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Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces

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Abstract: The purpose of this paper is to give some new fixed point theorems for contractive type mappings in fuzzy metric spaces. The results presented improve and generalize some recent result**.** The result is a genuine generalization of the corresponding result of S.Sedghi et al. (2010).

Keywords: Fuzzy metric space, t-norm, g-convergent, coupled common fixed point.
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Introduction:

Since Zadeh [1] introduced the concept of fuzzy sets, many authors have extensively developed the theory of fuzzy sets and applications. George and Veeramani [2, 3] gave the concept of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space which have very important applications in quantum particle physics particularly in connection with both string and infinity theory.

Bhaskar and Lakshmikantham [4], Lakshmikantham and Ćirić [5] discussed the mixed monotone mappings and gave some coupled fixed point theorems which can be used to discuss the existence and uniqueness of solution for a periodic boundary value problem. Sedghi et al. [6] gave a coupled fixed point theorem for contractions in fuzzy metric spaces, and Fang [7] gave some common fixed point theorems under contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Many authors [8–11] have proved fixed point theorems in (intuitionistic) fuzzy metric spaces or probabilistic metric spaces.

In this paper, using similar proof as in [7], we give a new common fixed point theorem under weaker conditions than in [6] and give an example which shows that the result is a genuine generalization of the corresponding result in [6].

Preliminaries:

We start by recalling two definitions.

Definition 2.1. Let X be a non empty set The mappings $F: X \times X \to X$ and $g: X \to X$ are said to commute if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Definition 2.2. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $g: X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

The mappings F and g have a common fixed if there exists $x \in X$ such that $x = gx = F(x, x)$.

Our main theorem states as follows.

Main Results:

Theorem 3.1: Let (X, M, T) be a complete fuzzy metric space satisfying (FM-6) with T a g-convergent t-norm . Let F : X × X → X and g : X → X be two mappings such that, for some k ϵ (0,1),

$$\mathrm{M}(\mathrm{F}(\mathrm{x}, \mathrm{y}), \mathrm{F}(\mathrm{u}, \mathrm{v}), \mathrm{Kt}) \geq \mathrm{Min}\{\mathrm{M}(\mathrm{gx}, \mathrm{gu}, \mathrm{t}), \mathrm{M}(\mathrm{gy}, \mathrm{gv}, \mathrm{t}), \mathrm{M}(\mathrm{gx}, \mathrm{gv}, \mathrm{t}), \mathrm{M}(\mathrm{gy}, \mathrm{gu}, \mathrm{t})\}.$$

for all x, y, u ,v ϵ X , t ≥ 0. Suppose that F (X × X) ⊂ g(x) and that g is continuous and commutes with F. If there exists a > 0 and x_0, y_0 ϵ X such that

$$\sup_{t>0} t^a \left(1 - M(gx_0, F(x_0, y_0), t)\right) < \infty \quad \text{and}$$

$$\sup_{t>0} t^a \left(1 - M(gy_0, F(y_0, x_0), t)\right) < \infty$$

then F and g have a unique common fixed points in X .

We note that if (x_0, y_0) is a coupled coincidence point of F and g, then the conditions

$$\sup_{t>0} t^a \left(1 - M(gx_0, F(x_0, y_0), t)\right) < \infty \quad \text{and}$$

$$\sup_{t>0} t^a \left(1 - M(gy_0, F(y_0, x_0), t)\right) < \infty \quad \text{are satisfied.}$$

Proof: Let x_0, y_0 be as in the statement of the theorem . Since F(X × X) ⊂ g(x), we can choose $x_1, y_1 \epsilon X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing in this way one can construct two sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ in X with the properties

$gx_{n+1} = F(x_n, y_n)$, $gy_{n+1} = F(y_n, x_n)$ for all $n \in N$

We divide the proof into five steps.

Step 1. We show that $\{gx_n\}_{n\in N}$ and $\{gy_n\}_{n\in N}$ are Cauchy sequences.

