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# Numerical Solution of Nonlinear Black – Scholes Equation by Accelerated Genetic Algorithm

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## Abstract

In this paper we using an accelerated genetic algorithm to find the numerical solution of the nonlinear versions of the standard Black–Scholes partial differential equation with stochastic volatility (transaction coast) for European call option . We study this equation with different models of volatility and comparison these solutions with the solution of linear model of Black-Scholes equation without transaction coast .

***Keywords:*** *Nonlinear Black-Scholes Equation , Accelerated Genetic Algorithm , Option Pricing*

## 1.Introduction

An option is the right to buy or sell a risky asset at a pre specified fixed price within a specified period. An option is a financial instrument that allows to make a bet on rising or falling values of an underlying asset. An option is a contract between two parties about trading the asset at a certain future time , [1 , 2 ]. One party is the writer, often a bank, who fixes the terms of the option contract and sells the option. The other party is the holder, who purchases the option, paying the market price, which is called premium .There are numerous different types of options, which are not all of interest to this paper. we concentrate on standard options, also known as vanilla options. Options have a limited life time. The maturity date $T$ fixes the time horizon. At this date the rights of the holder expire, and for later times ($t > T$) the option is worthless. There are two basic types of option: The call option gives the holder the right to *buy* the underlying for an agreed price $K$ by the date $T$. The put option gives the holder the right to *sell* the underlying for the price $K$ by the date $T$. The previously agreed price $K$ of the contract is called strike or exercise price. Not every option can be exercised at any time $t \le T$. For European options exercise is only permitted at expiration $T$. American options can be exercised at any time up to and including the expiration date. Both types are traded in each continent. The value of the option will be denoted by $V$. The value $V$ depends on the price per share of the underlying, which is denoted $S$. This letter $S$ symbolizes stocks. The variation of the asset price $S$ with time $t$ is expressed by $S_t$ or $S(t)$. The value of the option also depends on the remaining time to expiry $T - t$. That is, $V$ depends on time $t$. The dependence of $V$ on $S$ and $t$ is written $V(S, t)$. The value $V(S, T)$ of a call option at expiration date $T$ is given by :

$$V(S_T, T) = \begin{cases} 0 & \text{incase} S_T \le K \text{ (option expires worthless)} \\ S_T - K & \text{in case } S_T > K \text{ (option is exercised)} \end{cases}$$

Hence

$$V(S_T, T) = \max\{S_T - K, 0\} .$$

For a European put, exercising only makes sense in case $S < K$. The payoff $V(S, T)$ of a put at expiration time $T$ is

$$V(S_T, T) = \begin{cases} K - S_T & \text{incase} S_T < K \text{ (option is exercised)} \\ 0 & \text{in case } S_T \ge K \text{ ( option expires worthless)} \end{cases}$$

Hence

$$V(S_T, T) = max\{K - S_T, 0\} ,$$

or

$$V(S_T, T) = (K - S_T)^+$$

**Option pricing theory**: the Black–Scholes option pricing model was made by [2] and [4]. The solution of the famous (linear) Black–Scholes equation [3].

$$0 = V_t + \frac{1}{2}\sigma^2 V_{SS} + rSV_S - rV \qquad (1)$$

where $S := S(t) > 0$ and $t \in (0, T)$, provides both the price for a European option and a hedging portfolio that replicates the option assuming that [1] :

The price of the asset price or underlying derivative $S(t)$ follows a Geometric Brownian motion $W(t)$, meaning that $S$ satisfies the following stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

where the trend or drift $\mu$ (measures the average rate of growth of the asset price), the volatility $\sigma$ (measures the standard deviation of the returns) and the riskless interest rate $r$ are constant for $0 \leq t \leq T$ and no dividends are paid in that time period. Due to transaction costs [5 , 6] , where both the volatility $\sigma$ and the drift $\mu$ can depend on the time $t$, the stock price $S$ or the derivatives of the option price $V$ itself. In this paper we will be concerned with several transaction cost models from the most relevant class of nonlinear Black–Scholes equations for European and American options with a constant drift $\mu$ and a non constant volatility $\tilde{\sigma}^2 = \tilde{\sigma}^2(t, S, V_S, V_{SS})$ . Under these assumption (1) becomes the following nonlinear Black–Scholes equation, which we will consider for European options:

$$0 = V_t + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS} + rSV_S - rV \qquad (2)$$

where $S > 0$ and $t \in (0, T)$.

In order to make the model more realistic, we will consider a modification of (2) for American options, where $S$ pays out a dividend $qSdt$ in a time step $dt$:

$$0 = V_t + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS} + (r - q)SV_S - rV \qquad (3)$$

where $S > 0$ , $t \in (0, T)$ and the dividend yield $q$ is constant.

## 2. Volatility models with transaction costs

### 2.1. Leland's model

Leland's idea of relaxing the hedging conditions is to trade at discrete times [5], which refer to reduce the expenses of the portfolio adjustment. He assumes that the transaction cost, $\frac{\kappa}{2}|\Delta|S$ , is proportional to the monetary value of the assets bought or sold. Here, $\kappa$ denotes the round trip transaction cost per unit dollar of the transaction and $\Delta$ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price $S$. Leland then deduces that the option price is the solution of the nonlinear Black–Scholes equation (2) with the modified volatility

$$\tilde{\sigma}^2 = \sigma^2\left(1 + Le\, sign(V_{SS})\right) \qquad (4)$$

where $\sigma$ represents the original volatility and $Le$ the *Leland number* given by:

$$Le = \sqrt{\frac{2}{\pi}} \frac{\kappa}{\sigma\sqrt{\delta t}}$$

where $\delta t$ denotes the transaction frequency. The price of the discrete option converges to a Black–Scholes price with the modified volatility of the form,[6].

$$\tilde{\sigma}^2 = \sigma^2\left(1 + Le\sqrt{\frac{\pi}{2}}\, sign(V_{SS})\right) \qquad (5)$$

Just like Leland, Boyle and Vorst assume convexity of V, so that $\tilde{\sigma}^2 = \sigma^2 \left(1 + Le\sqrt{\frac{\pi}{2}}\right)$ and (2) turns into a linear equation.

