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Fixed Point Results In Fuzzy Menger Space With Rational Expression

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Abstract:

This paper presents some common fixed point theorems for occasionally weakly compatible mappings with rational expression in Fuzzy menger metric spaces.

Keywords: Occasionally weakly compatible mappings, Fuzzy menger metric space,Weak compatible mapping, Semi-compatible mapping, Implicit function, common fixed point.
Subject Classification: AMS (2000) 47H25

1. INTRODUCTION

The study of fixed point theorems in Menger spaces is an active area of research. Now it is extended in the form of Fuzzy Menger space. The theory of probabilistic metric spaces was introduced by Menger [2] in 1942 and since then the theory of probabilistic metric spaces has developed in many directions, especially in nonlinear analysis and applications. In 1966, Sehgal [5] initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then several generalizations of fixed point theorems in probabilistic metric space have been obtained by several authors including Sehgal and Bharucha-Reid [6], Sherwood [7], and Istratescu and Roventa [1]. The study of fixed point theorems in probabilistic metric spaces is useful in the study of existence of solutions of operator equations in probabilistic metric space and probabilistic functional analysis. The development of fixed point theory in probabilistic metric spaces was due to Schweizer and Sklar[4]. Singh et al. [9] introduced the concept of weakly commuting mappings in probabilistic metric spaces. In 2005, Mihet [3] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. Shrivastav et al.[8] proved fixed point result in fuzzy probabilistic metric space. The purpose of the present paper is to prove a common fixed point theorem for four mappings via occasionally weakly compatible mappings in fuzzy menger metric spaces satisfying contractive type implicit relations with rational coordinate.

2.PRELIMINARY NOTES

Let us define and recall some definitions:

Definition 2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair (X,F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collections of all distribution functions $F_\alpha \in R$ for all $\alpha \in [0,1]$. For x, y ∈ X we denote the distribution function $F_\alpha(x,y)$ by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all $\alpha \in [0,1]$ assumed to satisfy the following conditions:

(a) $F_{\alpha(x,y)}(u) = 1 \ \forall\, u > 0$ iff $x = y$,
(b) $F_{\alpha(x,y)}(0) = 0 \ \forall\, x, y$ in X,
(c) $F_{\alpha(x,y)} = F_{\alpha(y,x)} \ \forall\, x, y$ in X,
(d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1$ then $F_{\alpha(x,z)}(u+v) = 1 \ \forall\, x, y, z$ in X and $u, v > 0$.

Definition 2.2 A commutative, associative and non-decreasing mapping $t: [0,1] \times [0,1] \to [0,1]$ is a t-norm if and only if $t(a,1)= a \ \forall a\in[0,1]$, $t(0,0)=0$ and $t(c,d) \ge t(a,b)$ for $c \ge a$, $d \ge b$.

Definition 2.3 A Fuzzy Menger space is a triplet (X,F_α,t), where (X,F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_{\alpha(x,z)}(u+v) \ge t\,(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}(v)) \quad \text{holds for all } x, y, z \text{ in } X\ u, v > 0 \text{ and } \alpha \in [0,1]$$

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition 2.4 Let (X,F_α,t) be a Fuzzy Menger space. If $x\in X$, $\varepsilon > 0$ and $\lambda \in(0,1)$, then (ε,λ) - neighborhood of x, called $U_x(\varepsilon,\lambda)$, is defined by

$$U_x(\varepsilon,\lambda) = \{y\in X: F_{\alpha(x,y)}(\varepsilon)>(1-\lambda)\}$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \alpha. \in [0,1] \text{and } \lambda\in(0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy Menger space (X,F_α,t) is a Housdroff space in (ε,λ)-topology.

Let (X,F_α,t) be a complete Fuzzy Menger space and $A\subset X$. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf_{x,y\in A} F_{\alpha(x,y)}(u) = 1$$

Definition 2.5 Let (X,F_α,t) be a Fuzzy Menger space and a sequence $\{x_n\}$ in (X,F_α,t) is said to be convergent to a point x in X if for every $\varepsilon>0$and $\lambda>0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $x_n \in U_x(\varepsilon,\lambda)$ for all $n \geq N$ or equivalently $F_\alpha(x_n, x; \varepsilon) > 1-\lambda$ for all $n \geq N$ and $\alpha\in[0,1]$.

Definition 2.6 Let (X,F_α,t) be a Fuzzy Menger space and a sequence $\{x_n\}$ in (X,F_α, t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F_\alpha(x_n,x_m; \varepsilon) > 1-\lambda\ \forall\ n, m \geq N$ for all $\alpha\in[0,1]$.