Indeed, let $\alpha > 0$ be such that $t^{\alpha}\left(1 - M\big(gy_{0,}F(y_0, x_0), t\big)\right) \leq \alpha$

and $t^{\alpha}\,(1 - M\big(gx_0\,, F\big(x_{0,}y_0\big), t\big)) \leq \alpha$

For all $t > 0$. Then $M\left(gx_{0,}\ gx_1, \frac{1}{t^n}\right) \geq 1 - \alpha(t^{\alpha})^n$ and $M\left(gy_{0,}\ gy_1, \frac{1}{t^n}\right) \geq 1 - \alpha(t^{\alpha})^n$ for every $t > 0$ and $n \in N$.

If $t > 0$ and $\varepsilon \in (0,1)$ are given , we choose μ in the interval (k,1) such that

$T_{i=n+1}^{\infty}\,(1 - \mu^q)^i) > 1 - \varepsilon$ and $\delta = \frac{K}{\mu}$. As $\delta\ \varepsilon\ (0,1)$, we can find $n_1(= n_1(t))$ such that $\sum_{n=n1}^{\infty} \delta^n\ < t.$

condition (2.1) implies that, for all $s > 0$,

$$M\big(gx_{1,}\ gx_{2,}ks\big) = M(F(x_0, y_0), F(x_1, y_1), s)$$
$$\geq Min\{M\big(gx_{0,}\ gx_1, S\big), M\big(gy_{0,}\, gy_1, s\big)\}$$

$$M\big(gy_{1,}\ gy_{2,}ks\big) = M(F(y_0, x_0), F(y_1,\ x_1), s)$$
$$\geq Min\{M(gy_0, gy_1, s), M(gx_0, gx_1, s)\}$$

$$M\big(gx_{2,}\ gx_{3\,,}ks\big) = M\big(F(x_1, y_1\ ), F\big(x_{2,}y_2\big), s\big)$$

$$\geq Min\{M\big(gx_{1,}\ gx_2\big), M\big(gy_{1,}gy_2\big), s\}$$

$$M\big(gy_{2,}\ gy_{3,}ks\big) = M(F(y_1, x_1), F(y_2, x_2), s)$$

$$\geq Min\{M\big(gy_{1,}\ gy_2, S\big), M\big(gx_{1\,,}gx_2, S\big)\}$$

It follows by including that

$$M\big(gx_{n,}\ gx_{n+1}, k^nS\big) \geq Min\{M(gx_0, gx_1, S), M(gy_{0,}gy_1, S), M(gx_1, gx_2, S), M(gy_{1,}gy_2, S)\}$$
$$M\big(gy_{n,}\ gy_{n+1}, k^nS\big) \geq$$
$$Min\{M(gy_0, gy_1, S), M\big(gx_{1,}gx_2, S\big), M(gy_1, gy_2, S), M(gx_{1,}gx_2, S)\}$$

for all n ϵ N. Then for all x ϵ X and n ϵ N we obtain

$$M(gx_n, gx_{n+m}, t) \geq M(gx_n, gx_{n+,m}, \textstyle\sum_{x=n1}^{\infty} \delta^i)$$

$$\geq M(gx_n, gx_{n+m} \textstyle\sum_{i=n}^{n+m-1} \delta^i)$$

$$\geq T_{i=n}^{n+m-1} M\left(gx_i\,, gx_{i+1}, S^i\right)$$

$$\geq T_{i=n}^{n+m-1}\left(Min\begin{Bmatrix} M\left(gx_0, gx_1, \frac{1}{\mu^i}\right), M\left(gy_0, gy_1, \frac{1}{\mu^i}\right) \\ M\left(gx_1, gx_2, \frac{1}{\mu^i}\right), M\left(gy_1, gy_2, \frac{1}{\mu^i}\right)\end{Bmatrix}\right)$$

