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Numerical method for pricing governing American options under fractional Black-Scholes model

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Abstract

In this paper we develop a numerical approach to a fractional-order differential linear complementarity problem (LCP) arising in pricing European and American options under a geometric Lévy process. The (LCP) is first approximated by a penalized nonlinear fractional Black-Scholes (fBS) equation. To numerically solve this nonlinear (fBS), we use the horizontal method of lines to discretize the temporal variable and the spatial variable by means of Crank-Nicolson method and a cubic spline collocation method, respectively. This method exhibits a second order of convergence in space, in time and also has an acceptable speed in comparison with some existing methods. We will compare our results with some recently proposed approaches.

Keywords: Geometric Lévy process, fractional Black-Scholes, Crank-Nicolson scheme, Spline collocation, Free Boundary Value Problem.

Introduction

Over the last few years, the financial markets are regarded as complex and nonlinear dynamic systems. A series of studies have found that many financial market time series display scaling laws and long-range dependence which imply that there exists arbitrage in financial markets.

In classical finance theory, absence of arbitrage is one of the most unifying concepts. However, behavioral finance and econophysics as well as empirical studies sometime propose models for asset price that are not consistent with this basic assumption. A case is the (fBS) model, which displays the long-range dependence observed in empirical data [1, 2]. The (fBS) model is a generalization of the Black-Scholes model, which is based on replacing the standard Brownian motion by a fractional Brownian motion in the Black-Scholes model. Since fractional Brownian motion (fBm) is not a semi-martingale [3], it has been shown that the (fBS) model admits arbitrage in a complete and frictionless market [4, 5].

As a generalization of the integer order differential equation, fractional differential equation is used to model important phenomena in various fields such as fluid flow, electromagnetic, acoustics, electrochemistry, cosmology, and material science. Recently, fractional partial differential equation was introduced more and more into financial theory. There are several alternative solution methods of the (fBS) problem. Wyss [6] gave the (fBS) equation with a time-fractional derivative to price European call option. Cartea and del-Castillo-Negrete [7] gave several fractional diffusion models of option prices in markets with jumps and priced barrier option using fractional partial differential equation. Jumarie [8-9] derived the time and space (fBS) equations and gave optimal fractional Merton's portfolio. [10] obtain a European call option pricing formula with transaction costs for the (fBS) model. [11] obtain the explicit option pricing of a bi-fractional Black-Merton-Scholes model with the Hurst exponent H in [1/2,1]. [12] proposed and analyzed a power penalty method for the numerical solution of the (fBS) equation governing American option pricing. In this paper we develop a numerical method for pricing governing American options under (fBS) model by using the cubic spline collocation method and the generalized Newton method. First, (fBS) model can be formulated as parabolic partial differential complementarity (PDC) problem with the boundary and payoff conditions, the (PDC) problem is approximated by a sequence of nonlinear equation problems by using the penalty method given in [12]. Then we apply the spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the cubic spline coefficients of a nonsmooth system $\varphi(Y)=Y$, where $\varphi:\mathbb{R}^m\to\mathbb{R}^m$. In order to solve the nonsmooth equation we apply the generalized Newton method (see [13, 14, 15], for instance). We prove that the cubic spline collocation method converges quadratically provided that a property coupling the penalty parameter λ and the discretization parameter h is satisfied.

The paper is organized as follows. In Section 2, we describe briefly the problem for American options in a (fBS) model. In Section 3, we present the penalty method to approximate the (fBS) problem by a sequence of second order boundary value problems. Then, we discuss time semi-discretization in Section 4. In Section 5, we will first construct a cubic spline to approximate the solution of the boundary problem and devote to the presentation of the generalized Newton method. We will then show the convergence of the cubic spline to the solution of the boundary problem and provide an error estimate. In order to validate the theoretical results presented in this paper, we present numerical tests on two known examples in Section 6. The obtained numerical results are compared to the ones given in [12]. Finally, a conclusion is given in Section 7.

2. Geometric Lévy processes

It is proposed in [16] that the underlying stock price S_t of an option follows the following geometric Lévy process:

$$d(\ln S_t) = (r-v)dt + dL_t$$

with the solution

$$S_T = S_t \exp((r-v)(T-t) + \int_t^T dL_u),$$

Where T is a future known date, r is the risk-free rate, v is convexity adjustment so that the expectation of S_T becomes $E[S_T] = \exp(r(T-t))S_t$ and dL_t is the increment of a lévy process under the equivalent martingale measure (EMM). Boyarchenko and levendorskii [17] proposed the use of a modified lévy-stable (LS) (lévy-α -stable) process to model the dynamics of securities. This modification introduces a damping effect in the tails of the LS distribution, which is known as KoBoL process. Carr, Geman, Madan and Yor [18] proposed a process, known as the CGMY process, including both positive and negative jumps. In this paper, we are concerned with options based on finite moment log-stable (FMLS) processes proposed in [21].

