Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.6, 2015
www.iiste.org

An Application of Spline and Piecewise Interpolation to Heat Transfer (Cubic Case)

Chikwendu, C. R.[1], Oduwole, H. K.[*2] and Okoro S. I.[3]

[1,3]Department of Mathematics, Nnamdi Azikiwe University, Awka, Anambra State, Nigeria

[2]Department of Mathematical Sciences, Nasarawa State University, Keffi, Nasarawa State, Nigeria

[*1]E-mail of the corresponding author: [1]rosecc2k2@yahoo.com [2]kenresearch@yahoo.com[3]puritysam@yahoo.com

Abstract

An Application of Cubic spline and piecewise interpolation formula was applied to compute heat transfer across the thermocline depth of three lakes in the study area of Auchi in Edo State of Nigeria. Eight temperature values each for depths 1m to 8m were collected from the lakes. Graphs of these temperatures against the depths were plotted. Cubic spline interpolation equation was modelled. MAPLE 15 software was used to simulate the modelled equation using the values of temperatures and depths in order to obtain the unknown coefficients of the variables in the 21 new equations. Three optimal equations were found to represent the thermocline depth for the three lakes. These equations were used to obtain the thermocline gradients $\left(\frac{dT}{dz}\right)$ and subsequently to compute the heat flux across the thermocline for the three lakes. Similar methods were used for cubic piecewise interpolation. The analytical and numerical results obtained for the computation of thermocline depth and temperature were presented for the three lakes with relative error analysis. Absolute Relative Error$|\in_{aS}|$ for Analytic solution with Cubic Spline Interpolation was 0.41%, 0.70% and 0.74%, while Absolute Relative Error$|\in_{aP}|$ for Analytic solution with Cubic Piecewise Interpolation are 0.82%, 2.11% and 1.48% respectively. Comparative analysis showed that the results obtained with cubic spline interpolation method had less percentage error than the cubic piecewise interpolation method.

Keywords: Cubic spline, Interpolation, Heat transfer, Thermocline

1. Introduction

In the mathematical field of numerical analysis, interpolation is a method of constructing new data point within the range of a discrete set of known data points. In a more informal language, interpolation means a guess at what happens between two values already known. In Engineering and Environmental sciences application, data collected from the field are usually discrete, therefore a more analytically controlled function that fits the field data is desirable and the process of estimating the outcomes in between these desirable data points is achieved through interpolation (Crochiere and Rabiner, 1983).

There are two main uses of interpolation. The first use is in reconstructing the function $f(x)$ when it is not given explicitly and/or only the values of $f(x)$ and certain order derivatives at a set of points, called nodes are known. Secondly interpolation is used to replace the function $f(x)$ by an interpolating polynomial $p(x)$ so that many common operations like the determination of roots, differentiation, integration etc. which are intended for the function $f(x)$ may be performed using the interpolant $p(x)$ (Jain *et al*.; 2007)

Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called spline. Spline interpolation is often preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline (Hazewinkel, 2001). Spline interpolation avoids the problem of Runge's phenomenon, in which oscillation occurs between points when interpolating using high degree polynomials (Kim, 2005; Turner, 1989).

Cubic Spline interpolation is a special case of spline interpolation that is used very often to avoid the problem of Runge's phenomenon. This method gives an interpolating polynomial that is smoother and has smaller error than other interpolating polynomials such as Lagrange polynomial and Newton polynomial (Jain *et al*.; 2007).

Current literatures exist showing the application of Spline and cubic interpolation. The optimal point for a given group of oil wells was found by Jamal (2001) using the application of Lagrange multiplier, an approach that depend on the functional relationship between oil production and gas injection rates. Jamal (2001) used two type of function (quadratic and rational functions) to fit the gas injection against oil production rate. Ntherful (2013) worked on optimal Spline based Gas-left Allocation using Lagrange's multiplier in an oil production field. After fitting the gas injection and oil output rates of the wells with cubic spline functions, the optimal rates of gas injection and oil output in each of the wells are determined using the Lagrange multiplier method. It was found that the total optimum oil production rate for data fitting with spline based function was higher than the total optimum oil production rate for data fitting with rational function. The optimal value of the spline based function was found to be twice that of rational function.

Nelson *et al.* (1997), worked on parametric studies on thermally stratified chilled water storage system. In their work, they analyze the stratification decay in thermally stratified vertical cylindrical cool storage system using a one dimensional conjugate heat conduction model. They found out that the thermoclines degrade due to the heat transfer from the ambient, thermal diffusion in the storage tank, axial wall conduction and mixing due to admission of the fluid in the storage tank during charging and discharging. Another application of heat transfer was found in the construction of a natural cubic spline of the heat capacity of gadolinium using experimental measurement at fixed values of magnetic induction and the least square curve fitting was used to obtain the approximation function of the heat capacity of gadolinium (Siddikov, n.d).