$$\geq T_{i=n}^{n+m-1}(1-\alpha\mu^{\alpha i})$$

If we choose l_0 ϵ N such that $\alpha\mu^{al_0} \leq \mu^a$, then

$1-\alpha(\mu^a)^{n+l_0} \geq 1-(\mu^a)^{n+1}$ for all n,

Thus $M(gx_{n+l_0}, gx_{n+l_0+m}, t) \geq T_{l=n+1}^{\infty}\ (1-(\mu^a)^i > 1-\varepsilon,$

for every n $\geq$ n_1 and m ϵ N, hence $\{gx_n\}$ is a Cauchy sequence.

Similarly one can show that $\{gy_n\}$ is a Cauchy sequence.

Step 2. We prove that g and F have a coupled coincidence point. Since X is complete, thus exist x, y ϵ x such that $\lim_{n\to\infty} F(x_{n,}y_n) = \lim_{n\to\infty} g(x_n) = x, \lim_{n\to\infty} F(y_{n,}x_n) = \lim_{n\to\infty} g(y_n) = y$

Since F & g are compatible.

$$\lim_{n\to\infty}(gF(x_{n,}y_n)\,, F\big(g(x_n), g(y_n)\big), t) = 1,$$

$$\lim_{n\to\infty}(gF(y_{n,}x_n)\,, F\big(g(y_n), g(x_n)\big), t) = 1.$$

for all t > 0. Next we prove that

$$g(x) = F(x,y), g(y) = F(y,x)$$

from the continuity of g it follows that $\lim_{n\to\infty} ggx_n = gx$ and $\lim_{n\to\infty} ggy_n` = gy$ as F and g comments.

$ggx_{n+1} = gF(x_n, y_n) = F(gx_{n,}gy_n)$ and $ggy_{n+1} = gF(y_n, x_n) = F(gy_{n,}gx_n)$

Consequently, for all t > 0 and n ϵ N,

$$M(gx, F(x, y), kt) \geq M(ggx_{n+1,}F(x, y), kt), M(gx, ggx_{n+1\,,}k\, t)$$

$$= M(gF(x_n, y_n), F(x, y), kt), M(gx, ggx_{n+1,}kt)$$

$$\geq Min \begin{Bmatrix} M(gF(x_{n,}y_n\ ), F(gx_{n,}gy_n\ ), \underline{k}t), M(ggx_{n,}gx, kt) \\ M(ggy_{n,}gy, kt), M(gx, ggx_{n+1}, k\, t) \end{Bmatrix}$$

Since g & F are compatible with the continuity of g, we get $M(gx, F(x, y), \underline{k}t) \geq 1$ which implies that $gx = F(x, y)$.

Similarly we can get $gy = F(y, x)$.

Step 3. We prove that$gx = y$ and $gy = x$.

Obtained from

$M(gx, gy_{n+1}, kt) = M(F(x, y), F(y_{n,}x_n),$

Letting n→∞ in the inequality

$M(gx, gy_{n+1}, kt) \geq Min\{(gx, gy_{n,}t), M(gy, gx_{n,}t)$

We get, $M(gx, y, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}$

and similarly we can get,

$M(gy, x, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}$

Thus, $Min\{M(gx, y, t), M(gy, x, t)\} \geq Min\left\{M\left(gx, y, \frac{t}{k^n}\right), M\left(gy, x, \frac{1}{k^n}\right)\right\}.$

By this way, we can get for all n ϵ N

$Min\{M(gx, y, t), M(gy, x, t)\} = 1$ for all t > 0.

It follows that $M(gx, y, t) = M(gy, x, t) = 1$ for t > 0.

We can get that $gx = y \ and \ gy = x$, as claimed.

Step 4. We prove that $x = y$.