A time-dependent random variable X_t is a lévy process, if and only if it has independent and stationary increments with the following log-characteristic function in lévy-Khintchine representation

$$\ln E[e^{i\xi X_t}] = t\psi(\xi) = mit\xi - \frac{1}{2}\sigma^2 t\xi^2 + t\int_{IR^*}(e^{i\xi x} - 1 - i\xi h(x))W(dx),$$

where $i = \sqrt{-1},\ m \in IR$ is the drift rate, $\sigma \geq 0$ is the (constant) volatility, $h(x)$ is a truncation function, W is the lévy measure satisfying

$$\int_{IR} \min(1, x^2)W(dx) < \infty,$$

and $\psi(\xi)$ is the characteristic exponent of the lévy process which is a combination of a drift component, a Brownian motion component and a jump component. These three components are determined by the lévy -Khintchine triplet (m, σ, W). In lévy's process, $W(dx) = w_{LS}(x)$, where $w_{LS}(x)$ is the lévy density given by

$$w_{LS}(x) = \begin{cases} Dq|x|^{-1-\alpha} & \forall x < 0, \\ Dq\ x^{-1-\alpha} & \forall x > 0, \end{cases}$$

For a constant $\alpha \in (0,2]$, where $\alpha > 0,\ p, q \in [-1,1]$, satisfying $p + q = 1$. The characteristic exponent of the LS process is

$$\psi_{LS}(\xi) = -\frac{\sigma^\alpha}{4\cos(\alpha\pi/2)}[(1-s)(i\xi)^\alpha + (1+s)(-i\xi)^\alpha] + im\xi,$$

Where α and σ are respectively the stability index and scaling parameter, $s = p - q$ is the skewness parameter satisfying $-1 \leq s \leq 1$, and m is a location parameter. When $s = 1$ (resp. $s = -1$) the random variable X is maximally skewed to the left (resp.right).When $\alpha = 2$ and $s = 0$, it becomes the Gaussian case. A particular characteristic of the FMLS process is that it only exhibits downwards jumps, while upwards movements have continuous paths. The characteristic exponent of the LS process with $s = -1$, is

$$\psi_{FMLS}(\xi) = \frac{1}{2}\sigma^{\alpha}\sec(\frac{\alpha\pi}{2})(-i\xi)^{\alpha},$$

where $v = 0.5\sigma^{\alpha}\sec(\alpha\pi/2)$ is the convexity adjustment of the random walk.

In [18], the authors derived a fractional Black-Scholes (fBS) equation for the European option valuation based on the FMLS process and a Fourier transform. Let $V(x,t)$ be the value of a European option whose underlying stock price satisfies (1) and $\hat{V}(x,t)$ is the Fourier transform of $V(x,t)$. It has been shown in [18] that $\hat{V}(x,t)$ satisfies

$$\frac{\partial \hat{V}(\xi,t)}{\partial t} = [r + i\xi(r-v) - \psi(-\xi)]\hat{V}(\xi,t), \tag{1}$$

Different choices of dL_t and the convexity adjustment v will result in different fPDEs from (1). The authors in [18] also derived the fPDEs under CGMY and KoBoL processes, which are both useful damped Lévy process. Because the expected value of the stock price diverges when the distribution of the random variable X_t exhibits algebraic tails, the power-law truncation does not suitable for derivative pricing, whereas for FMLS it is not an issue. Therefore, we assume the risk-neutral asset price S_t follows the FMLS process in this paper.

It has been shown in [16] that under the transformation $x = \ln S_t$, V the Fourier inverse transform of $\hat{V}$ in (1), satisfies the following fBS equation:

$$\begin{cases} \mathcal{L}_{fbs}V &= 0, \quad (x,t)\in\Omega, \\ V(x_{min},t) &= V_0(t), \quad t\in[0,T], \\ V(x_{max},t) &= V_1(x), \quad t\in[0,T], \\ V(x,T) &= V^*(x), \quad x\in\Omega_x, \end{cases} \tag{2}$$

where V_0, V_1, and V^* are given functions satisfying the compatibility conditions $V_0(T) = V^*(x_{\min})$ and $V_1(T) = V^*(x_{\max})$. ${}_{x_{\min}}D_x^{\alpha}V$ denotes the $\alpha-$th derivative of V with $\alpha\in(0,2)$ a constant, $x_{\min} << 0$ and $x_{\max} > 0$ are two constants representing the lower and upper bounds for x, and

$$a = -r - \frac{1}{2}\sigma^{\alpha}\sec(\frac{\alpha\pi}{2}), \qquad b = -\frac{1}{2}\sigma^{\alpha}\sec(\frac{\alpha\pi}{2}).$$

$$\mathcal{L}_{fbs}V = -\frac{\partial V}{\partial t} + a\partial_x V - b[{}_{x_{\min}}D_x^{\alpha}V] + rV,$$

For brevity, we will only consider, in the rest of this paper, Vanilla put options, which has the payoff function $V^*(x) = \max(K - e^x, 0)$ where K is the strike price of the option satisfying

$$0 < K < e^{x_{\max}} - e^{x_{\min}}.$$

Note that the original fBS equation derived in [18] is defined on the infinite domain $(-\infty,\infty)$. However, we truncate $(-\infty,\infty)$ into a finite one in (2) for the purpose of computation. There are various representations of the fractional derivative ${}_{x_{\min}}D_x^{\alpha}W(x)$ such as those of Riemann-Liouville (RL) and Grüwald-Letnikov (GL) [20]. One representation is

$${}_{x_{\min}}D_x^{\alpha}W(x) = \frac{W(x_{\min})}{\Gamma(1-\alpha)(x-x_{\min})^{\alpha}} + \frac{W'(x_{\min})}{\Gamma(2-\alpha)(x-x_{\min})^{\alpha-1}} + \frac{1}{\Gamma(2-\alpha)}\int_{x_{\min}}^{x}\frac{W''(\xi)}{(x-\xi)^{\alpha}}d\xi,$$

for $x > x_{\min}$ and $\alpha\in(0,2)$ where $\Gamma(.)$ denotes the Gamma function. Clearly, this representation is singular at $x_{\min}$ unless $W(x_{\min}) = 0 = W'(x_{\min})$. It turns out that both of these conditions are satisfied (up to a truncation error) under a transformation when $x_{\min} << 0$. We will leave this discussion in the next section after we introduce the American option pricing model.

