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A Fixed Point Theorems for a Contractive Condition of Integral Type by Using Altering Distance Functions

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Abstract

In this article, we prove some fixed point theorems in metric space by using altering distance function. Our result are generalization of many previously known results

***Key words*: -** Metric space, fixed point, Common fixed point, Altering Distance function.

Introduction

A new category of contractive fixed point problem was introduced by M.S. Khan , M. Swalech and S.Sessa [10]. In this work, they introduced the concept of altering distance function which is a control function that alters distance between two points in a metric space.

2 Preliminary

Definition 2.1: If T is a mapping of a complete metric space (X, d) into itself satisfying the condition:

$$d(Tx, Ty) \leq kd(x, y)$$

For all $x, y \in X$ and for some $k, 0 \leq k < 1$ then T has a unique fixed point. A mapping satisfying above condition is called contraction mapping.

Definition 2.2: The function $\psi: [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non-decreasing.

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.3: If $\xi: [0, \infty) \rightarrow [0, \infty)$ is subadditive on each $[a, b] \subset [0, \infty)$ then

$$\int_0^{a+b} \xi(t)dt \leq \int_0^a \xi(t)dt + \int_0^b \xi(t)dt.$$

Lemma 2.4: Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$$

If $\{x_n\}$ is not Cauchy sequence in X, then there exist an $\epsilon_0 > 0$ and sequence of integer positive $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon_0, \ d\left(x_{m(k)-1}, x_{n(k)}\right) < \epsilon_0.$$

Main Results

Theorem 3.1: Let (X, d) be a complete metric space, let $S: X \rightarrow X$ be a mapping which satisfies the following condition:

$$\psi(\int_0^{d(Sx,Sy)} \xi(t)dt) \leq \psi\left(M(x, y)\right) - \varphi\left(M(x, y)\right) \qquad (3.1.1)$$

For each $x, y \in X$ with $a, b, c > 0$ such that $a + b + 2c < 1$, where ψ, φ are altering distance functions, and

$$M(x, y) = a\int_0^{d(x,y)} \xi(t)dt$$

$$+b\int_0^{\frac{d(y,Sy)[1+d(x,Sx)]}{1+d(x,y)}} \xi(t)dt \qquad +c\int_0^{\frac{d(x,Sx).+d(y,Sy)}{1+d(x,y).d(y,Sx)}} \xi(t)dt \qquad (3.1.2)$$

Where $\xi: R^+ \rightarrow R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$$\int_0^{\epsilon_0} \xi(t)dt > 0.$$

Then S has a unique fixed point $z_0 \in X$.

Proof: Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as follow:

$$x_{n+1} = Sx_n \text{ for each } n \geq 0.$$

Now

$$M(x_{n-1}, x_n) = a\int_0^{d(x_{n-1},x_n)} \xi(t)dt \quad +b\int_0^{\frac{d(x_n,Sx_n)[1+d(x_{n-1},Sx_{n-1})]}{1+d(x_{n-1},x_n)}} \xi(t)dt$$
$$+c\int_0^{\frac{d(x_{n-1},Sx_{n-1}).+d(x_n,Sx_n)}{1+d(x_{n-1},x_n).d(x_n,Sx_{n-1})}} \xi(t)dt$$
$$= a\int_0^{d(x_{n-1},x_n)} \xi(t)dt + b\int_0^{d(x_n,x_{n+1})} \xi(t)dt \quad +c\int_0^{d(x_{n-1},x_n)+d(x_n,x_{n+1})} \xi(t)dt$$

By sub-additivity of ξ, we get

$$M(x_{n-1}, x_n) \leq (a+c)\int_0^{d(x_{n-1},x_n)} \xi(t)dt \quad +(b+c)\int_0^{d(x_n,x_{n+1})} \xi(t)dt$$

From (3.1.1), we have

$$\psi\left(\int_0^{d(x_n,x_{n+1})} \xi(t)dt\right) = \psi\left(\int_0^{d(Sx_{n-1},Sx_n)} \xi(t)dt\right)$$
$$\leq \psi(M(x_{n-1},x_n)) - \varphi(M(x_{n-1},x_n))$$
$$\leq \psi\begin{pmatrix}(a+c)\int_0^{d(x_{n-1},x_n)} \xi(t)dt \\ +(b+c)\int_0^{d(x_n,x_{n+1})} \xi(t)dt\end{pmatrix} \quad -\varphi(M(x_{n-1},x_n))$$

Since ψ is non-decreasing, we get

$$\int_0^{d(x_n,x_{n+1})} \xi(t)dt \leq \frac{a+c}{(1-b-c)}\int_0^{d(x_{n-1},x_n)} \xi(t)dt$$

Continuing this process, we get in general

$$\int_0^{d(x_n,x_{n+1})} \xi(t)dt \leq k^n\int_0^{d(x_0,x_1)} \xi(t)dt \qquad \text{Let } k = \frac{a+c}{(1-b-c)} < 1$$

Taking $n \to \infty$, we get

$\lim_{n\to\infty}\int_0^{d(x_n,x_{n+1})} \xi(t)dt = 0$ .Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$ (3.1.3)