This research work focuses on the application of spline cubic interpolation and piecewise cubic interpolation on heat transfer in selected lakes. The thermal stratification of the lakes informs the temperature difference at each layer of the lake. However, the values of the interval between the thermocline depth, temperatures at these intervals and the heat flux will be used to derive equations using cubic spline interpolation and piecewise cubic interpolation. The results from these equations will then be used to obtain the thermocline depths, thermocline temperatures and the thermocline heat flux for the selected lakes. A comparative analysis of the values obtained using the cubic spline interpolation and piecewise cubic interpolation will be made to determine the method with the least percentage error.

2. An Overview of Approximation and Interpolation Theory

Curve fitting is the process of finding a curve that could best fit a given set of data (Won *et al*.; 2005). There are two approaches of fitting a curve from a set of data points. The first approach called collocation is the case where the curve is made to pass through all data points. This approach is used either when the data is known to be accurate or the data are generated from the evaluation of some complicated function at discrete set of points. Such function could be polynomial, trigonometric or exponential functions. The second approach is when a given curve is made to represent the general trend of the data. This approach is useful when there are more data points than the number of unknown coefficient or when the data appear to have a significant error or noise (Singiresu, 2002).

Interpolation is used to estimate the value of a function between known data points without knowing the actual function. Two main broad categories of interpolation exists; global and piecewise interpolation (Henrici, 1982). Global interpolation methods use a single equation that maps all the data points into an nth order polynomial. These methods result in smooth curves, but in many cases they are prone to severe oscillation and overshoot at intermediate points. Piecewise interpolation method uses a polynomial of low degree between each pair of known data points. If a first degree polynomial is used, it is called linear interpolation, for second and third degree polynomial; it is called quadratic and cubic spline respectively. The higher the degree of the spline, the smoother the curve spline of degree m, will have continuous derivative up to $(m-1)$ at the data points.

Interpolating a set of data points can be done using polynomial, spline function or Fourier series. However polynomial interpolation is commonly used and many numerical methods are based on polynomial approximations.

For a given set of $(N+1)$ data points $(x_o, y_o), (x_1, y_1), \dots, (x_N, y_N)$, it is of interest to find N^{th} order polynomial function that can match these data points. The N^{th} polynomial function is given as

$$P_N = a_o + a_1x + a_2x^2 + \ .\ .\ . + a_Nx^N \qquad (2)$$

The coefficients can be obtained by solving a set of algebraic equations

$a_o + a_1x_o + a_2x_o^2 + ... + a_Nx_o^N = y_o$

$a_o + a_1x_1 + a_2x_1^2 + ... + a_Nx_1^N = y_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_o + a_Nx_N + a_2x_N^2 + ... + a_Nx_N^N = y_N$

As the number of data points increases, so also that of the unknown variables and equations, the resulting system of equations may not be so easy to solve. However, there are a number of alternative forms of expressing an interpolating polynomial beyond the familiar format stated above. Among them are Lagrange, Newton's forward and backward difference, and Hermite interpolations.

According to Singiresu (2002), the errors of a single polynomial tend to increase drastically as its order n becomes large. The higher order polynomial often introduces unnecessary oscillation and wiggles and due to this, polynomial interpolation will not be always accurate. To avoid this, information from more data points will be used and at the same time keeping the function true to the data behaviour is the objective of spline interpolation. The most common spline and piecewise interpolation used are linear, quadratic and cubic respectively. To obtain a smoother curve, cubic splines are frequently recommended, because they provide the simplest representation that exhibits the desired appearance of smoothness. They are generally well behaved and continuous up to the second order derivative at the data points. Even though cubic splines are less prone to oscillation or overshooting due to instability inherent in higher order polynomial than global polynomial equations, they do not prevent it. Thus, to avoid these oscillations, it is common to divide the interval into sub-interval and approximate the function using low degree polynomial on each sub-interval (Kruger, n.d).