Unlike a European option whose value is determined by (2), the value of an American option is governed by a linear complementarity problem involving the fBS operator $\mathsf{L}_{\mathsf{fbs}}$ and a constraint on the value of the option. Since the closed form of the solutions can rarely be found in practice, numerical approximations to this complementarity problem are usually sought, which in turn needs efficient and accurate numerical methods.

3. Penalty problem

As mentioned before, it is known that the value V of an American option satisfies the following linear complementarity problem [21, 22]:

$$\begin{cases} \mathsf{L}_{\mathsf{fbs}} V \geq 0 \\ V(x,t) - V^*(x) \geq 0 \\ \mathsf{L}_{\mathsf{fbs}} V(x,t).(V(x,t) - V^*(x)) = 0 \end{cases}$$

for $(x,t) \in \Omega_x \times [0,T)$ with the boundary and payoff conditions

$$V(x_{\min},t) = K,\ V(x_{max},t) = 0,\ V(x,T) = V^*(x),$$

Introducing a transformation $u(x,t) = (1 - \frac{e^x - e^{x_{\min}}}{e^{x_{\max}} - e^{x_{\min}}})K - V(x,t),$ the option pricing problem can be formulated as the following parabolic partial differential complementarity problem ([12]):

$$\begin{cases} \mathsf{L}_{\mathsf{fbs}} u(x,t) \leq f(x) \\ u(x,t) - u^*(x) \leq 0 \\ (\mathsf{L}_{\mathsf{fbs}} u(x,t) - f(x)).(u(x,t) - u^*(x)) = 0 \end{cases} \quad (3)$$

satisfying the boundary and payoff conditions

$$\begin{cases} u(x_{\min},t) = 0 = u(x_{\max},t), & x \in [0,T), \\ u(x,T) = u^*(x), & x \in \Omega_x, \end{cases}$$

$$f(x) = -\frac{(a-b)K}{e^{x_{\max}} - e^{x_{\min}}} e^x + rK(1 - \frac{e^x - e^{x_{\min}}}{e^{x_{\max}} - e^{x_{\min}}}),$$

$$u^*(x) = \begin{cases} (1 - \frac{K}{e^{x_{\max}} - e^{x_{\min}}})e^x + K\frac{e^{x_{\min}}}{e^{x_{\max}} - e^{x_{\min}}}, & x_{\min} \leq x \leq \ln K, \\ (1 - \frac{e^x - e^{x_{\min}}}{e^{x_{\max}} - e^{x_{\min}}})K, & \ln K < x \leq e^{x_{\max}}. \end{cases}$$

$${}_{x_{\min}}D_x^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_{x_{\min}}^{x} \frac{u_{xx}(\xi,t)}{(x-\xi)^{\alpha-1}} d\xi,$$

when $x_{\min} << 0$. This is the Caputo's representation of the α -th fractional derivative of u for $\alpha \in (1,2)$.
It is well known that problem (3) admits a unique solution $u \in \{v \in H_0^{\alpha/2}(\Omega_x) : v \leq u^*\}$ (see [12]).

Let λ be a real positive number. The penalty problem is given by the following boundary value problem (see [24, 25]):

$$\begin{cases} \dfrac{\partial u_\lambda(x,t)}{\partial t} - \mathrm{L}u_\lambda(x,t) &= I(t,u_\lambda(x,t)), \quad (x,t)\in\Omega, \\ u_\lambda(x,T) &= u^*(x), \quad x\in\Omega_x, \\ u_\lambda(x_{min},t) &= 0, \quad t\in[0,T], \\ u_\lambda(x_{max},t) &= 0, \quad t\in[0,T], \end{cases} \tag{4}$$

where

$$\Omega=\Omega_x\times[0,T], \text{ and } \Omega_x=[x_{min},x_{max}]$$
$$L=a\partial_x V - b[{}_{x_{\min}}D_x^\alpha V]+rV$$
$$I(t,u_\lambda(x,t))=\lambda[u_\lambda(x,t)-u^*(x)]_+^{1/k}-f(x),$$
$[z]_+=\max\{0,z\}$ for any function z, and $\lambda>1$ and $k>0$ are parameters.

Then problem (4) admits a unique solution $u_\lambda(t)\in H^{\alpha/2}(\Omega_x)$ for $t\in[0,T)$ a.e. (see [12]).
We have the interesting properties.

Theorem (see [12]) Let u and u_λ be the solutions to problem (3) and (4), respectively.

If $\dfrac{\partial u}{\partial t}\in L^{k+1}(\Omega)$, then there exists a constant $c>0$, independent of λ, such that

$$\|u_\lambda-u\|_{L^\infty(0,T;L^2(\Omega_x))}+\|u_\lambda-u\|_{L^2(0,T;H_0^{\alpha/2}(\Omega_x))}\le c\lambda^{-k/2}, \tag{5}$$

where k and λ are the parameters used in (4).

