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Common Fixed Point Theorem for R-Weakly Commuting Pairs of Mappings In Fuzzy Metric Space

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Abstract

In this paper, we prove common fixed point theorem for R- weakly commuting mapping in fuzzy metric space, finally we established results in fuzzy metric space by taking different inequality in order to reduce the minimum value.

Introduction : The concept of a fuzzy set was first introduced by Zadeh L.A.[22] and fuzzy metric spaces have been introduced by Kramosil and Michalek[7] and George and Veersamani[3] modified the notion of fuzzy metric with help of continuous t-norms. Recently many have proved fixed point theorems involving fuzzy sets[1-6,8-12,14-16] Balasubramaniam P., Muralishankar S.R. and Pant R.P.[1] proved the open problem of Rhoades[17] on the existence of a contractive definition which generals a fixed point but does not force the mapping to be continuous at the fixed point possesses an affirmative answer. Namdeo,Shrivastava and solanki[13] proved common fixed point theorem for four mappings in fuzzy metric space, we generalized the result of Namdeo,Shrivastava and solanki[13] by using new condition in fuzzy metric space.

Definition-1: The 3-tuple (X, S, *) is said to be a S-Fuzzy Metric Space if X is an arbitrary Set, * is a continuous t-norm and S is a Fuzzy set on X^2 x (0, ∞). satisfying the following conditions: i. S(x, y, t) > 0, ii. S(x, y, t) = 1 if and only if x=y, iii. S(x, y, t) = S(y, x, t), iv. S(x, y, t) * S(y, z, s) ≤ S(x, z, t+s),
v. S(x, y, .); (0, ∞) → [0, 1] is continuous for all x, y, z ∈ X and t, s > 0.
Definition-2: A sequence $\{x_n\}$ in a fuzzy metric Space (X, S, *) is a Cauchy Sequence if and only if for each ε > 0 , t > 0 there
exists n_0 ∈N such that $S(x_n, x_m, t) > 1- ε$ for all n, m≥ n_0.
Definition-3: A sequence $\{x_n\}$ in a fuzzy metric Space (X, S, *) is converges to x if and only if for each ε > 0 , t > 0 there exists
n_0 ∈ N such that $S(x_n, x, t) > 1- ε$ for all n≥ n_0.
Definition-4: Fuzzy metric Space (X, S, *) is said to be complete if every Cauchy Sequence in (X, S, *) is a convergent sequence.
Definition-5: Two mappings f and g of a fuzzy metric space (X, S, *) in to itself are said to be weakly commuting if S(fgx, gfx,t) ≥ S(fx, gx, t) for each x in X.
Definition-6: The mappings f and g of a fuzzy metric space (X, S, *) in to itself are said to be R-weakly commuting, provided there exists some positive real numbers R such that S(fgx, gfx, t) ≥ S(fx, gx, t/R) for each x in X.
Definition-7: The mappings F and G of a fuzzy metric space (X, S, *) in to itself are said to be compatible iff $S(FGx_n, GFx_n, t)$ →1 For all t > 0, whenever $\{x_n\}$ is a sequence in X such that Fx_n, Gx_n →y for some y in X.
Definition-8: Let A and B be self mappings of a fuzzy metric space (X, S, *) ,we will call A and B to be reciprocally continuous If $\lim_{n\to\infty} ABx_n$ = Ap and $\lim_{n\to\infty} BAx_n$= Bp whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$ =p for some p in X.
If A and B are continuous then they are obviously reciprocally continuous. But the converse need not be true.

Theorem-1: Let A, B, M and N be self maps of a complete fuzzy metric space (X, S, *) with continuous t –norm * defined by a*b = min {a,b}, a,b∈[0,1] satisfying the following conditions: i. A(x) ⊂N(x), B(x) ⊂ M(x), ii. [A, M], [B, N] are pointwise R-weakly commuting pairs of maps. iii. [A, M] or [B, N] is compatible pair of reciprocally continuous maps. iv. For all x, y in X,
k ∈ [0,1] t>0, $S^2(Ax, By, kt) \geq \max\{ S^2(Mx, Ny, t), S^2(Ax, Mx, t), S^2(By, Ny, t), S^2(Bx, My, t), \frac{(S^2(Ax,Nx,t)+S^2(Bx,Nx,t)}{2} \}$, v. For all x, y in X, $\lim_{t\to\infty} S(x, y, t) \to 1$.
Then A, B, M and N have a unique common fixed point in X.
Proof: Let x_0∈ X be arbitrary. Construct a sequence $\{y_n\}$ such that

$y_{2n-1} = Nx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Mx_{2n} = Bx_{2n-1}$ n=1,2,3,……….