2.1 Mathematical Treatments

Give a function f on $[a, b]$ and nodes $a = x_0 < x_1 <\ldots< x_n = b,$ a cubic spline interpolant f satisfied the following conditions (Kim, 2005 and Jain *et al*. 2007):

1. f is a cubic polynomial on each subinterval $[x_i, x_{i+1}]$
2. $S_i(x_i) = f_i(x_i) \; for \; i = 0,1,2,..n$ (i.e the spline matches function values)
3. $S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = f_{i+1}$ for $i = 0,1, \ldots .n-2$ (i.e the spline is continuous)
4. $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ for $i = 0,1, \ldots .n-2$ (the spline C^1)
5. $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$ for $i = 0,1, \ldots .n-2$ (i.e the spline C^2)
6. (i).$S_i''(x_0) = S_i^{ii}(x_n) = 0$ (Natural Spline resulting from free boundary condition).
   (ii). $S_i'(x_0) = f_i'(x_0) \; and \; S_i'(x_n) = f_i'(x_n)$ (Clamped end condition resulting from clamped boundary condition)

In the derivation of the cubic spline interpolation polynomial, we follow the above conditions one after the other. The objective of the cubic spline is to derive a third-order polynomial for each interval between knots as represented by a general cubic spline polynomial function given below

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (2.1)$$

The first condition is that the cubic spline must pass through all data points.

Hence we have

$$f_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (2.2a)$$

$$\Rightarrow f_i = a_i + b_i(x_i - x_i) + c_i(x_i - x_i)^2 + d_i(x_i - x_i)^3 \quad (2.2b)$$

Hence $f_i = a_i$ (2.3)

Therefore the constant in each cubic must be equal to the value of the dependent variable at the beginning of the interval. Substituting (2.3) in (2.1), we have

$$S_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (2.4)$$

Next we apply the third condition that each of the cubic spline must join the knot.

For $(i + 1)$ knot,we have

$$S_{i+1}(x) = f_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3$$

$$\Longrightarrow f_{i+1} = f_i + b_i h_i + c_i h_i^2 + d_i {h_i}^3 \quad (2.5)$$

$where\ h_i = x_{i+1} - x_i$

Applying the fourth condition, the first derivation at the interior nodes must be equal.
Hence differentiating (2.4)yields

$$S_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2 \quad (2.6)$$

The equivalent of the derivative at the interior node $(i + 1)$ is given as

$$b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1} \quad (2.7)$$

Applying the fifth condition, the second derivative at the interior nodes must also be equal.
Hence differentiating equation (2.6) yields

$$S_i''(x) = 2c_i + 6d_i(x - x_i) \quad (2.8)$$

The equivalent of the second derivative at the interior node $(i + 1)$is given as

$$c_i + 3\ d_i h_i = c_{i+1} \quad (2.9)$$

Solving (2.9)for d_i, yields

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \quad (2.10)$$

Substituting (2.10) into (2.5) yields

$$f_{i+1} = f_i + b_i h_i + \frac{{h_i}^2}{3}(2c_i + c_{i+1}) \quad (2.11)$$

Substituting (2.10) into (2.7) yields

$$b_{i+1} = b_i + h_i(c_i + c_{i+1}) \quad (2.12)$$

Equation (2.11) can be solved for b_i to get

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}), \quad (2.13)$$

Reducing the index by 1, equation (2.12) and (2.13)can be rewritten as

$$b_{i-1} = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i), \tag{2.14}$$

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i) \tag{2.15}$$

Substituting equation (2.13) and (2.14) in (2.15) yields

$$\frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) + h_{i-1}(c_{i-1} + c_i)$$

Further simplification yields

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3\left(\frac{f_{i+1} - f_i}{h_i}\right) - 3\left(\frac{f_i - f_{i-1}}{h_{i-1}}\right) \tag{2.16}$$

Using divided difference notation, $f[x_i, x_j] = \frac{f_i - f_j}{x_i - x_j}$

Equation (2.16)can be written as

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3(f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \tag{2.17}$$

Equation (2.17) can be written for the interior knots, $i = 2, 3, \ldots, n-2$, which result in $(n-3)$ simultaneous tridiagonal equations with $(n-1)$ unknown coefficient $c_1, c_2, \ldots, c_{n-1}$. Using the last condition (boundary condition), the second derivative at the first node (that is equation (2.8)) can be set to zero, given as

$$S_i''(x) = 2c_i + 6d_i(x - x_i) = 0 \implies c_i = 0 \tag{2.18}$$

Similarly the same evaluation can be made at the last node, given as

$$S_{n-1}''(x_n) = 2c_{n-1} + 6d_{n-1}h_{n-1} = 0 \tag{2.19}$$

Recalling (2.9), equation (2.19) became

$$c_{n-1} + 3d_{n-1}h_{n-1} = c_n = 0 \tag{2.20}$$

Thus to impose a zero second derivative at the last node, we set $c_n = 0$

Rewriting equation (2.17) in matrix form, we have

$$\begin{bmatrix} 1 & . & . & . & . & . & . & . \\ h_1 & 2(h_1 + h_2) & h_2 & . & . & . & . & . \\ . & h_2 & 2(h_1 + h_2) & h_3 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ . & . & . & . & . & . & . & 1 \end{bmatrix} . \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ . \\ . \\ . \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 3(F[x_3, x_2] - F[x_2, x_1]) \\ . \\ . \\ . \\ . \\ 3(F[x_n, x_{n-1}] - F[x_{n-1}, x_{n-2}]) \\ 0 \end{bmatrix} \tag{2.21}$$

From (2.20) we can now solve for c’s. Equation (2.10) and (2.13) can then be used to determine the remaining coefficient b and d.