4. Time discretization and description of the Crank-Nicolson scheme

Discretize the time variable by setting $t^m=m\Delta t$ for $m=0,1,...,M$, in which $\Delta t=T/M$ and then define

$$u_\lambda{}^m(x)=u_\lambda(x,t^m), \quad m=0,1,...,M.$$

Now by applying the Crank-Nicolson scheme on (4), we arrive at the following equation

$$\frac{u_\lambda{}^{m+1}-u_\lambda{}^m}{\Delta t}-\frac{1}{2}\mathrm{L}(u_\lambda{}^{m+1}+u_\lambda{}^m)=\frac{1}{2}\Big(I(t_{m+1,}u_\lambda{}^{m+1})+I(t_{m,}u_\lambda{}^m)\Big)$$

One way is to replace $u_\lambda{}^{m+1}$ with $u_\lambda{}^m$ in the linear terms. This leads to the following modified system:

$$(1-\frac{\Delta t}{2}\mathrm{L})u_\lambda{}^{m+1}=(1+\frac{\Delta t}{2}\mathrm{L})u_\lambda{}^m+\frac{\Delta t}{2}\Big(I(t_{m+1,}u_\lambda{}^{m+1})+I(t_{m,}u_\lambda{}^m)\Big) \tag{6}$$

For $m=0,1,...,M$. The final price of the American option at time level m will be of the form:

$$\begin{cases} (1-\frac{\Delta t}{2}\mathsf{L})u_\lambda{}^{m+1} &= J_\lambda{}^m(.,u_\lambda{}^{m+1}), \quad \forall x \in \Omega_x, \\ u_\lambda{}^0(x) &= u^*(x), \quad \forall x \in \Omega_x, \\ u_\lambda{}^{m+1}(x_{min}) &= 0, \quad 0 \le m < M, \\ u_\lambda{}^{m+1}(x_{max}) &= 0, \quad 0 \le m < M. \end{cases} \tag{7}$$

Where, for any $m \ge 0$ and for any $x \in \Omega_x$, we have

$$J_\lambda{}^m(u_\lambda{}^{m+1}) = (1+\frac{\Delta t}{2}\mathsf{L})u_\lambda{}^m + \frac{\Delta t}{2}\left(I(t_{m+1,}u_\lambda{}^{m+1}) + I(t_{m,}u_\lambda{}^m)\right)$$

$u_\lambda{}^{m+1}$ is solution of (7), at the $(m+1)$ th-time level.
The following theorem proves the order of convergence of the solution $u_\lambda{}^m$ to $u_\lambda(x,t)$.

Theorem 3.1([26]) suppose that $\Delta t\left(\left\|\widetilde{L}\right\|_\infty + \lambda\right) < 2.$ Then, problem (7) is second order convergent .i.e.

$$\left\|u_\lambda(x,t_m) - u_\lambda{}^m\right\|_\infty \le C(\Delta t)^2, \tag{8}$$

where $\widetilde{L}$ the linear operator associated with the differential operator L,
For any $m \ge 0$, problem (7) has a unique solution and can be written on the following form:

$$\begin{cases} (1-\frac{\Delta t}{2}\mathsf{L})u_\lambda(x) &= J(.,u_\lambda(x)), \quad \forall x \in \Omega_x, \\ u_\lambda(x_{min}) &= 0, \\ u_\lambda(x_{max}) &= 0, \end{cases} \tag{9}$$

In the sequel of this paper, we will focus on the solution of problem (9).

5. Spatial discretization and cubic spline collocation method

Let $\|.\|$ the Euclidean norm on R^{n+1} and $S^{(k)}$ the k^{th} derivative of a function S.

In this section we construct a cubic spline which approximates the solution u_λ of problem (9), in the interval $\Omega_x \subset \mathsf{R}$.

Let $\Theta = \{x_{min} = x_{-3} = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = x_{n+3} = x_{max}\}$ be a subdivision of the interval Ω_x. Without loss of generality, we put $x_i = a + ih$, where $0 \le i \le n$ and $h = (x_{max} - x_{min})/h.$ Denote by $\mathsf{S}_4(\Omega_x,\Theta) = \mathsf{P}_3^2(\Omega_x,\Theta)$ the space of piecewise polynomials of degree less than or equal to 3 over the subdivision Θ and of class C^2 everywhere on Ω_x. Let B_i, $i = -3,\cdots,n-1$, be the B-splines of degree 3 associated with Θ. These B-splines are positives and form a basis of the space $S_4(\Omega_x,\Theta)$.

Consider the local linear operator Q_3 which maps the function u_λ onto a cubic spline space $\mathsf{S}_4(\Omega_x,\Theta)$ and which has an optimal approximation order. This operator is the discrete C^2 cubic quasi-interpolant (see [15]) defined by

$$Q_3 u_\lambda = \sum_{i=-3}^{n-1} \mu_i(u_\lambda) B_i,$$

where the coefficients $\mu_j(u_\lambda)$ are determined by solving a linear system of equations given by the exactness of Q_3 on the space of cubic polynomial functions $\mathsf{P}_3(\Omega_x)$. Precisely, these coefficients are defined as follows:

$$\begin{cases} \mu_{-3}(u_\lambda) = u_\lambda(x_0) = u_\lambda(x_{min}), \\ \mu_{-2}(u_\lambda) = \dfrac{1}{18}(7u_\lambda(x_0) + 18u_\lambda(x_1) - 9u_\lambda(x_2) + 2u_\lambda(x_3)), \\ \mu_j(u_\lambda) = \dfrac{1}{6}(-u_\lambda(x_{j+1}) + 8u_\lambda(x_{j+2}) - u_\lambda(x_{j+3})), \text{for } j = -1,\dots,n-3, \\ \mu_{n-2}(u_\lambda) = \dfrac{1}{18}(2u_\lambda(x_{n-3}) - 9u_\lambda(x_{n-2}) + 18u_\lambda(x_{n-1}) + 7u_\lambda(x_n)), \\ \mu_{n-1}(u_\lambda) = u_\lambda(x_n) = u_\lambda(x_{max}). \end{cases}$$