Now using (iv) we have

$S^2(y_{2n+1}, y_{2n+2}, kt) = S^2(Ax_{2n}, Bx_{2n+1}, kt)$

$\geq \max\{S^2(Mx_{2n}, Nx_{2n+1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n+1}, Nx_{2n+1}, t),$

$S^2(Bx_{2n+1},Mx_{2n+1},t), \frac{(S^2(Ax_{2n},Nx_{2n},t)+S^2(Bx_{2n},Nx_{2n},t)}{2}\}$

$\geq \max\{ S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n+2}, y_{2n+1}, t)$

$S^2(y_{2n+2}, y_{2n+1}, t), \frac{(S^2(y_{2n+1},y_{2n},t)+S^2(y_{2n+1},y_{2n},t)}{2}\}$

$\geq \max\{ S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n+2}, t) \}$

$\geq S^2(y_{2n}, y_{2n+1}, t).$

So,

$S(y_{2n+1}, y_{2n+2}, kt) \geq S(y_{2n}, y_{2n+1}, t)$ (1.1)

Further using (iv) we have

$S^2(y_{2n}, y_{2n+1}, kt) = S^2(Bx_{2n-1}, Ax_{2n}, kt)$

$= S^2(Ax_{2n}, Bx_{2n-1}, kt)$

$\geq \max\{ S^2(Mx_{2n}, Nx_{2n-1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n-1}, Nx_{2n-1}, t),$

$S^2(Bx_{2n-1},Mx_{2n-1}, t), \frac{(S^2(Ax_{2n},Nx_{2n},t)+S^2(Bx_{2n},Nx_{2n},t)}{2}\}$

$\geq \max\{ S^2(y_{2n}, y_{2n-1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n}, y_{2n-1}, t)$

$S^2(y_{2n}, y_{2n-1}, t), \frac{(S^2(y_{2n+1},y_{2n},t)+S^2(y_{2n+1},y_{2n},t)}{2}\}$

$\geq \max\{S^2(y_{2n}, y_{2n-1}, t), S^2(y_{2n+1}, y_{2n}, t),\}$

$\Rightarrow S(y_{2n}, y_{2n+1}, kt) \geq S(y_{2n-1}, y_{2n}, t)$ (1.2)

Using (1.1) and (1.2) we have

$S(y_n, y_{n-1}, (1-k)t/k) \geq S(y_{n-1}, y_{n-2}, (1-k)t/k^2)$

$\geq S(y_{n-2}, y_{n-3}, (1-k)t/k^3)$

$\geq S(y_0, y_1, (1-k)t/k^n) \to 1$ as $n\to\infty$

Hence for $t > 0$, $k, \lambda \in (0, 1)$ we can choose $n_0 \in N$ such that

$S(y_n, y_{n-1}, (1-k)t/k) \geq 1- \lambda$, $n \geq n_0$ (1.3)

To prove that $\{y_n\}$is a Cauchy sequence we claim (1.4) is true for all $n \geq n_0$ and for every $m \in N$

$S(y_n, y_{n+m}, t) \geq 1- \lambda$ (1.4)

From (1.1) (1.2) and (1.3) we have

$S(y_n, y_{n+1}, t) \geq S(y_n, y_{n-1}, t/k)$

$\geq S(y_n, y_{n-1}, (1-k)t/k)$

$\geq 1- \lambda$

Thus result (1.4) is true for m = 1. Further suppose (1.4) is true for m.

Then we shall show that it is also true for m+1.

Using (1.1) (1.2) and definition for t – norm we have

$S(y_n, y_{n+m+1}, t) \geq S(y_{n-1}, y_{n+m}, t/k),$

$\geq \min\{S(y_n, y_{n-1}, (1-k)t/k), S(y_n, y_{n+m}, t)\}$

$\geq 1- \lambda$

Thus (1.4) is true for m+1 and so it is true for every $m \in N$ therefore $\{y_n\}$ is a Cauchy Sequence. Since (X, S, *) is complete so $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}$ $\{Mx_{2n}\}\{Bx_{2n-1}\}$ and $\{Nx_{2n-1}\}$ also converges to z.