Definition 2.7 A Fuzzy Menger space (X,F_α,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha\in[0,1]$.

Definition 2.8 Let (X,F_α,t) be a Fuzzy Menger space. Two mappings f, g :$X\to X$ are said to be weakly comptable if they commute at coincidence point for all $\alpha\in[0,1]$.

Lemma 1 Let $\{x_n\}$ be a sequence in a Fuzzy Menger space (X,F_α,t), where t is continuous and $t(p,p) \geq p$ for all $p\in[0,1]$, if there exists a constant k(0,1) such that for all $p > 0$ and $n\in N$

$$F_\alpha(x_n,x_{n+1}; kp) \geq F_\alpha(x_{n-1},x_n; p),$$

for all $\alpha\in[0,1]$ then $\{x_n\}$ is cauchy sequence.

Lemma 2 If (X,d) is a metric space, then the metric d induces, a mapping F_α: $XxX\to L$ defined by $F_\alpha(p, q) = H_\alpha(x- d(p, q))$, p, q $\in$ R for all $\alpha\in[0,1]$. Further if t: $[0,1] \times [0,1]\to [0,1]$ is defined by $t(a,b) = \min\{a,b\}$, then (X,F_α,t) is a Fuzzy Menger space. It is complete if (X,d) is complete. **Definition 2.9]:** Let (X,F_α,t) be a Fuzzy Menger space. Two mappings f, g: $X\to X$ are said to be compatible if and only if $F_{\alpha\ (fgxn,gfxn)}(t)\to 1$ for all $t > 0$ whenever $\{x_n\}$ in X such that $fx_n,gx_n\to z$ for some $z \epsilon X$.

Definition 2.10: Two self mappings f and g of a Fuzzy Menger space (X,F_α,t) are said to be pointwise R-weakly commuting if given $x \epsilon X$, there exists $R > 0$ such that

$F_{\alpha\ (fgx,gfx)}(t) \geq F_{\alpha\ (fx,gx)}(t/R)$ for $t > 0$ and $\alpha\in[0,1]$.

Definition 2.11: Let X be a set, f, g be self maps of X. A point x in X is called a coincidence point of f and g iff fx = gx . We shall call w = fx = gx a point of coincidence of f and g.

Definition 2.12: A pair of maps f and g is called weakly compatible pair if they commute at coincidence points.

Definition 2.13: Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition2.14. A function $\phi:[0,\infty) \to[0,\infty)$ is said to be a ϕ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to\infty$ as $t \to\infty$,

(iii) ϕ is left continuous in $(0,\infty)$ and

(iv) ϕ is continuous at 0.

Lemma 2.15: Let (X,F_α,t) be a Fuzzy Menger space and a sequence $\{x_n\}$ in (X,F_α,t) where t is continuous If there exists a constant h ε (0, 1) such that $F_\alpha(x_n, x_{n+1}; kt) \geq F_\alpha(x_{n-1}, x_n;t)$, n ε **N,** then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.16: Let X be a set, f, g be owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3. MAIN RESULTS:

Theorem 3.1: Let (X, F_α,t) be a complete Fuzzy Menger space and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$F_{\alpha\ (px,qy)}(ht) \geq \varphi(\min\{ F_{\alpha\ (fx,gy)}(t), F_{\alpha\ (fx,px)}(t), F_{\alpha\ (qy,gy)}(t),$

$[F_{\alpha\,(px,gy)}(t)+ F_{\alpha\,(fx,px)}(t)]/ F_{\alpha\,(fx,px)}(t), [F_{\alpha\,(qy,fx)}(t)+ F_{\alpha\,(qy,gy)}(t)]/ F_{\alpha\,(qy,gy)}(t)\})$ …… …..(3.1)

for all x,y ∈ X , $\phi \in \Phi$ for all 0< t <1, then there exists a unique point w ∈ X such that pw = fw = w and a unique point z ∈ X such that qz = gz =z .Moreover, z = w, so that there is a unique common fixed point of p, f, q and g.