It is well known (see e.g. [16], chapter 5) that there exist constants k_i, $i = 0,1,2,3,$ such that, for any function $u_\lambda \in C^4(\Omega_x)$,

$$\left\| u_\lambda{}^{(i)} - Q_3 u_\lambda{}^{(i)} \right\|_{\Omega_x} \le k_i h^{4-i} \left\| u_\lambda{}^{(4-i)} \right\|_{\Omega_x}, \quad i = 0,1,2,3, \tag{10}$$

By using the boundary conditions of problem (9), we obtain

$$\mu_{-3}(u_\lambda) = Q_3 u_\lambda(x_{min}) = u_\lambda(x_{min}) = 0 \text{ and } \mu_{n-1}(u_\lambda) = Q_3 u_\lambda(x_{max}) = u_\lambda(x_{max}) = 0.$$

Hence $S = \sum_{j=-2}^{n-2} \mu_j(u_\lambda) B_j$.

From equation: (10), we can easily see that the spline S satisfies the following equation

$$QS^{(3)} + PS^{(2)}(x_j) + RS^{(1)}(x_j) + LS^{(0)}(x_j) = J_\lambda(x_j, u_\lambda) + O(h^3), \ j = 0,\dots,n \tag{11}$$

The goal of this section is to compute a cubic spline collocation $\tilde{S}p = \sum_{j=-3}^{n-1} \tilde{c}_j B_j$, $i = 1,\dots,H$ which satisfies the equation (9) at the points τ_j, $j = 0,\dots,n+2$ with $\tau_0 = x_0$, $\tau_j = (x_{j-1} + x_j)/2$, $j = 1,\cdots,n$, $\tau_{n+1} = x_{n-1}$ and $\tau_{n+2} = x_n$.

Then, it is easy to see that $\tilde{c}_{-3} = \tilde{c}_{n-1} = 0$. Hence $\tilde{S} = \sum_{j=-2}^{n-2} \tilde{c}_j B_j$, and the coefficients $\tilde{c}_j$, $j = -2,\dots,n-2$ satisfy the following collocation conditions:

$$Q\tilde{S}^{(3)}(\tau_j) + P\tilde{S}^{(2)}(\tau_j) + R\tilde{S}^{(1)}(\tau_j) + L\tilde{S}^{(0)}(\tau_j) = J_\lambda(\tau_j, u_\lambda), \ j = 1,\dots,n+1, \tag{12}$$

Taking $\tilde{C}_\lambda = [\tilde{c}_{-2},\dots,\tilde{c}_{n-2}]^T$ and $C_\lambda = [\mu_{-2}(u_\lambda),\dots,\mu_{n-2}(u_\lambda)]^T$,

and using equations (11) and (12), we get:

$$\left(QA_h^{(3)} + PA_h^{(2)} + RA_h^{(1)} + LA_h^{(0)}\right) C_\lambda = F + E \tag{13}$$

and

$$\left(QA_h^{(3)} + PA_h^{(2)} + RA_h^{(1)} + LA_h^{(0)}\right) \tilde{C}_\lambda = F_{\tilde{C}}, \tag{14}$$

with

$$F=[\frac{1}{\Delta t}J_{\lambda}(\tau_1,u_{\lambda}(\tau_1)),...,\frac{1}{\Delta t}J_{\lambda}(\tau_{n+1},u_{\lambda}(\tau_{n+1}))]^T,$$
$$F_{\tilde{C}_{\lambda}}=[\frac{1}{\Delta t}J_{\lambda}(\tau_1,\tilde{S}_{\lambda}(\tau_1)),...,\frac{1}{\Delta t}J_{\lambda}(\tau_{n+1},\tilde{S}_{\lambda}(\tau_{n+1}))]^T,$$
$$Q=\left(diag(\frac{b}{2\Gamma(2-\alpha)}I_{j,j})A\right)_{j=1,...,n+1},$$
$$A=\left(tridiag\inf(a_{i,j})\right)_{i,j=1,...,n+1},$$
$$P=\left(diag(\frac{b}{2\Gamma(2-\alpha)}I_{j,j})B\right)_{j=1,...,n+1},$$
$$B=\left(tridiag\inf(b_{i,j})\right)_{i,j=1,...,n+1},$$
$$R=\left(diag(\frac{-a}{2}I_{j,j})B\right)_{j=1,...,n+1},$$
$$G=\left(diag((\frac{1}{\Delta t}-\frac{r}{2})I_{j,j})B\right)_{j=1,...,n+1},$$
$$E=[O(\frac{h^3}{\Delta t}),...,O(\frac{h^3}{\Delta t})]^T\in\mathbb{R}^{n+1},$$
$$A_h^{(k)}=(B_{-3+p}^{(k)}(\tau_j))_{1\le j,p\le n+1},\quad k=0,1,2,3$$

It is well known that $A_h^{(k)}=\frac{1}{h^k}A_k$ for $k=0,1,2,3$ where matrices A_0, A_1, A_2 and A_3 are independent of h.

Then, relations (13) and (14) can be written in the following form

$$QA_3(I+U+V+W)C_{\lambda}=h^3F+h^3E, \tag{15}$$

$$QA_3(I+U+V+W)\tilde{C}_{\lambda}=h^3F_{\tilde{C}}, \tag{16}$$

with

$$U=h(QA_3)^{-1}PA_2, \tag{17}$$

$$V=h^2(QA_3)^{-1}RA_1, \tag{18}$$

$$W=h^3(QA_3)^{-1}LA_0. \tag{19}$$

The results of this work are basically based on the invertibility of the matrix A_3. Then, in order to prove that A_3 is invertible we give the flowing lemma.