Alternatively the above scheme for a_i, b_i, c_i and d_i can be found by solving a total of $(4n-4)$ equations with $(4n-4)$ unknowns using matrix method of solving a system of equations. This is gotten from the condition 1 through 6 in section 2.1. From conditions 1 and 2, the spline function goes through the first and last point of the interval, yielding $2(n-1)$ equations as shown in equations (2.3) and (2.5) respectively.

Also from condition 4 of section 2, the first derivative is continuous at each interior point, yielding $(n-2)$ equations as shown in equation(2.6).

Another condition (the fifth condition) of section 2 requires that the second derivative is continuous at each interior point which leads to $(n-2)$ equations as shown in equation (2.8).

The sixth condition of section 2.1, that the second derivative at the end knots is zero yields $\left(\frac{n}{4}\right)$ equations. Finally this gives a total of $(4n-4)$ equations with $(4n-4)$ unknown. This can be solve using $Ax=b$ as shown below

$$\begin{bmatrix} Tz_1(a_1) & Tz_1(b_1) & Tz_1(c_1) & Tz_1(d_1) & 0 & 0 & 0 & 0 & & \cdots & & \cdots & 0 & 0 \\ Tz_{1*}(a_{1*}) & Tz_{1*}(b_{1*}) & Tz_{1*}(c_{1*}) & Tz_{1*}(d_{1*}) & 0 & 0 & 0 & 0 & & \cdots & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & Tz_2(a_2) & & & & & & \cdots & & 0 & 0 \\ & & & & Tz_{2*}(a_{2*}) & & & & & & & & & \\ & \cdots & & & & & & & Tz_3(a_3) & & \cdots & & 0 & 0 \\ & & & & & & & & Tz_{3*}(a_{3*}) & & & & & \\ & & & & \vdots & & & & & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & & & & & & \vdots \\ & & & & & & & & & & \cdots & & \cdots & \\ & & & & & & & & \vdots & & & & & \\ & & & & \vdots & & & & & & & & & \\ & \cdots & & & & & & & & T^1 z(a) & & & & \\ & & & & & & \vdots & & & & & & & \\ & & & & & & & & & & & T^{11} z(a) & & \\ 0 & 0 & 0 & 0 & & & & & \cdots & & & & & \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ \vdots \\ a_7 \\ b_7 \\ c_7 \\ d_7 \end{bmatrix} = \begin{bmatrix} T_1(z) \\ \vdots \\ T_2(z) \\ \vdots \\ \\ T_7''(z) \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (2.22)$$

For the purpose of computational advantage, we also use the newton’s divided difference interpolation to obtain the interpolating polynomial of cubic piecewise for the three sites as shown in equation (2.22).

Alternatively we can use the piecewise cubic interpolation formula, given below (Jain *et al.* 2007)

$$P_{i,3}(x) = l_{i,o}(x)f(x_o) + l_{i,1}(x)f(x_{i+1}) + l_{i,2}(x)f(x_{i+2}) + l_{i,3}(x)f(x_{i+3}) \quad (2.23)$$

where

$$l_{i,o} = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})}, \quad l_{i,1} = \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})}, \quad l_{i,2} = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})}$$

and $l_{i,3} = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})}$

For interpolating the polynomial of cubic piecewise, the Newton's divided difference interpolation formula for the solution in term of four points(x_0, f_0), (x_1, f_1), (x_2, f_2), and (x_3, f_3) is given as

$$f_n(x) = f_0 + \Delta x_0 f[x_0, x_1] + \Delta x_0 \Delta x_1 f[x_0, x_1, x_2] + \Delta x_0 \Delta x_1 \Delta x_2 f[x_0, x_1, x_2, x_3] \qquad (2.24)$$

where $\Delta x_0 = x - x_0$, $\Delta x_1 = x - x_1$, $\Delta x_2 = x - x_2$, $f[x_0, x_1] = \frac{f(x_0) - f(x_0)}{x_1 - x_0}$

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

2.2 Determination of the Temperature of the thermocline Depth

The location of the thermocline depth is defined as the inflection point of the temperature-depth curve, that is the point at which $\frac{d^2T}{dz^2} = 0$. We calculate the thermocline depth of the lakes in the study area using $\frac{d^2T}{dz^2} = 0$

2.3 Determination of heat flux across the thermocline

According to Fourier's law of heat conduction, heat flows from the regions of high temperature to low temperature. For the one-dimensional case, this can be expressed mathematically as $q = -k\frac{dT}{dx}$
Where $q(x) =$ heat flux (w/m^2) $k =$ coefficient of thermal conductivity $W/(m.k)$, $T =$ Temperature(k) $x =$ Distance(m).