### 2.2 Barle's and Soner's model

Barles and Soner derived a more complicated model by following the utility function approach of Hodges and Neuberger [7], that was further developed by Davis et al. in [8]. They use an exponential utility function and prove – using the theory of stochastic optimal control [9] that as ϵ and $\kappa$ go to *0*, *V* is the unique (viscosity) solution of (2) where

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + \psi\left(e^{r(T-t)}a^2 S^2 V_{SS}\right)\right) \qquad (6)$$

With $a = \frac{\kappa}{\sqrt{\epsilon}}$ and $\psi(x)$ denotes the solution to the following nonlinear ordinary differential equation

$$\psi'(x) = \frac{\psi(x) + 1}{2\sqrt{x\psi(x)} - x} \quad , x \neq 0 \qquad (7a)$$

with the initial condition

$$\psi(0) = 0 \qquad (7b)$$

The analysis of this ordinary differential equation by Barles and Soner in [8] implies that

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1 \quad and \quad \lim_{x\to-\infty} \psi(x) = -1 \qquad (8)$$

This property allows the treatment of the function $\psi(.)$ as the identity for large arguments and therefore the volatility becomes

$$\tilde{\sigma}^2 = \sigma^2\left(1 + e^{r(T-t)}a^2 S^2 V_{SS}\right) \qquad (9)$$

The ODE (7) is solved with function *ode45* in *Matlab* and the solution was shown in Fig.1

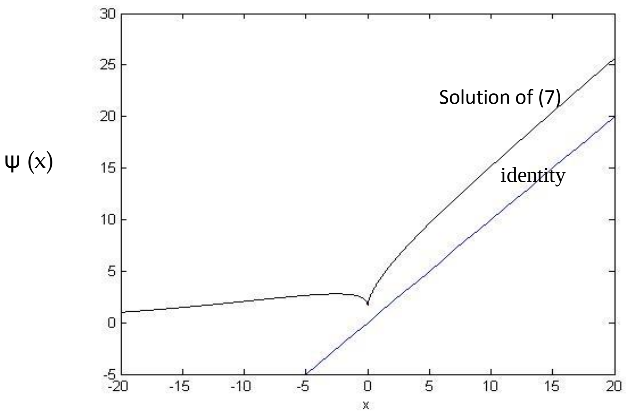



Fig.1 Solution of equ. (7) and the identity equ. (9)

### 2.3 Risk adjusted pricing methodology

This model proposed in [10], the optimal time-lag δ*t* between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected and the modified volatility is now of the form

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + 3\left(\frac{C^2 M}{2\pi} S V_{SS}\right)^{\frac{1}{3}}\right) \qquad (10)$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ the risk premium measure.

Note that these nonlinear models are all consistent with the linear model if the additional parameters for transaction costs vanish $(Le\ , \psi(.), M)$ . We will study these models that is equations (2) and (3) where the volatility is given by the equations (4), (6), (9) and (10) , for both European and American Call options. The European Call option is the solution to (2) on $0 \leq S < \infty\ , 0 \leq t \leq T$ , with the following terminal and boundary conditions:

$$
\begin{aligned}
&V(S, T) = (S - K)^{+} \ \ for\ 0 \leq S < \infty \\
&V(0, t) = 0\ for\ 0 \leq t \leq T \\
&V(S,t) \sim S - Ke^{r(T-t)}
\end{aligned} \tag{11}
$$

For the American Call option the domain is divided into two regions by the free boundary $S_f(t)$, the stopping region $S_f(t) < S < \infty\ , 0 \leq t \leq T$ , where the option is exercised or dead with $V(S, t) = S - K$ and the continuation region $0 \leq S \leq S_f(t)\ ,\ 0 \leq t \leq T$ , where the option stays alive and (2) is valid under the following terminal and boundary conditions ,[13]:

$$
\begin{aligned}
&V(S, T) = (S - K)^{+} && for\ \ 0 \leq S \leq S_f(t) \\
&V(0, t) = 0 && for\ \ 0 \leq t \leq T \\
&V(S_f(t), t) = S_f(t) - K && for\ \ 0 \leq t \leq T \\
&V_S(S_f(t), t) = 1 && for\ \ 0 \leq t \leq T \\
&S_f(T) = max(K\ , rK/q).
\end{aligned} \tag{12}
$$

The general an exact analytical solution leading to a closed expression is not known neither for European nor for American options in a market with transaction costs [3]. The focus of this paper is the numerical solution of the problem. Therefore an accelerated Genetic Algorithm will be specified and used to solved this problem. The different volatility models will be compared to each other.

## 3. Numerical Solutions of Nonlinear Black-Scholes Equation

In this section we solve the nonlinear Black-Scholes model numerically without transformation . We used an accelerated genetic algorithm to solved it.