Lemma 5.1 (de Boor [27]) Let $S\in\mathbb{S}_{k+1}$ such that $S=0$ on $\lfloor x_{p-1},x_p\rfloor\cup\lfloor x_q,x_{q+1}\rfloor$ where $p<q$. If S

admits r zeros in $\lfloor x_p, x_q \rfloor$ then $r \le p - q - (k+1)$.

Proposition 5.2 The matrix A_3 is invertible.

Proof: Let $D = [d_1, \cdots, d_{n+1}]^T$ be a vector of R^{n+1} such that $A_3 D = 0$. If we put

$$S(x) = \sum_{j=-2}^{n-2} d_j B_j ,$$

then we have $S(a) = S(b) = 0$ and $S'''(t_i) = 0$ for any $i = 1, \cdots, n+1$. Since $S \in S_4(I, \tau)$ then $S''' \in S_1(I, \tau)$. If we assume that $S''' \ne 0$ in $[x_0, x_n]$, then using the above lemma and the fact that S''' has $n+1$ zeros in $[x_0, x_n]$, we conclude that $n+1 \le n-1$, which is impossible. Therefore $S''' = 0$ for each $x \in \Omega_x$. This means that the function S is a piecewise linear polynomial in Ω_x. Since $S(x_{\min}) = S(x_{\max}) = 0$, then we obtain $S(x) = 0$ for any $x \in \Omega_x$. Consequently $D = 0$ and the matrix A_3 is invertible. □

In order to determine the bounded of $\| C_\lambda - \tilde{C}_\lambda \|_\infty$, we need the following Lemma.

Lemma 3.1 If $h^2 \rho < \frac{\Delta t}{2}$, then $I + U + V + W$ is invertible, where $\rho = \| (QA_3)^{-1} \|_\infty$.

Proof: From the relation (17), we have

$$\| U \|_\infty \le h \| (QA_3)^{-1} \|_\infty \| PA_2 \|_\infty$$

$$\| V \|_\infty \le h^2 \| (QA_3)^{-1} \|_\infty \| RA_1 \|_\infty$$

For h sufficiently small, we conclude

$$\| U \|_\infty < \frac{1}{6}. \tag{19}$$

$$\| V \|_\infty < \frac{1}{6}. \tag{19}$$

From the relation (18) and $\| A_0 \|_\infty \le 1,$ we have

$$\| W \|_\infty \le h^3 \| (QA_3)^{-1} \|_\infty \| GA_0 \|_\infty$$

$$\le h^3 \| (QA_3)^{-1} \|_\infty \| G \|_\infty$$

$$\le h^3 \frac{\mathrm{r}\rho}{2} + \frac{h^3 \rho}{\Delta t}$$

For h sufficiently small, we conclude that $h^3 \frac{\mathrm{r}\rho}{2} < \frac{1}{6}$. Then

$$\| W \|_\infty < \frac{1}{6} + \frac{h^3 \rho}{\Delta t}. \tag{20}$$

As $\dfrac{h^3\rho}{\Delta t} < \dfrac{1}{2}$. So, $\| U+V+W \| \leq \| U \| + \| V \| + \| W \| < 1$, and therefore $I+U+V+W$ is invertible.

Proposition 5 *Assume that the penalty parameter* ε *and the discretization parameter* h *satisfy the following relation:*

$$\frac{\lambda}{2} h^3 \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty < 1. \tag{13}$$

Then there exists a unique cubic spline which approximates the exact solution u_λ of problem (9).

Proof. From relation (12), we have $\tilde{C}_\lambda = h^3(I+U+V+W)^{-1}(QA_3)^{-1}F_{\tilde{C}_\lambda}$. Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a function defined by

$$\varphi(Y) = h^3(I+U+V+W)^{-1}(QA_3)^{-1}F_{\tilde{Y}}. \tag{14}$$

To prove the existence of cubic spline collocation it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of $\mathbb{R}^{n+1}$. Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq h^3 \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty \| F_{Y_1} - F_{Y_2} \|_\infty$$

Using relation (8) and the fact that $\sum_{j=-2}^{n-2} B_j \leq 1$, we get

$$| J_\lambda(t_i, S_{Y_1}(t_i)) - J_\lambda(t_i, S_{Y_2}(t_i)) | \leq \frac{\lambda}{2} \Delta t \, | S_{Y_1}(t_i) - S_{Y_2}(t_i) | \leq L_\lambda \| Y_1 - Y_2 \|_\infty, \tag{15}$$

Then we obtain

$$\| F_{Y_1} - F_{Y_2} \|_\infty \leq L_\lambda \| Y_1 - Y_2 \|_\infty .$$

From relation (15), we conclude that

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq \frac{\lambda}{2} h^3 \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty \| Y_1 - Y_2 \|_\infty,$$

Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq k \| Y_1 - Y_2 \|_\infty,$$

With $k = \dfrac{\lambda}{2} h^3 \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty$.

Hence the function φ admits a unique fixed point. □

In order to calculate the coefficients of the cubic spline collocation given by the nonsmooth system

$$\tilde{C}_\lambda = \varphi(\tilde{C}_\lambda), \tag{16}$$

we propose the generalized Newton method defined by

$$\tilde{C}_\lambda^{(k+1)} = \tilde{C}_\lambda^{(k)} - (I_{n+1} - V_k)^{-1}(\tilde{C}_\lambda^{(k)} - \varphi(\tilde{C}_\lambda^{(k)})), \tag{17}$$

where I_{n+1} is the unit matrix of order $n+1$ and V_k is the generalized Jacobian of the function $\tilde{C}_\lambda \mapsto \varphi(\tilde{C}_\lambda)$, (see [28], for instance).