The temperature gradient is important in its own right because it can be used in conjunction with Fourier's law to determine the heat flux across the thermocline. Heat flux is defined as the amount of heat transferred per unit area per unit time from or to a surface (John and John, 2008). It is calculated using the formula $= -\alpha\rho c\frac{dT}{dz}$,
where $J =$ heat flux$[J/(m^2.s)]$, $\alpha =$ an eddy diffusion coefficient $\left(10^{-3}\frac{m^2}{s}\right)$, $\rho = density\left(\cong \frac{1000kg}{m^3}\right)$
and $c =$ specific heat $[\cong 4200J/(kgK)]$

From the twenty one equations (21) obtained (i.e. in this study seven equations each for the three lakes sampled), three optimal equations that represents the thermocline depth for the three lakes were chosen. These optimal equations were differentiated and used to obtain the thermocline gradient$\left(\frac{dT}{dz}\right)$. These gradients were substituted into the heat flux equation to obtain the values of the heat flux across the thermocline for the three lakes.

2.4 Determination of cubic spline equations using Maple 15 software

The maple 15 software was launch and it displayed the dialogue box. From the dialog box, access was granted into the worksheet. Values of the temperatures and their corresponding depths for the various lakes were inputted into the worksheet alongside the degree and end point of the spline interpolation. The entire entries were then made to run in order to obtain twenty one cubic spline interpolation equations (seven equations for each lake) using the linear Algebra package.

2.5 Absolute Relative Error between the Analytic solution and the two methods

The absolute relative approximate error $|\in_a|$ obtained between the analytic solution of the temperature results with the Cubic Piecewise Interpolation and Cubic Spline Interpolation error is given below:

The error for Analytic solution with the Cubic Spline Interpolation is given by

$$|\epsilon_{aSpline}| = \left|\frac{p^{**}(z) - p(z)}{p^{**}(z)}\right| \times 100$$

Where $p^{**}(z)$ = Temperature obtained by Analytic Solution

$p(z)$ = Temperature estimate obtained by Cubic Spline Interpolation

The error for Analytic solution with the Cubic Piecewise Interpolation is given by

$$|\epsilon_{aPiecewise}| = \left|\frac{p^{**}(z) - p^{*}(z)}{p^{**}(z)}\right| \times 100$$

Where $p^{**}(z)$ = Temperature obtained by Analytic Solution

$p^{*}(z)$ = Temperature estimate obtained by Cubic Piecewise Interpolation

3. Research Method

3.1 The Study Area

The study area is Edo State of Nigeria. Edo State is an inland state in western Nigeria. Its capital is Benin City. It is bounded in the north and east by Kogi State, in the south by Delta State and in the west by Ondo State. The State was created on the 27th August 1991 when Bendel State was split into Edo and Delta State. Three lakes were chosen all located in Auchi. Auchi is a town in Edo State, Nigeria. It serves as the administrative headquarters of Etsako West Local Government Area. According to the 1991 population census of Nigeria, Auchi population was estimated at 42,610.

3.2 Materials

Table 3.1 Showing Laboratory materials use in the Research work

S/N	Materials/Equipment	Description	Uses
1.	Laboratory Thermometer	Calibrated mercury in glass thermometer with a sensitive bulb.	Measurement of temperature in the water body during solar radiation
2.	Calibrated Rope	A rope calibrated in meters with 1m for each length was attached to the thermometer.	The calibrated rope was used to determine the depth of the water body in the selected lake sites.
3.	Maple 15 Software	Mathematical symbolic software.	Maple software is use in a broad spectrum concept from Calculus, Algebra, introductory interpolation to higher degree of interpolations

3.3 Methodology

Three lakes were chosen for this research work and they are located around Auchi in Edo State. A mercury in glass thermometer was attached to a rope graduated from 1.0m – 8.0m, and lowered into each of these lakes at different time. For each lake, the temperature reading at 1m was taken after 10 minutes interval. A similar procedure was adopted for lengths ranging from 2 to 8m depth of the lakes after 10 minutes each 1m for each of the lakes. The temperatures and their corresponding depths were recorded.