### 3.1 Genetic Algorithm and grammatical evaluations

Genetic algorithms are simulations of evolution process based on sexual and asexual reproduction, natural selection, mutation, and so on. However ,genetic algorithms are probabilistic optimization methods which are based on the principles of evolution. Grammatical evolution is an evolutionary algorithm that can produce code in any programming language[11 ,12 , 13]. The algorithm requires as inputs the *Backus–Naur Form* (BNF) grammar definition of the target language and evaluate fitness function. We applied the grammar developed in [19] with adding the function (square root (sqrt)) to the set of functions in this grammar to find the numerical solutions of nonlinear Black-Scholes equation with different models of volatility. Further details about grammatical evolution can be found in [14, 15, 16, 17,18].

#### 3.1.1 Technique of the algorithm

The algorithm has the following steps :

### 3.1.2 Initialization

In the initialization phase the values for mutation rate and selection rate are set.

### 3.1.3 Fitness evaluation

We express the PDE's in the following form:

$$f\left(x, y, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y), \frac{\partial^2 u}{\partial x^2}(x,y), \frac{\partial^2 u}{\partial y^2}(x,y)\right) = 0 \ , \ x \in [x_0, x_1] \ , \ y \in [y_0, y_1]$$

The associated Dirichlet boundary conditions are expressed as:

$$u(x_0, y) = f_0(y) \ , u(x_1, y) = f_1(y) \ , \ u(x, y_0) = g_0(y) \ , \ u(x, y_1) = g_1(y)$$

The steps for the fitness evaluation of the population are the following:

1. Choose $N^2$ equidistant points in the box $[x_0, x_1] \times [y_0, y_1]$ , $N_x$ equidistant points on the boundary
   at $x = x_0$ and at $x = x_1$ , $N_y$ equidistant points on the boundary at $y = y_0$ and at $y = y_1$
2. For every chromosome $i$

- Construct the corresponding model $M_i(x, y)$, expressed in the grammar described in [19].
- Calculate the quantity

$$E(M_i) = \sum_{j=0}^{N^2} \left(f\left(x_j, y_j, \frac{\partial}{\partial x} M_i(x_j, y_j), \frac{\partial}{\partial y} M_i(x_j, y_j), \frac{\partial^2}{\partial x^2} M_i(x_j, y_j), \frac{\partial^2}{\partial y^2} M_i(x_j, y_j)\right)\right)^2$$

- Calculate an associated penalty $P_i(M_i)$ . The penalty function P depends on the boundary conditions and it has the form:

$$P_1(M_i) = \sum_{j=1}^{N_x} (M_i(x_0, y_j) - f_0(y_j))^2$$
$$P_2(M_i) = \sum_{j=1}^{N_x} (M_i(x_1, y_j) - f_1(y_j))^2$$
$$P_3(M_i) = \sum_{j=1}^{N_y} (M_i(x_j, y_0) - g_0(x_j))^2$$
$$P_4(M_i) = \sum_{j=1}^{N_y} (M_i(x_j, y_1) - g_1(x_j))^2$$

- Calculate the fitness value of the chromosome as:

$$v_i = E(M_i) + P_1(M_i) + P_2(M_i) + P_3(M_i) + P_4(M_i)$$

### 3.1.4 Genetic operators

The genetic operators that are applied to the genetic population are the initialization, the crossover and the mutation as shown in, [ 16 , 19]. The parents are selected via *tournament selection*.

### 3.1.5 Termination control

The genetic operators are applied to the population creating new generations, until a maximum number of generations or the best chromosome in the population has fitness better than a preset threshold.

### 3.2.Technical of the accelerated method

To make the method is faster to arrived the approximation solution of the nonlinear Black-Scholes partial differential equations by the following :

1- insert the boundary conditions of the problem as a part of chromosomes in the our population of the problem, the algorithm gives the best approximation solution in a few generations.
2- insert a part of exact solution of the linear model of Black-Scholes as a part of a chromosome in the population.
3- insert the vector of numerical solution of the linear model of Black-Scholes where obtained by the above algorithm in [19 ] as a chromosome in the our population of the problem.

### 3.3. Applications of the Algorithm

We applied the our method to find numerical solution for nonlinear Black-Scholes partial differential equation (2) with different volatility models (4) , (6) , (9) , (10) for European call option. We used 20% for the replication rate (hence the crossover probability is set to 80%) and 2% for the mutation rate. We investigated the importance of these two parameters by performing experiments using sets of different values. Each experiment was performed 30 times . As one can see the performance is somewhat dependent on these parameters, but not critically. The population size was set to 200 and the length of each chromosome to 64. The size of the population is a critical parameter. Too small a size weakens the method's effectiveness. Too big a size renders the method slow. Hence since there is no first principals estimation for the the population size, we resorted to an experimental approach to obtain a realistic determination of its range. It turns out that values in the interval [0 , 250] are proper. We used fixed length chromosomes instead of variable - length to avoid creation of unnecessary large chromosomes which will render the method inefficient. The length of the chromosomes is usually depended on the problem to be solved. The maximum number of generations allowed was set to 100 and the preset fitness target for the termination criteria was 0.01. From the conducted experiments, we have observed that the maximum number of generations allowed must be greater for difficult problems . The value of N was set *64*. depending on the problem. In all experiments we use *Mat lab R2010a* ,And use the function (*randi*) to generate the random integers with normal distribution where used to generation the population .