Proposition 3.1 Assume that the penalty parameter λ and the discretization parameter h satisfy the following relation

$$\frac{\lambda}{2} h^3 \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty < 1. \tag{18}$$

Then the cubic spline $\tilde{S}$ converges to the solution u_λ. Moreover the error estimate is order $O(h^3)$

Proof: We pose $\upsilon = \| (I+U+V+W)^{-1}(QA_3)^{-1} \|_\infty$. From the relation (10), we have

$$C_\lambda - \tilde{C}_\lambda = h^3 (I+U+V+W)^{-1}(QA_3)^{-1}[(F - F_{\tilde{C}_\lambda}) + E].$$

Since E is of order $O(1/\Delta t)$, then there exists a constant K_1 such that

$$\|E\|_\infty \leq K_1 / \Delta t,$$

Hence, we have

$$\| C_\lambda - \tilde{C}_\lambda \| \leq h^3 \upsilon \left(\| F - F_{\tilde{C}_\lambda} \| + K_1 \right) \tag{19}$$

On the other hand we have

$$| J_\lambda(\tau_i, u_\lambda(\tau_i)) - J_\lambda(\tau_i, \tilde{S}_\lambda(\tau_i)) | \leq \frac{\lambda}{2} \Delta t \, | u_\lambda(\tau_i)) - \tilde{S}_\lambda(\tau_i) |,$$

$$\leq \frac{\lambda}{2} \Delta t \, | u_\lambda(\tau_i)) - S_\lambda(\tau_i) | + \frac{\lambda}{2} \Delta t \, | S_\lambda(\tau_i) - \tilde{S}_\lambda(\tau_i) |,$$

From relation (10), there exists a constant such that

$$\|u_\lambda - Q_3 u_\lambda\|_\infty \leq K_2 h^3 \| u_\lambda^{(3)} \|_\infty,$$

Using the fact that

$$\left| Q_3 u_\lambda - \tilde{S} \right| \leq \left\| C_\lambda - \tilde{C}_\lambda \right\|_\infty \sum_{j=-2}^{n-2} B_j \leq \left\| C_\lambda - \tilde{C}_\lambda \right\|_\infty,$$

Then, we obtain

$$\left| F - F_{\tilde{C}} \right| \leq \frac{1}{\Delta t} \left| J_\lambda(\tau_1, u_\lambda(\tau_i)) - J_\varepsilon(\tau_i, \tilde{S}_\lambda(\tau_i)) \right|$$

$$\leq \frac{\lambda}{2} \left\| C_\lambda - \tilde{C}_\lambda \right\|_\infty + \frac{\lambda}{2} K_2 h^3 \left\| u_\lambda^{(3)} \right\|_\infty$$

By using relation (18) and assumption (19) it is easy to see that

$$\left\| C_\lambda - \tilde{C}_\lambda \right\|_\infty \leq \frac{h^3 \upsilon}{1 - \frac{\lambda}{2} h^3 \upsilon} \left(K_2 h^3 \frac{\lambda}{2} \| u_\lambda^{(3)} \|_\infty + K_1 \right),$$

$$\leq 2\upsilon\left(K_2 h^3 \frac{\lambda}{2} \| u_\lambda^{(3)} \|_\infty + K_1\right) h^3,$$

By using relation (19), assumption (18) and relation

$$\| (I+U+V+W)^{-1} \|_\infty < \frac{1}{1\text{-}\| U+V+W \|_\infty} = cte,$$

It is easy to see that

$$\left\| C_\lambda - \widetilde{C}_\lambda \right\|_\infty \leq cte\left(K_2 h^3 \frac{\lambda}{2} \| u_\lambda^{(3)} \|_\infty + K_1\right) h^3,$$

We have

$$\left\| u_\lambda - \widetilde{S}_\lambda \right\|_\infty \leq \left\| u_\lambda - Q_3 u_\lambda \right\|_\infty + \left\| Q_3 u_\lambda - \widetilde{S}_\lambda \right\|_\infty,$$

Then from relations, we deduce that $\left\| u_\lambda - \widetilde{S}_\lambda \right\|_\infty$ is $O(h^3)$.

6. Numerical examples

In this section we verify experimentally theoretical results obtained in the previous section. If the exact solution is known, then at time $t \leq T$ the maximum error E^{max} can be calculated as:

$$E^{max} = \max_{x\in[x_{min}, x_{max}], t\in[0,T]} | S^{M,N}(x,t) - u(x,t) |.$$

Otherwise it can be estimated by the following double mesh principle:

$$E^{max}_{M,N} = \max_{x\in[x_{min}, x_{max}], t\in[0,T]} | S^{M,N}(x,t) - S^{2M,2N}(x,t) |,$$

where $S^{M,N}(x,t)$ is the numerical solution on the $M+1$ grids in space and $N+1$ grids in time, and $S^{2M,2N}(x,t)$ is the numerical solution on the $2M+1$ grids in space and $2N+1$ grids in time.