Indeed from $M(gx_{n+1,}gy_{n+1,}\ kt) = M(F(x_{n,}y_n), F(y_{n,}x_n), kt$

$$\geq Min\{M(gx_{n,}gy_{n,}t), M(gy_{n,}gx_{n,}\ t), M(gy_{n,}gx_{n,}t)$$

$$M(x, y, kt) \geq Min\{M(x, y, t), M(y, x, t) \qquad (t > 0)$$

It follows that $M(x, y, kt) \geq M(x, y, t)$ for all t > 0 and so x = y

Step 5. We prove that the fixed point is unique. Let z , w be common fixed points for F and g. Then from (2.1) we obtain

$$M(F(z,z), F(w,w), kt) \geq Min\{M(gz, gw, t), M(gz, gw, t), M(gz, gw, t), M(gz, gw, t) \ (t > 0)$$

That is $M(z, w, kt) \geq M(z, w, t)$ for all t > 0, implying$z = w$.

Our next theorem shows that ,if the t-norm T is of Hadzic- type, then the conditions

$\sup_{t>0} t^{\alpha} \ (1 - M(gx_{0,}F(x_{0,}\ y_0), t)) < \infty$ and $\sup_{t>0} t^{\alpha} \ (1 - M(gy_{0,}F(y_{0,}\ x_0), t)) < \infty$

Can be dropped.

Theorem 3.2: Let (X, M, T) be a complete fuzzy metric space satisfying (FM-6) with $T \in H$. Let F : X × X → X and g : X → X be two mappings such that, for some k ϵ (0,1),

$$\mathrm{M(F(x,y), F(u,v), Kt) \geq Min\{M(gx,gu,t), M(gy,gv,t), M(gx,gv,t), M(gy,gu,t)\}}.$$

for all x, y, u ,v ϵ X , t > 0. Suppose that F (X × X) ⊂ g(x) and g is continuous and commutes with F. Then F and g have a unique common fixed point in X.

Proof: We only have to verify step 1 in Theorem 2.3, that is to prove that $\{gx_n\} \ and \ \{gy_n\}$are Cauchy sequence.

Let $t > 0 \ and \ \varepsilon \in (0,1)$ be given. Since T is a t-norm of Hadzic- type then exists $\mu > 0$ such that $T^k(1 - \mu) > 1 - \varepsilon \ for \ all \ k \in N$.

By (FM-6), we can find $s > 0$ such that

$M(gx_0, gx_1, s) > 1 - \mu$ *and* $M(gy_0, gy_1, s) > 1 - \mu$

Let $n_0 \in N$ be such that $t > \sum_{i=n_0}^{\infty} k^i s$.

As in Step 1 in the proof of theorem 2.3 it can be proved that

$$M(gx_{n,}\ \ gx_{n+1}, k^n S) \geq Min \begin{Bmatrix} M(gx_0, gx_1, S), M(gy_{0,} gy_1, S) \\ M(gx_1, gx_2, S), M(gy_{1,} gy_2, S) \end{Bmatrix} > 1 - \mu$$

and $$M(gy_{n,}\ \ gy_{n+1}, k^n S) \geq Min \begin{Bmatrix} M(gy_0, gy_1, S), M(gx_{0,} gx_1, S) \\ M(gy_1, gy_2, S), M(gx_{1,} gx_2, S) \end{Bmatrix} > 1 - \mu$$

for all $n \in N$. Therefore, for all $n \geq n_0$ and all $m \in N$the following inequalities hold:

$$M(gx_{n,}\ \ gx_{n+1}, t) \geq M\left(gx_{n,}\ \ gx_{n+1}, \sum_{i=n_1}^{\infty} k^i S\right) \geq M\left(gx_{n,}\ \ gx_{n+m}, \sum_{i=n}^{n+m-1} k^i S\right)$$

$$\geq T_{i=n}^{n+m-1} M(gx_i, gx_{i+1}, k^i s) \geq T_{i=n}^{n+m-1}(1 - \mu) > 1 - \varepsilon.$$

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