\times[\alpha,\beta], R\times R)$ with values in a Banach lattice E with order continuous norm, is representable in the from

$$L(h)=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} h(t,u)df(t)df(u)$$

$$L(w)=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} w(t,u)dg(t)dg(u)\,,$$

where f,g is a nondecreasing functions from $[\alpha,\beta]\times[\alpha,\beta]$ into $E\times E$.

Proof: It is clear that it suffices to prove the result for the interval $[0,1]$ .The double sequence $(a_{kl},b_{kl})_{k,l\in N}$ defined by the formula $a_{kl}=L(t^k+u^l)$, $b_{kl}=L(t^k+u^l)$ is completely monotone.

In fact, we have

$$\Delta^n a_{kl}=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}a_{k+i,l+j}=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}L(t^{k+i}+u^{l+j})=L(t^k(1-t)^n+u^l(1-u)^n)\ \geq 0$$

$$\Delta^m b_{kl}=\sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}b_{k+i,l+j}=\sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}L(t^{k+i}+u^{l+j})=L(t^k(1-t)^m+u^l(1-u)^m)\ \geq 0\,.$$

By the preceeding theorem, there exists a nondecreasing functions f,g such that

$$L(t^k+u^l)=\int_0^1\int_0^1(t^k+u^l)df(t)df(u)\quad,\qquad L(t^k+u^l)=\int_0^1\int_0^1(t^k+u^l)dg(t)dg(u)$$

and , by Weierstrass theorem this equality extends to every continuous functions. We recall that a positive linear mapping from a Banach lattice into a normed vector lattice is automatically continuous ([2,Theorem 12.3]). ◙

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