We present two examples to better illustrate the use of the Geometric Lévy process approach and the proposed pricing methodology in concrete situations. These examples are concerned with model American put option pricing problem

6.1. Example 1

Consider the following continuous LCP:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - {}_0D_x^{1,5} u(x,t) \geq f(x), \quad 0 < t \leq 1, \quad 0 < t < 2, \\ u(x,t) \geq xt, \\ \left(\dfrac{\partial u(x,t)}{\partial t} - {}_0D_x^{1,5} u(x,t) - f(x)\right).\left((u(x,t) - xt\right) = 0 \end{cases}$$

satisfying the boundary and payoff conditions

$$\begin{cases} u(0,t) = 0, \quad u(2,t) = 8t, \quad 0 < t < 1, \\ u(x,0) = 0, \quad x \in [0,2], \end{cases}$$

where $f(x) = x^3 - t\frac{\Gamma(4)}{\Gamma(2.5)}x^{1.5}.$

The solution to the unconstrained problem is x^3t. The damping parameter and the stopping criterion in the Newton's method are chosen to be 0.02 and 0.0001 respectively. The space and time intervals [0, 2] and [0, 1] are divided uniformly into M ($\Delta t = 2/M$) and N ($h = 1/N$) subintervals respectively for positive integers M and N.

The comparison of the maximum error values between the method developed in this paper with the one developed in [12] will be taken at five different values of the number of space steps $M = 2N = 5\times 2^l$ with $l = 1,2,3,4,5.$ We conduct experiments on different values of N, M. Table 1 show values of the maximum error (max_error) obtained in our numerical experiments and the one obtained in [12]. We see that the values of maximum error obtained by our method improve the ones in [12].

Table 1. Numerical results for different values of Δt and h.

$N = 2M = 5\times 2^l$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
max_error in [12]	1.0894×10^{-1}	2.5670×10^{-2}	5.9231×10^{-3}	1.3179×10^{-3}	2.6188×10^{-4}
Our max_error	8.1316×10^{-3}	2.0441×10^{-3}	5.3318×10^{-4}	1.5708×10^{-4}	8.7373×10^{-5}

6.2. Example 2

American put option with parameters:

$$S_{\max} = 100,\ S_{\min} = 100,\ \ T = 1,\ K = 50,\ S_{\max} = 100\ r = 0.05\ \sigma = 0.25.$$

To investigate the rates of convergence of the method in both λ and k, we choose a fixed uniform mesh for the solution domain $(\ln(0.1), \ln(100))\times(0,1)$ in (x,t) with $N = 100$ $M = 104.$ We also choose $\alpha = 1.5.$ Again the exact solution to this problem is unknown and we thus use the numerical solution with $\lambda = 10^{10}$ and $k = 1$ as our 'exact' or reference solution denoted as problem (4) corresponding to the problem are solved on the aforementioned uniform mesh (i.e., $N = 100$ $M = 104$) for a sequence of values of λ when k is fixed. Table 2 show values of the maximum error (max_error) obtained in our numerical experiments and the one obtained in [12]. We see that the values of maximum error obtained by our method improve the ones in [12] for the chosen values of λ and k.

Table 2. Numerical results for different values of λ.

$\lambda = 40\times 2^{n-k}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
k=1						
max_error in [12]	8.44×10^{-2}	4.31×10^{-2}	2.18×10^{-2}	1.09×10^{-2}	5.49×10^{-3}	2.75×10^{-3}
Our max_error	1.61×10^{-3}	1.14×10^{-3}	8.19×10^{-3}	5.88×10^{-3}	4.24×10^{-4}	3.08×10^{-4}
k=2						
max_error in [12]	4.24×10^{-2}	1.09×10^{-2}	2.75×10^{-3}	6.91×10^{-4}	1.73×10^{-4}	4.32×10^{-5}
Our max_error	1.14×10^{-3}	8.19×10^{-4}	5.88×10^{-4}	4.24×10^{-5}	3.08×10^{-5}	2.26×10^{-6}
k=3						
max_error in [12]	7.89×10^{-2}	1.09×10^{-2}	1.38×10^{-3}	1.73×10^{-4}	2.17×10^{-5}	2.70×10^{-6}
Our max_error	0.81×10^{-3}	5.88×10^{-4}	4.24×10^{-4}	3.08×10^{-5}	2.26×10^{-6}	1.68×10^{-7}
k=4						
max_error in [12]	2.82×10^{-1}	4.11×10^{-2}	2.75×10^{-3}	1.74×10^{-4}	1.09×10^{-5}	6.84×10^{-7}
Our max_error	5.88×10^{-2}	4.24×10^{-3}	3.08×10^{-4}	2.26×10^{-5}	1.68×10^{-6}	1.28×10^{-8}
k=5						
max_error in [12]	5.73×10^{-1}	2.53×10^{-1}	2.11×10^{-2}	6.95×10^{-4}	2.18×10^{-5}	6.86×10^{-7}
Our max_error	2.56×10^{-2}	5.85×10^{-2}	7.09×10^{-3}	8.12×10^{-5}	1.43×10^{-6}	5.29×10^{-8}

7. Conclusion

In this paper we have presented the American options whose underlying stock prices follow a geometric Lévy process, this problem is approximated by a sequence of nonlinear equation problems by using the penalty method given in [24, 25], and its time discretization scheme. Then, we have developed and analyzed a cubic spline collocation method and the generalized Newton method for approximating solutions of the semi-discret problem. We have shown the convergence of the method provided that the penalty and discret parameters satisfy the relation (13). Moreover we have provided an error estimate of order $O(h^3)$ with respect to the maximum norm $\| \|_\infty$. Numerical experiment was performed on one known model to validate the convergence and efficiency of the method. The computational results show that the proposed numerical method is an efficient alternative method to the one proposed in [12].

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