Proof: Let the pairs { p,f } and {q,g } be owc, so there are points x,y ∈ X such that px =fx and qy = gy. We claim that px = qy. If not, by inequality (3.1)

$$F_{\alpha\,(px,qy)}(ht) \geq \varphi(\min\{ F_{\alpha\,(fx,gy)}(t), F_{\alpha\,(fx,px)}(t), F_{\alpha\,(qy,gy)}(t),$$

$$[F_{\alpha\,(px,gy)}(t)+ F_{\alpha\,(fx,px)}(t)]/ F_{\alpha\,(fx,px)}(t),[F_{\alpha\,(qy,fx)}(t)+ F_{\alpha\,(qy,gy)}(t)]/ F_{\alpha\,(qy,gy)}(t)\})$$

$$= \varphi(\min\{ F_{\alpha\,(px,qy)}(t), F_{\alpha\,(px,px)}(t), F_{\alpha\,(qy,qy)}(t),$$

$$[F_{\alpha\,(px,qy)}(t)+ F_{\alpha\,(px,px)}(t)]/ F_{\alpha\,(px,px)}(t), [F_{\alpha\,(qy,px)}(t)+ F_{\alpha\,(qy,qy)}(t)]/ F_{\alpha\,(qy,qy)}(t)\})$$

$$= \varphi(F_{\alpha\,(px,qy)}(t)) = F_{\alpha\,(px,qy)}(t)$$

Therefore px = qy, i.e. px = fx = qy = gy. Suppose that there is an another point z such that pz = fz then by (1) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.16 w is the only common fixed point of p and f. Similarly there is a unique point z∈ X such that z =qz = gz. Assume that w ≠ z. We have

$$F_{\alpha\,(w,z)}(ht) = F_{\alpha\,(pw,fz)}(ht)$$
$$\geq \varphi(\min\{ F_{\alpha\,(fw,gz)}(t), F_{\alpha\,(fw,pw)}(t), F_{\alpha\,(qz,gz)}(t),$$
$$[F_{\alpha\,(pw,gz)}(t)+ F_{\alpha\,(fw,pw)}(t)]/ F_{\alpha\,(fw,pw)}(t),[F_{\alpha\,(qz,fw)}(t)+ F_{\alpha\,(qz,gz)}(t)]/ F_{\alpha\,(qz,gz)}(t)\})$$
$$\geq \varphi(\min\{ F_{\alpha\,(w,z)}(t), F_{\alpha\,(w,w)}(t), F_{\alpha\,(z,z)}(t), [F_{\alpha\,(w,z)}(t)+ F_{\alpha\,(w,w)}(t)]/ F_{\alpha\,(w,w)}(t),$$
$$[F_{\alpha\,(z,w)}(t)+ F_{\alpha\,(z,z)}(t)]/ F_{\alpha\,(z,z)}(t)\})$$
$$= \varphi(F_{\alpha\,(w,z)}(t)) = F_{\alpha\,(w,z)}(t)$$

Therefore we have z = w by lemma 2.16 and z is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (3.1)

Theorem3.2 Let (X, F,t) be a complete Fuzzy Menger space and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F_{\alpha\,(px,qy)}(ht) \geq \phi\,\{ F_{\alpha\,(fx,gy)}(t), F_{\alpha\,(fx,px)}(t), F_{\alpha\,(qy,gy)}(t),[F_{\alpha\,(px,gy)}(t)+ F_{\alpha\,(fx,px)}(t)]/ F_{\alpha\,(fx,px)}(t),$$
$$[F_{\alpha\,(qy,fx)}(t)+ F_{\alpha\,(qy,gy)}(t)]/ F_{\alpha\,(qy,gy)}(t)\} \ldots\ldots\ldots.(3.2)$$

for all x,y ∈ X and $\phi:[0, 1]^5 \to [0, 1]$ such that ϕ (t,1,1,t,t)> t for all 0< t <1, then there exists a unique common fixed point of p, f, q and g.

Proof: Let the pairs { p,f } and {q,g } be owc, so there are points x,y ∈ X such that px =fx and qy = gy. We claim that px = qy. By inequality (2)we have

$$F_{\alpha\,(px,qy)}(ht) \geq \phi\,(\{F_{\alpha\,(fx,gy)}(t), F_{\alpha\,(fx,px)}(t), F_{\alpha\,(qy,gy)}(t),[F_{\alpha\,(px,gy)}(t)+ F_{\alpha\,(fx,px)}(t)]/ F_{\alpha\,(fx,px)}(t),$$
$$[F_{\alpha\,(qy,fx)}(t)+ F_{\alpha\,(qy,gy)}(t)]/ F_{\alpha\,(qy,gy)}(t)\})$$
$$= \phi(\{ F_{\alpha\,(px,qy)}(t), F_{\alpha\,(px,px)}(t), F_{\alpha\,(qy,qy)}(t),[F_{\alpha\,(px,qy)}(t)+ F_{\alpha\,(px,px)}(t)]/ F_{\alpha\,(px,px)}(t),$$
$$[F_{\alpha\,(qy,px)}(t)+ F_{\alpha\,(qy,qy)}(t)]/ F_{\alpha\,(qy,qy)}(t)\})$$
$$= \phi\,(\{F_{\alpha\,(px,qy)}(t), 1, 1, F_{\alpha\,(px,qy)}(t), F_{\alpha\,(px,qy)}(t)\})$$
$$> F_{\alpha\,(px,qy)}(t)$$