Suppose [A, M] is a compatible pair of reciprocally continuous maps. Then by the definition of reciprocally continuous maps,

$AMx_{2n} \to Az$ and $MAx_{2n} \to Mz$ And then the compatibility of A and M yields,

$\lim_{n\to\infty} S(AMx_{2n}, MAx_{2n}, t) = 1$ i.e. $S(Az, Mz, t) = 1$. Hence $Az = Mz$, Since $A(x) \subset N(x)$, There exists a point w in X such that $Az = Nw$

Using (iv) we have,

$S^2(Az, Bw, kt) \geq \max\{S^2(Mz, Nw, t), S^2(Az, Mz, t), S^2(Bw, Nw, t),$

$S^2(Bw, Mw, t), \frac{S^2(Az,Nz,t)+S^2(Bz,Nz,t)}{2}\}$

$$\geq \max \{ S^2(Az, Az, t), 1, S^2(Bw, Az, t), S^2(Bw, Mz, t), \frac{S^2(Az,Nz,t)+S^2(Bz,Nz,t)}{2}\}$$

Or

$S^2(Az, Bw, kt) \geq 1$

Which implies that Az=Bw, thus Mz = Az = Nw = Bw.

Point-wise R-weakly commutativity of A and M implies that there exists R > 0 such that S(AMz, MAz, t) ≥S(Az, Mz, t/R) = 1

i.e. AMz = MAz and AAz = AMz = MAz = MMz

Similarly pointwise R-weakly commutativity of B and N implies that

BBw = BNw = NBw = NNw

Now by (iv) we have

$$S^2(AAz, Az, kt) = S^2(AAz, Bw, kt)$$
$$\geq \max \{ S^2(MAz, Nw, t),S^2(AAz, MAz, t), S^2(Bw, Nw, t), S^2(Bw, Mw, t), \frac{S^2(AAz,NAz,t)+S^2(BAz,NAz,t)}{2}\}$$
$$\geq \max \{S^2(MAz,Nw, t),1, S^2(Az, Az, t), S^2(Bw, Mw, t), \frac{S^2(AAz,NAz,t)+S^2(BAz,NAz,t)}{2}\}$$

Or

$S^2(AAz, Az, kt) \geq 1$

⇒AAz = Az thus Az = AAz = MAz

Thus Az is a common fixed point of A and M. Again by (iv) we have

$$S^2(Az, BBw, kt) \geq \max \{ S^2(Mz, NBw, t), S^2(Az, Mz, t), S^2(BBw, NBw, t), S^2(BBw, MBw, t), \frac{S^2(Az,Nz,t)+S^2(Bz,Nz,t)}{2}\}$$
$$\geq \max \{ S^2(Mz, NBw, t), 1, S^2(BBw, NBw, t), S^2(BBw, MBw, t), \frac{S^2(Az,Nz,t)+S^2(Bz,Nz,t)}{2}\}$$

Or $S^2(Az, BBw, kt) \geq 1$

⇒Az = BBw thus Az = BBw = Bw.

Thus Bw(=Az) is a common fixed point of B and N and hence Az is a common fixed point of A, B, M and N.

To prove Uniqueness, let Az_1 be another common fixed point of A, B, M and N. Then we have

$$S^2(Az, Az_1, kt) = S^2(AAz, BAz_1, kt)$$
$$\geq \max \{ S^2(MAz, NAz_1, t), S^2(AAz, MAz, t), S^2(BAz_1, NAz_1, t), S^2(BAz_1, MAz_1, t), \frac{S^2(AAz,NAz,t)+S^2(BAz,NAz,t)}{2}\}$$
$$\geq \max \{ S^2(Az, Az_1, t), S^2(Az, Az, t), S^2(Az_1, Az_1, t), S^2(Az_1, Az_1, t), \frac{S^2(Az,Az,t)+S^2(Az,Az,t)}{2}\}$$
$$\geq \max \{ S^2(Az, Az_1, t), 1, 1,1,1\}$$

or $S^2(Az, Az_1, kt) \geq 1$

Thus $Az = Az_1$

Thus Az is a unique common fixed point of A, B, M and N.

Conclusion

Theorem 1 extends the generalize results Balasubramaniam and Muralishankar S., Pant R.P.[1] on the existence of a contractive.

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