#### 3.3.1 European Call Option

Appling accelerated genetic algorithm method to find approximation solution for nonlinear Black-Schole equation (2) with terminal and boundary conditions (11) and volatilities models (4), (6) , (9) , (10) . We found the following approximations solutions as the best approximations after 30 runs for each model. For all calculations we used the following parameters:
$r = 0.1$ , $\sigma = 0.2$ , $K = 100$ , $T = 1$ (one year) ,$C = 30$ , $M = 0.01$ , $a = 0.02$ , $\kappa = 0.001$
With volatility model (4) (Leland's model) we found the best numerical solution at generation 20 :

$$Gp20(S,t) = S - 10\sqrt{S}/e^{\frac{2t}{9}} \qquad (13)$$

With volatility model (6) (Barle's and Soner's) we found the best numerical solution at generation 27 :

$$Gp27(S,t) = \frac{SlogS}{10} + 5S\left(\frac{0.4t}{16}\right) - 440\left(\frac{1}{\sqrt{S}}e^{-\frac{0.05t}{40}}\right) + 6t \qquad (14)$$

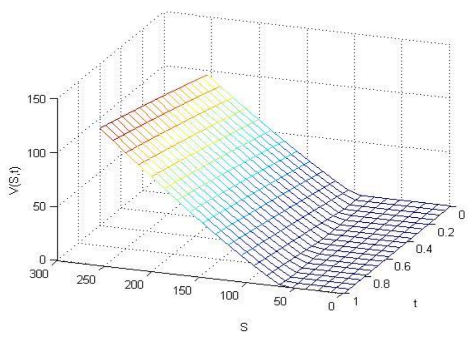
With volatility model (9) (identity model) we found the best numerical solution at generation 15 :

$$Gp15(S,t) = S - 5(4e^{-\frac{2t}{11}}logS + 1) \qquad (15)$$

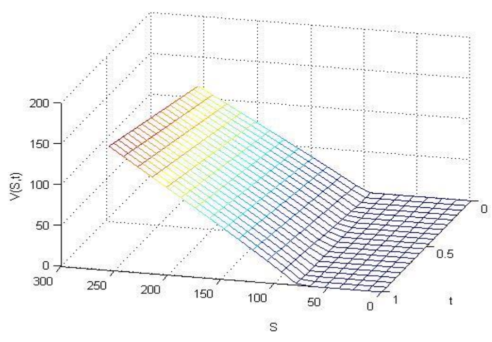
With volatility model (10) (RAPM model) we found the best numerical solution at generation 10 :

$$Gp10(S,t) = S - e^{\frac{-S}{75}} - 100e^{-0.4t} - t \qquad (16)$$

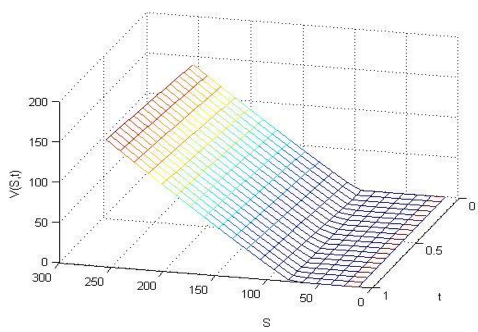
These solutions are shown in Fig.2 blow .



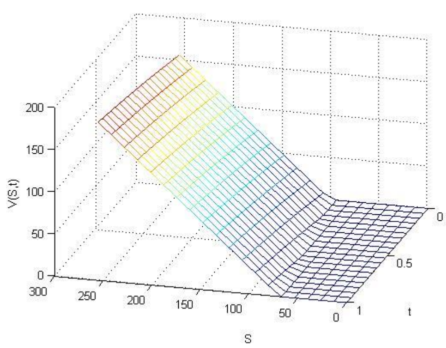

(a) Leland model eq.(13)




(b) Barle's and Soner's model eq. (14)




© identity model eq. (15)




(d) RAPM model eq. (16)

Fig.2 numerical solutions of nonlinear models of Black-Scholes equation

## 4. Comparison the study

In this section we compare the different approximation solutions of the nonlinear Black-Scholes equation (2) with transaction cost models to the model without transaction costs (Linear Black-Scholes equation (1)) and to each other to find the effect of transaction costs modeled by the volatilities (4), (6), (9) and (10) . Because there is no exact solution of equation (2),[3] . In the following sub sections we plot the difference $max(V_{nonlinear}(S, t) - V_{linear}(S, t),0)$ between the price of the European Call option with transaction costs and the price of the European Call without transaction costs. And the results were compared with figures of [3] .As expected the numerical results indicate an economically significant price deviation between the standard (linear) Black–Scholes model and the nonlinear models.

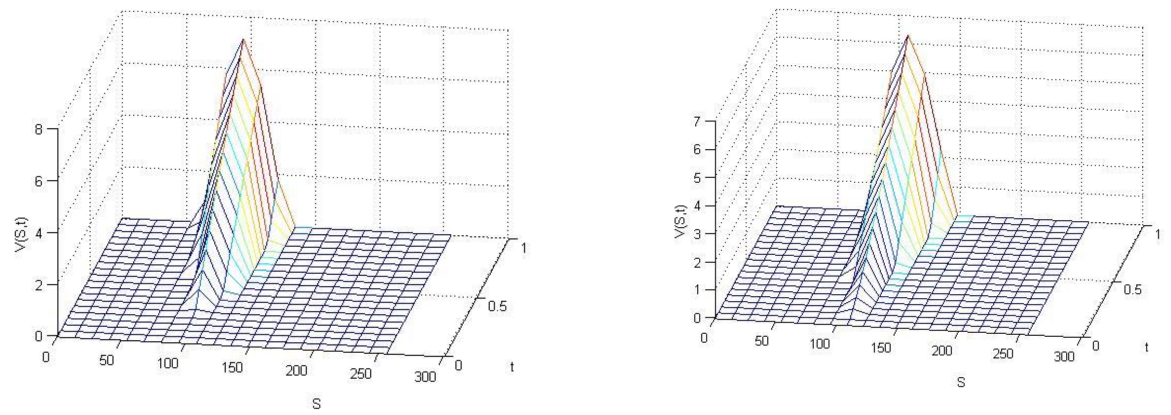
4.1 **Comparison with the exact solution of linear model (1)**