This a contradiction, therefore px = qy, i.e. px = fx = qy = gy. Suppose that there is a another point z such that pz = fz then by (2) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.16 w is the only common fixed point of p and f. Similarly there is a unique point z∈ X such that z =qz = gz. Thus z is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (3.2).

Corollary 3.3: Let (X, F,t) be a complete Fuzzy Menger and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F_{\alpha\,(px,qy)}(ht) \geq \min\{ F_{\alpha(\,fx,gy)}(t), F_{\alpha(\,px,fx)}(t), F_{\alpha(\,qy,gy)}(t),$$

$[F_{\alpha\,(px,gy)}(t)+F_{\alpha\,(fx,px)}(t)]/F_{\alpha\,(fx,px)}(t),[F_{\alpha\,(qy,fx)}(t)+F_{\alpha\,(qy,gy)}(t)]/F_{\alpha\,(qy,gy)}(t)\}$ ……………..(3.3)

for all x,y ∈ X and t > 0, then there exists a unique common fixed point of p, f, q and g.

Proof : Let the pairs { p,f } and {q,g } be owc, so there are points x,y ∈ X such that px =fx and qy = gy. We claim that px = qy. By inequality (3.3)we have

$$F_{\alpha(px,qy)}(ht) \geq \min\{F_{\alpha\,(fx,gy)}(t), F_{\alpha(px,fx)}(t), F_{\alpha\,(qy,gy)}(t),$$
$$[F_{\alpha\,(px,qy)}(t)+F_{\alpha\,(px,px)}(t)]/F_{\alpha\,(px,px)}(t),[F_{\alpha\,(qy,px)}(t)+F_{\alpha\,(qy,qy)}(t)]/F_{\alpha\,(qy,qy)}(t)\}$$
$$= \min\{F_{\alpha\,(px,qy)}(t),F_{\alpha\,(px,px)}(t), F_{\alpha(qy,qy)}(t),$$
$$[F_{\alpha\,(px,qy)}(t)+F_{\alpha\,(px,px)}(t)]/F_{\alpha\,(px,px)}(t),[F_{\alpha\,(qy,px)}(t)+F_{\alpha\,(qy,qy)}(t)]/F_{\alpha\,(qy,qy)}(t)\}$$
$$\geq \min\{F_{\alpha\,(px,qy)}(t), 1, 1, F_{\alpha(px,qy)}(t), F_{\alpha(px,qy)}(t)\}$$
$$\geq F_{\alpha\,(px,qy)}(t)$$

Thus we have px = qy, i.e. px = fx = qy = gy. Suppose that there is an another point z such that pz = fz then by (3) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.16 w is the only common fixed point of p and f. Similarly there is a unique point z∈ X such that z =qz = gz. Thus w is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (3.3).

4.REFERENCES

[1] V.I. Istrˇatescu and I. Sˇacuiu, Fixed point theorem for contraction mappings on probabilistic metric spaces, Rev. Roumaine Math. Pures. Appl.**18** (1973), 1375–1380.

[2] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537.

[3] D. Mihet, A generalization of a contraction principle in probabilistic metric spaces, Part II, Int. J. Math. Math. Sci. **2005** (2005), 729–736.

[4] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland (Amsterdam, 1983).

[5] V.M. Sehgal, Some fixed point theorems in function analysis and probability,Ph.D dissertation, Wayne State Univ. Michigan (1966).

[6] V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, Math. Systems Theory **6** (1972), 97–102.

[7] H. Sherwood, Complete probabilistic metric spaces, Z. wahrscheinlichkeits theorie and verw. Grebiete **20** (1971), 117–128.

[8] R. Shrivastav,S. Nath , V. Patel and V. Dhagat,weak and semi compatible maps in Fuzzy Probabilistic metric space using implicit relation, IJMA 2(6), 2011, 958-963.

[9] S.L. Singh, B.D. Pant and R. Talwar, Fixed points of weakly commuting mappings on Menger spaces, Jnanabha **23** (1993), 115–122.