In obtaining the cubic piecewise interpolation polynomials, we use the Newton's divided difference method for two sets of points $[1,2,3,4]$ and $[4,5,6,7]$ in the data from the study area for each site. Also the Thermocline depth for each site were also computed.

4. Results and Discussion

Table 4.1 shows the results of the values of temperatures at various depths that were determined from the three lakes selected for the study around Auchi in Edo State. For each of the lakes the temperature reading at 1 m each was taken after 10 minutes interval and their corresponding depths were recorded after 1 m each until a depth of 8 m was reached

Table 4.1 Temperature and depth taken for three (3) lakes around Auchi

Lake 1		**Lake 2**		**Lake 3**	
Depth (z) m	Temperature (^{0}C)	Depth (z) m	Temperature (^{0}C)	Depth (z) m	Temperature (^{0}C)
1.0	30	1.0	35	1.0	34
2.0	30	2.0	35	2.0	34
3.0	30	3.0	34	3.0	34
4.0	28	4.0	32	4.0	30
5.0	23	5.0	26	5.0	23
6.0	19	6.0	22	6.0	21
7.0	18	7.0	21	7.0	20
8.0	18	8.0	21	8.0	20

Figures 4.1, 4.2 and 4.3 are graphical representation of the analytical results presented in Table 4.1. The graph shows plot for the temperature against depth for the three lake site respectively. The graphs shows that with increasing depths of lakes there was decreasing temperature. However, the heat transfer vis a vis the temperature at the region between 4 and 5 m for the three lakes was highest indicating the region of thermocline thickness and thermocline depths for the three lakes. The thermocline depth were found to be 4.7m at 24.50℃ for the first lake, 4.6m at 28.50℃ for the second lake and 4.4m at 27℃ for the third lake. Meanwhile, the graphs also shows other regions above thermocline thickness known as the epilimnion and the regions below the thermocline thickness known as the hypolimnion.

Table 4.2 shows the summary of Analytical and numerical result obtained from data analyzed from the three sites. The thermocline depth for Lake 1, Lake 2 and Lake 3 were found to be $4.73m$, $4.64m$ and $4.38m$ respectively which indicates that the thermocline depths varies with lakes. In addition, the percentage error between the analytical and numerical results for temperature (cubic spline interpolation) were found to be 0.41%, 0.70% and 0.74% for the three sites respectively. The percentage error between the analytical results for temperature and numerical result (cubic piecewise interpolation) were found to be 0.82%, 2.11% and 1.48% for Lake 1, Lake 2 and Lake 3 respectively. These errors are indications that cubic spline interpolation have lower error than cubic piecewise interpolation for heat transfer across the thermocline depth (cubic case).

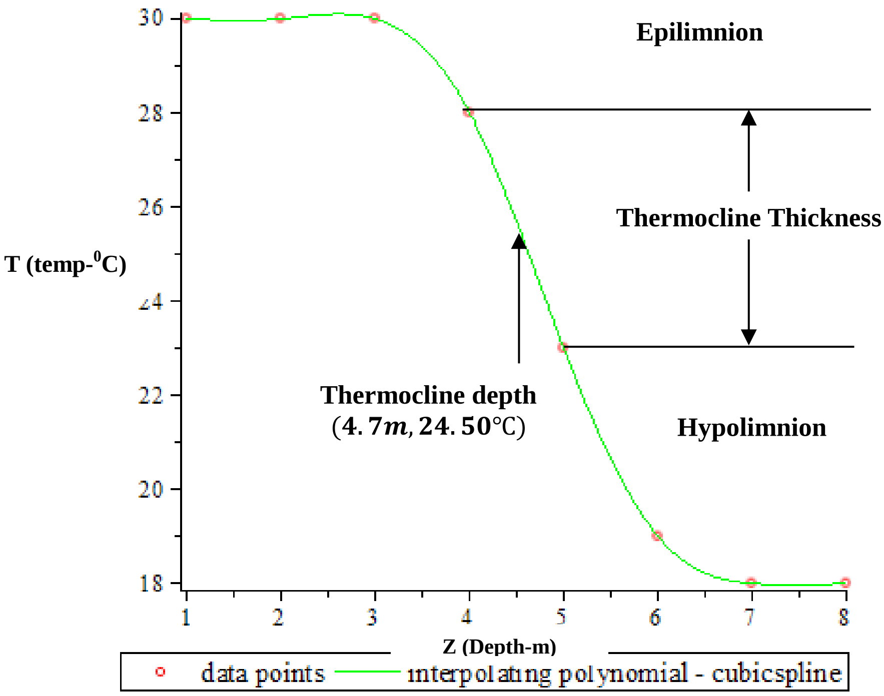

Figure 4.1 Cubic Spline Interpolation using Natural Boundary Condition for Lake 1