We show the difference between these solutions in previous section and the exact solution of the linear Black-Scholes equation as shown in Fig.3.where the exact solution for European call potion is:

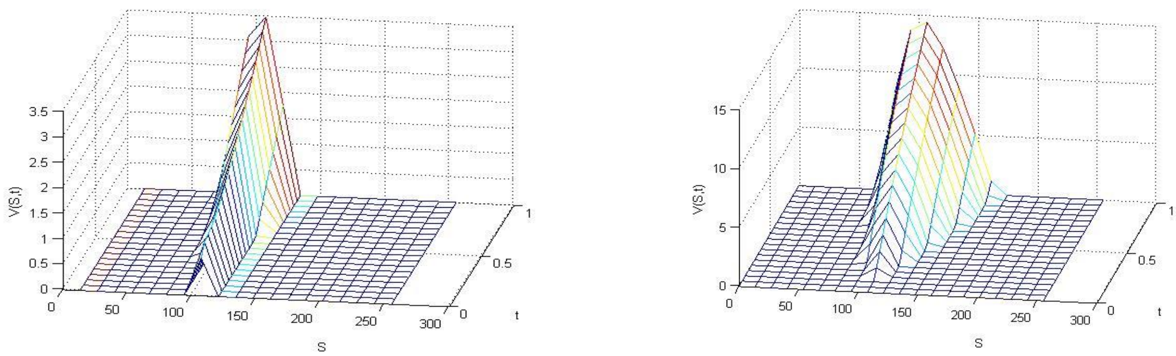
$$V_c(S,t) = S\emptyset(d1) - Ke^{-r(T-t)}\emptyset(d2)$$

Where

$$d1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right).(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d2 = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right).(T-t)}{\sigma\sqrt{T-t}} = d1 - \sigma\sqrt{T-t}$$




(a) the identity vs. Exact of linear. (b) Leland's model vs. Exact of linear.




© Barle's and Soner's model vs. Exact of linear. (d) RAPM model vs. Exact of linear.

Fig.3 The effect of transaction costs (nonlinear model vs linear).

Table.1 shows the comparison of the computation between the different solutions of model (2)
with vitalities at (4) , (6) , (9) and (10) with the exact solution $V_c(S,t)$ of linear model (1) for European call option.

***Table.1 Comparison of the solutions of nonlinear models vs. exact solution of linear***

| *S* | *t* | *Max(Gp20(S,t)-V<sub></sub>c(S,t),0)* | *Max(Gp27(S,t)-Vc(S,t),0)* | *Max(Gp15(S,t)-Vc(S,t),0)* | *Max(Gp10(S,t)-Vc(S,t),0)* |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 13.42105 | 0.9473 | 0 | 0 | 0 | 0 |
| 26.84210 | 0.8947 | 0 | 0 | 0 | 0 |
| 40.26315 | 0.8421 | 0 | 0 | 0 | 0 |
| 53.68421 | 0.7894 | 0 | 0 | 0 | 0 |
| 67.10526 | 0.7368 | 0 | 0 | 0 | 0 |
| 80.52631 | 0.6842 | 1.9979 | 0 | 0 | 1.5589 |
| 93.94736 | 0.6315 | 3.7304 | 1.9437 | 1.3680 | 7.5771 |
| 107.3684 | 0.5789 | 1.5764 | 2.8688 | 2.0368 | 8.1355 |
| 120.7894 | 0.5263 | 0 | 0.15508 | 0 | 4.9544 |
| 134.2105 | 0.4736 | 0 | 0 | 0 | 0.2994 |
| 147.6315 | 0.4210 | 0 | 0 | 0 | 0 |
| 161.0526 | 0.3684 | 0 | 0 | 0 | 0 |
| 174.4736 | 0.3157 | 0 | 0 | 0 | 0 |
| 187.8947 | 0.2631 | 0 | 0 | 0 | 0 |
| 201.3157 | 0.2105 | 0 | 0 | 0 | 0 |
| 214.73684 | 0.1578 | 0 | 0 | 0 | 0 |
| 228.1578 | 0.1052 | 0 | 0 | 0 | 0 |
| 241.5789 | 0.0526 | 0 | 0 | 0 | 0 |
| 255 | 0 | 0 | 0 | 0 | 0 |

4.2 **Comparison with the numerical solution of linear model (1)**

We show the difference between these solutions of nonlinear models in previous section and the following numerical solutions of the linear Black-Scholes equation obtained in our previous paper [19 ].

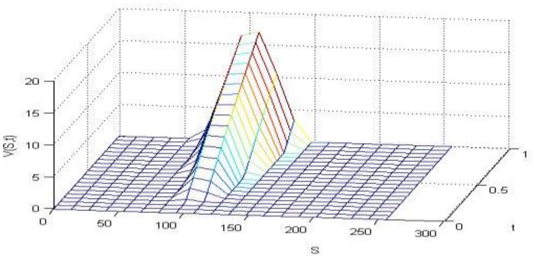
1- $V(S,t) = S.\sin(\ln\left(\frac{S}{K}\right).e^{-\frac{1}{2}\sigma^2(T-t)}$ (17)

2- $Gp4(S,t) = 0.5S(\ln\left(\frac{S}{K}\right) + \ln\left(\frac{S}{K}\right)e^{-\frac{1}{2}\sigma^2(T-t)})$ (18)