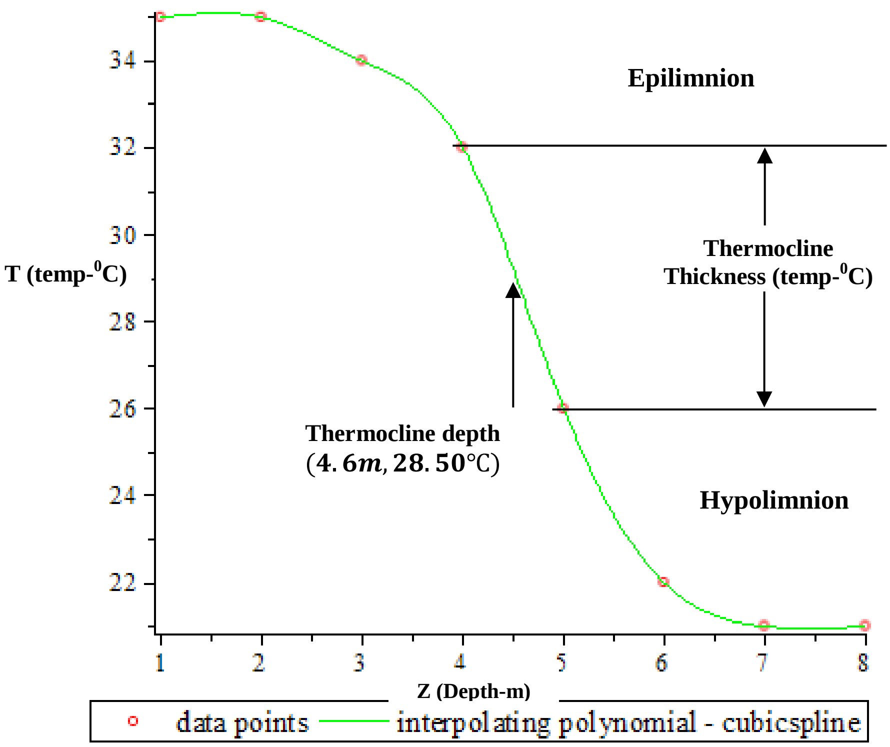

Figure 4.2 Cubic Spline Interpolation using Natural Boundary Condition for Lake 2

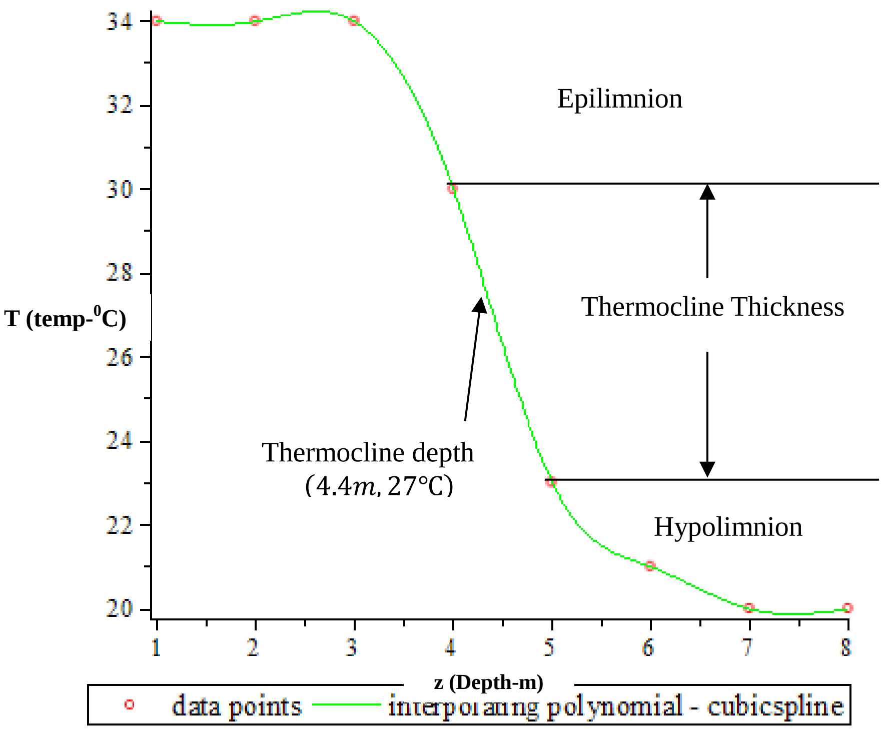

Figure 4.3 Cubic Spline Interpolation using Natural Boundary Condition for Lake

Table 4.2 Summary of Analytical and Numerical Results

S/N	Descriptions	Lake 1	Lake 2	Lake 3
1	Thermocline depth (Analytical solution by Graph)	$4.7m$	$4.6m$	$4.4m$
2	Thermocline depth (by $\frac{d^2T}{dz^2} = 0$)	$4.73m$	$4.64m$	$4.38m$
3	Temperature (Analytical solution by Graph)	24.50℃	28.50℃	27.00℃
4	Temperature (Cubic Spline Interpolation)	24.44℃	28.29℃	27.19℃
5	Temperature (Cubic Piecewise Interpolation)	24.34℃	27.98℃	26.56℃
6	Heat Flux (Heat Transfer)	$22{,}730.42J/(m^2.s)$	$27{,}632.19J/(m^2.s)$	$4{,}350.92\,J/(m^2.s)$
7	Absolute Relative Error$\|\epsilon_{aS}\|$ for Analytic solution with Cubic Spline Interpolation	0.41 %	0.70 %	0.74 %
8	Absolute Relative Error$\|\epsilon_{aP}\|$for Analytic solution with Cubic Piecewise Interpolation	0.82 %	2.11 %	1.48 %

The heat flux across the thermocline thickness were also computed to be $22{,}730.42J/(m^2.s)$, $27{,}632.19J/(m^2.s)$ and $4{,}350.92J/(m^2.s)$ respectively for Lake 1, Lake 2 and Lake 3 respectively. These are indications of the amount of heat transfer per unit area per unit time across the thermocline depth.

5. Conclusion

The results obtained from this work showed that the thermocline thickness as well as the thermocline depth vis-a-vis the heat transfer for the lakes were found to be in the interval of 4.0m and 5.0m depth. The cubic spline was derived and the result was use to approximate thermocline depth and temperature.

The cubic spline interpolation method showed less percentage error when compared to the analytical result, while the results obtained from the cubic piecewise interpolation method showed a higher percentage error when compared to the analytical results.

The effective determination of the thermocline depth will enable fishermen to know when to set out to catch fishes and also in generating income by the use of aquatic study. Maple software is highly recommended for use in handling any constraint of cubic case interpolation problems. Finally cubic spline interpolation method will produce better results when use to determined thermocline depth.

References

Crochiere, R.E. and Rabiner, L.R. (1983): http:// en.m.wikipedia.org/wiki/interpolation

Hazewinkel, Michiel, ed. (2001): "Spline interpolation", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

Henrici, P. (1982): *Essential of numerical analysis*, New York. John Wiley & Sons.

Jain, M.K., Iyenger, S.R.K., and Jain, R.K. (2007): Numerical Methods for Scientific and Engineering Computation. Fifth Edition. Published by New Age International (P) Limited, Publishers

Jamal, A. S. (2001): Production optimization via Lagrange multipliers. Unpublished master's thesis. Texas Tech University.

John, H.L IV and John, H.L V (208): A heat transfer textbook. Published by phlogiston press Cambridge, Massachusetts, U.S.A.

Kim, H.K, (2005): Spline and Piecewise Interpolation. Slightly modified 3/1/09, 2/28/06 Firstly written at March 2005. DM21869/Computational Numerical Analysis/16_cna.doc Available at http://bml.pusan.ac.kr BML/SME/PNU

Kruger, C. J. C. (n.d): *Constrained cubic spline interpolation for chemical engineering applications*.

Nelson J.E.B., Balakrishnan, A.R., and Murthy S.S (1997): Parametric studies on thermally stratified chilled water storage systems. Applied thermal Engineering 19(1999)89-115

Ntherful E.G (2013): Optimal Spline Based Gas-Left Allocation Using Lagrange's Multiplier. Unpublished Master's thesis. Kwame Nkrumah University of Science and Technology, Kumasi.

Siddikov, B. (n. d): Natural Cubic Interpolating Spline for the Heat Capacity of Gadolinium. Recent Research in Applied Mathematics and Economics. Ferris State University, U.S.A siddikob@ferris.edu.

Singiresu, S.R. (2002): *Applied numerical methods for engineers and scientists*. (M.J.Horton, Ed.). Tom Robbins prentice Hall, Inc. Upper Saddle River, New Jersey 07458.

Turner, P.R. (1989): Guide to Numerical Analysis. Published by Macmillan Education Ltd printed in Hong Kong.

Won, Y., Wenyu, C., Tae-Sang, C., & Morris, J.(2005): Applied numerical methods using matlab. John Wiley & Sons. Inc.
www.maplesoft.com/help