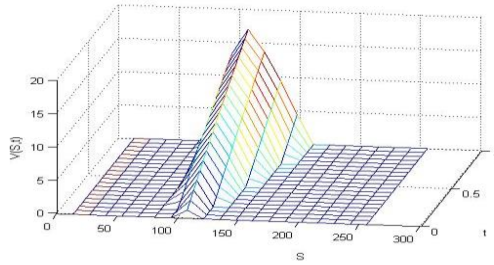
3- $Gp5(S,t) = V(S,t) = S - 100e^{-\frac{1}{9}(T-t)}$ (19)

**4.2.1 Comparison of solutions in equations (13) , (14) , (15) , (16) with solution in equation (17).**

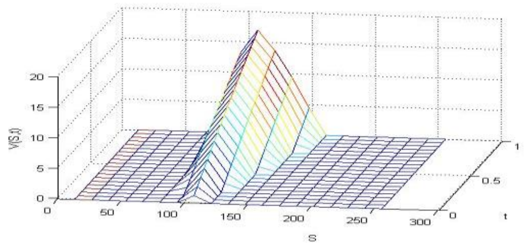
Fig.4 show the difference between these solutions.



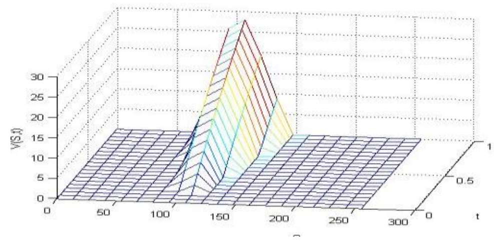

(a) the identity vs. linear model.




(b) Leland's model vs. linear model.




© Barle's and Soner's model vs. linear model




d) RAPM model vs. linear model.

Fig.4 The effect of transaction costs (nonlinear models vs. linear (17)).

(

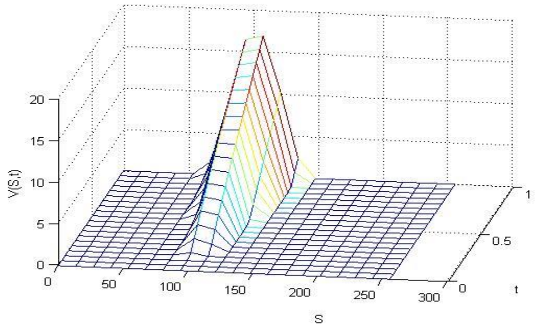
for European call option.

***Table.2 Comparison of the solutions of nonlinear models vs. numerical solution V(S,t) of linear***

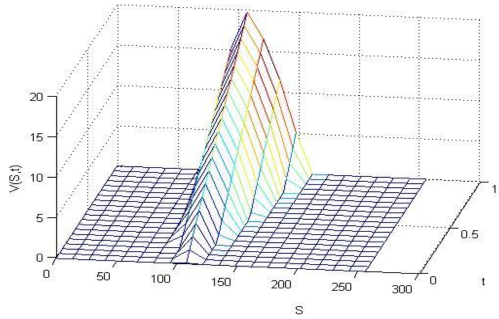
| S | t | Max(Gp20(S,t)-V(S,t),0) | Max(Gp27(S,t)-V(S,t),0) | Max(Gp15(S,t)-V(S,t),0) | Max(Gp10(S,t)-V(S,t),0) |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 13.42105 | 0.9473 | 0 | 0 | 0 | 0 |
| 26.84210 | 0.8947 | 0 | 0 | 0 | 0 |
| 40.26315 | 0.8421 | 0 | 0 | 0 | 0 |
| 53.68421 | 0.7894 | 0 | 0 | 0 | 0 |
| 67.10526 | 0.7368 | 0 | 0 | 0 | 0 |
| 80.52631 | 0.6842 | 3.4473 | 0 | 0 | 3.4431 |
| 93.94736 | 0.6315 | 9.7131 | 8.5247 | 7.9490 | 15.354 |
| 107.3684 | 0.5789 | 8.6359 | 11.3966 | 10.5646 | 15.788 |
| 120.7894 | 0.5263 | 1.9954 | 7.9823 | 7.6375 | 7.5164 |
| 134.2105 | 0.4736 | 0 | 4.1673 | 4.8895 | 0 |
| 147.6315 | 0.4210 | 0 | 0.0237 | 2.2825 | 0 |
| 161.0526 | 0.3684 | 0 | 0 | 0 | 0 |
| 174.4736 | 0.3157 | 0 | 0 | 0 | 0 |
| 187.8947 | 0.2631 | 0 | 0 | 0 | 0 |
| 201.3157 | 0.2105 | 0 | 0 | 0 | 0 |
| 214.73684 | 0.1578 | 0 | 0 | 0 | 0 |
| 228.1578 | 0.1052 | 0 | 0 | 0 | 0 |
| 241.5789 | 0.0526 | 0 | 0 | 0 | 0 |
| 255 | 0 | 0 | 0 | 0 | 0 |

### 4.2.2 Comparison of solutions in equations (13) , (14) , (15) , (16) with solution in equation (18).

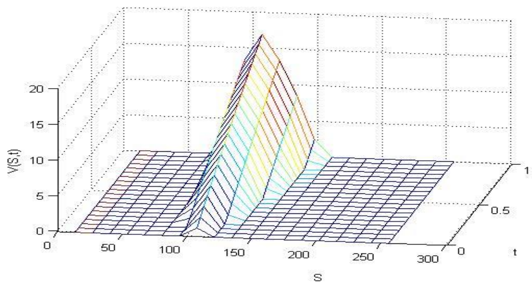
Fig.5 show the difference between these solutions.



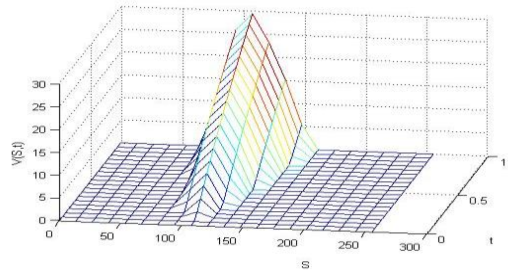

(a) the identity vs. linear model.




(b) Leland's model vs. linear model.




© Barle's and Soner's model vs. linear model




d) RAPM model vs. linear model.

Fig.5 The effect of transaction costs ( nonlinear model vs. linear (18)).

T (2) with vitalities (4) , (6) , (9) and (10) with the numerical solution (18) of linear model (1) for European call option.

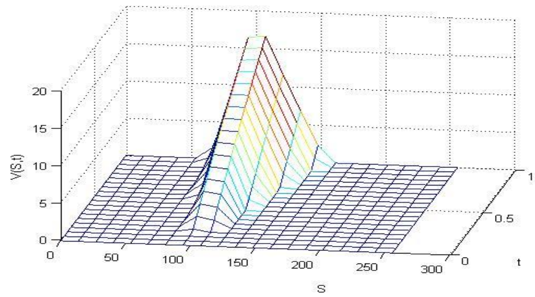
***Table.3 Comparison of the solutions of nonlinear models vs. numerical solution Gp4(S,t) of linear***

| *S* | *t* | *Max(Gp20(S,t)-Gp4(S,t),0)* | *Max(Gp27(S,t)-Gp4(S,t),0)* | *Max(Gp15(S,t)-Gp4(S,t),0)* | *Max(Gp10(S,t)-Gp4(S,t),0)* |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 13.42105 | 0.9473 | 0 | 0 | 0 | 0 |
| 26.84210 | 0.8947 | 0 | 0 | 0 | 0 |
| 40.26315 | 0.8421 | 0 | 0 | 0 | 0 |
| 53.68421 | 0.7894 | 0 | 0 | 0 | 0 |
| 67.10526 | 0.7368 | 0 | 0 | 0 | 0 |
| 80.52631 | 0.6842 | 3.4473 | 0 | 0 | 3.4431 |
| 93.94736 | 0.6315 | 9.7131 | 8.5247 | 7.9490 | 15.3546 |
| 107.3684 | 0.5789 | 8.6693 | 11.4299 | 10.5979 | 17.3561 |
| 120.7894 | 0.5263 | 0.3211 | 6.30798 | 5.9632 | 9.5439 |
| 134.2105 | 0.4736 | 0 | 0 | 0.0044 | 0 |
| 147.6315 | 0.4210 | 0 | 0 | 0 | 0 |
| 161.0526 | 0.3684 | 0 | 0 | 0 | 0 |
| 174.4736 | 0.3157 | 0 | 0 | 0 | 0 |
| 187.8947 | 0.2631 | 0 | 0 | 0 | 0 |
| 201.3157 | 0.2105 | 0 | 0 | 0 | 0 |
| 214.73684 | 0.1578 | 0 | 0 | 0 | 0 |
| 228.1578 | 0.1052 | 0 | 0 | 0 | 0 |
| 241.5789 | 0.0526 | 0 | 0 | 0 | 0 |
| 255 | 0 | 0 | 0 | 0 | 0 |

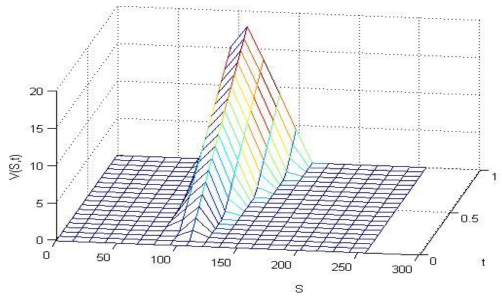
 

### 4.2.1 Comparison of solutions in equations (13) , (14) , (15) , (16) with solution in equation (19).

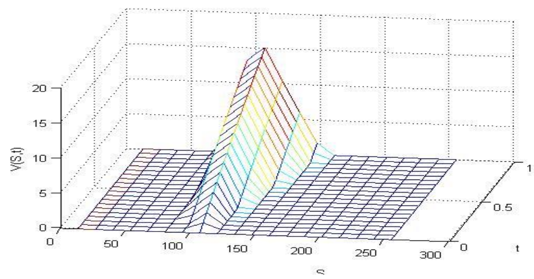
Fig.6 show the difference between these solutions.



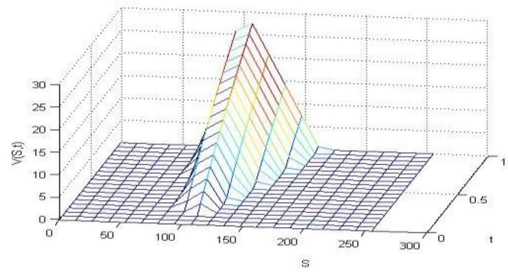

(a) the identity vs. linear model.




(b) Leland's model vs. linear model.




© Barle's and Soner's model vs. linear model




d) RAPM model vs. linear model.

Fig.6 The effect of transaction costs ( nonlinear model vs. linear (19)).

Table.4 shows the comparison of the computation between the different solutions of model (2) with vitalities (4) , (6) , (9) and (10) with the numerical solution (19) of linear model (1) for European call option.

***Table.4 Comparison of the solutions of nonlinear models vs. numerical solution Gp5(S,t) of linear***

| *S* | *t* | *Max(Gp20(S,t)-Gp5(S,t),0)* | *Max(Gp27(S,t)-Gp5(S,t),0)* | *Max(Gp15(S,t)-Gp5(S,t),0)* | *Max(Gp10(S,t)-Gp5(S,t),0)* |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 13.42105 | 0.9473 | 0 | 0 | 0 | 0 |
| 26.84210 | 0.8947 | 0 | 0 | 0 | 0 |
| 40.26315 | 0.8421 | 0 | 0 | 0 | 0 |
| 53.68421 | 0.7894 | 0 | 0 | 0 | 0 |
| 67.10526 | 0.7368 | 0 | 0 | 0 | 0 |
| 80.52631 | 0.6842 | 3.4473 | 0 | 0 | 3.4431 |
| 93.94736 | 0.6315 | 9.7131 | 8.5247 | 2.5691 | 15.354 |
| 107.3684 | 0.5789 | 8.6359 | 9.8720 | 5.4531 | 15.7882 |
| 120.7894 | 0.5263 | 1.9954 | 3.7781 | 2.8409 | 7.51642 |
| 134.2105 | 0.4736 | 0 | 0 | 0.4473 | 0 |
| 147.6315 | 0.4210 | 0 | 0 | 0 | 0 |
| 161.0526 | 0.3684 | 0 | 0 | 0 | 0 |
| 174.4736 | 0.3157 | 0 | 0 | 0 | 0 |
| 187.8947 | 0.2631 | 0 | 0 | 0 | 0 |
| 201.3157 | 0.2105 | 0 | 0 | 0 | 0 |
| 214.73684 | 0.1578 | 0 | 0 | 0 | 0 |
| 228.1578 | 0.1052 | 0 | 0 | 0 | 0 |
| 241.5789 | 0.0526 | 0 | 0 | 0 | 0 |
| 255 | 0 | 0 | 0 | 0 | 0 |

## 5. Conclusion

In this paper we found and studied properties of numerical solutions to nonlinear Black–Scholes equation arising in derivative asset analysis in markets with stochastic volatility. Using accelerated genetic algorithm method. We find this method was applicable to determined the numerical solutions of the nonlinear model of Black-Scholes with different models of volatility, and we comparison the results for these solutions with the result for the solution of the linear model of Black-Schole obtained in [ 19], and with the solution of nonlinear model obtained in [ 3 , 20 ]. We find our method was excellent for the computation.

## References

[1] Seydel, R. (2004) .Tools for Computational Finance. 2nd edition, Springer, Berlin.
[2] Black, F. , Scholes, M. (1973) .The pricing of options and corporate liabilities. J. Polit. Econ.
81 (3) 637–654.
[3] Ankudinova, J., Ehrhardt ,M. (2008) .On the numerical solution of nonlinear Black–Scholes equations.
Computers and Mathematics with Applications 56, 799–812
[3] Merton, R.C. (1973) .Theory of rational option pricing. Bell J. Econ. 4 (1) ,141–183.
[4] Tavella, D. , Randall,C. (2000) . Pricing Financial Instruments -The Finite Difference Method. John Wiley & Sons, Inc., New York.
[5] Leland, H.E. (1985) .Option pricing and replication with transactions costs. J. Finance 40 ,1283–1301.
[6] Barles , G. , Soner ,H.M. (1998) .Option pricing with transaction costs and a nonlinear Black–Scholes equation. Finance Stoch. 2 ,369–397.
[7] Hodges , S., Neuberger, A. (1989) .Optimal replication of contingent claims under transaction costs. Rev. Futures Markets 8 , 222–239.
[8] Davis ,M.H.A., Panas, V.G., Zariphopoulou,T. (1993) .European option pricing with transaction costs. SIAM J. Control. Optim. 31 (2) ,470–493.
[9] Fleming, W.H., Soner, H.M. (1993) .Controlled Markov Processes and Viscosity Solutions. Springer, New York.

.

[10] Jandacka, M., Sevcovic, D. (2005) .On the risk-adjusted pricing-methodology-based valuation of
vanilla options and explanation of the volatility smile. J. Appl. Math. (3), 235–258.

[ 11] Tsoulos, G.I. (2003) .Solving differential equations with genetic programming. P.O. Box 1186,
Ioannina 45110,

[12] Hussain, E.A. , Alrajhi ,Y.M. (2014) .Solution of partial differential equations using accelerated genetic algorithm. Int. J. of Mathematics and Statistics Studies,Vol.2, No.1, pp.55-69.

[13] Naur, P.( 1963) .Revised report on the algorithmic language ALGOL,.

[14] O'Neill, M. , Ryan,C. (1999) .Under the hood of grammatical evolution. Kluwer Academic Publishers.

[15] O'Neill, M. , Ryan, C. (2003) .Grammatical Evolution: Evolutionary Automatic Programming in a Arbitrary Language. Kluwer Academic Publishers,.

[16] O'Neill, M. , Ryan, C.( 2001) .Grammatical Evolution. IEEE Trans. Evolutionary Computation, Vol.
5, pp. 349-358.

[17] Sydow,A. (2009) .GMD-FIRST Genetic Algorithms and Genetic Programming Modern Concepts and
Practical Applications. Berlin, Germany.

[18] Goldberg,D.E.( 1989) .Genetic algorithms in search, Optimization and Machine Learning. Addison
Wesley. .

[19] Hussain, E.A. , Alrajhi ,Y.M. (2014). Accelerated Genetic Algorithm Solution of Linear Black –
Scholes Equation. (IJMCAR) ,Vol. 4, Issue 3, 33-46

[20] Ankudinova,J. (2008) .The numerical solution of nonlinear Black–Scholes equations. Master’s Thesis,
Technische Universitat Berlin.