Now, we will show that $\{x_n\}$ is a Cauchy sequence in X. suppose that $\{x_n\}$ is not a Cauchy sequence, which means that there is a constant $\epsilon_0 > 0$ such that for each positive integer k, there are positive integer $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon_0, d(x_{m(k)-1}, x_{n(k)}) < \epsilon_0$$

By triangle inequality

$$\epsilon_0 \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) \quad +d(x_{m(k)-1}, x_{n(k)}) \quad < d(x_{m(k)}, x_{m(k)-1}) + \epsilon_0$$

Letting $k \to \infty$ and using (3.1.3), we get

$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon_{0.}$ (3.1.4)

Similarly, we have

$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon_{0.}$ (3.1.5)

For $x = x_{m(k)}$ and $y = x_{n(k)}$ from (3.1.2), (3.1.3) and (3.1.4) we have,

$+\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = a\int_0^{\epsilon_0} \xi(t)dt$ (3.1.6)

From (3.1.1), we have

$$\psi\left(\int_0^{d(x_{m(k)+1},x_{n(k)+1})} \xi(t)dt\right) \quad = \psi\left(\int_0^{d(Sx_{m(k)},Sx_{n(k)})} \xi(t)dt\right)$$
$$\leq \psi\left(M(x_{m(k)}, x_{n(k)})\right) - \varphi\left(M(x_{m(k)}, x_{n(k)})\right)$$

Taking $k \to \infty$ and using (3.1.5), (3.1.6) and the continuity of ψ and φ, we get

$$\psi\left(\int_0^{\epsilon_0} \xi(t)dt\right) \leq \psi\left(a\int_0^{\epsilon_0} \xi(t)dt\right) \quad -\varphi\left(a\int_0^{\epsilon_0} \xi(t)dt\right) \quad \leq \psi\left(\int_0^{\epsilon_0} \xi(t)dt\right) - \varphi\left(a\int_0^{\epsilon_0} \xi(t)dt\right)$$

This leads to $\varphi\left(a\int_0^{\epsilon_0} \xi(t)dt\right) = 0$, and property of φ we get $\int_0^{\epsilon_0} \xi(t)dt = 0$.

This is contradiction. Thus $\{x_n\}$ is a Cauchy Sequence in (X, d), which is complete. Thus, there is $z_0 \in X$ such that

$$\lim_{n\to\infty} x_n = z_0 \quad ,$$

Setting $x = x_n$ and $y = z_0$ in (3.1.2) we have

$M(x_n, z_0) = a\int_0^{d(x_n,z_0)} \xi(t)dt \quad +b\int_0^{\frac{d(z_0,Sz_0)[1+d(x_n,Sx_n)]}{1+d(x_n,z_0)}} \xi(t)dt \quad +c\int_0^{\frac{d(x_n,Sx_n)+d(z_0,Sz_0)}{1+d(x_n,z_0).d(z_0,Sx_n)}} \xi(t)dt$

$= a\int_0^{d(x_n,z_0)} \xi(t)dt + b\int_0^{\frac{d(z_0,Sz_0)[1+d(x_n,x_{n+1})]}{1+d(x_n,z_0)}} \xi(t)dt$

$+c\int_0^{\frac{d(x_n,x_{n+1})+d(z_0,Sz_0)}{1+d(x_n,z_0).d((z_0,x_{n+1})}} \xi(t)dt$

Let $n \to \infty$ and using (3.1.3) and (3.1.7), we get

$\lim_{n\to\infty} M(x_n, z_0) = (b+c)\int_0^{d(z_0,Sz_0)} \xi(t)dt$ (3.1.8)

From (3.1.1) we have

$\psi\left(\int_0^{d(x_{n+1},Sz_0)} \xi(t)dt\right) = \psi\left(\int_0^{d(Sx_n,Sz_0)} \xi(t)dt\right)$

$\leq \psi(M(x_n, z_0)) - \varphi(M(x_n, z_0))$

Using (3.1.7) and (3.1.8), we get

$\psi\left(\int_0^{d(z_0,Sz_0)} \xi(t)dt\right) \leq \psi\left((b+c)\int_0^{d(z_0,Sz_0)} \xi(t)dt\right) \quad -\varphi\left((b+c)\int_0^{d(z_0,Sz_0)} \xi(t)dt\right)$

$\leq \psi\left(\int_0^{d(z_0,Sz_0)} \xi(t)dt\right) \quad -\varphi\left((b+c)\int_0^{d(z_0,Sz_0)} \xi(t)dt\right)$

Which implies$\varphi\left((b+c)\int_0^{d(z_0,Sz_0)} \xi(t)dt\right) = 0$, so $d(z_0, Sz_0) = 0$, that is $Sz_0 = z_0$.

Uniqueness: Let y_0, z_0 be two fixed point of S such that $y_0 \neq z_0$.

Putting $x = y_0$ and $y = z_0$ in (3.1.2) we have

$M(y_0, z_0) = a\int_0^{d(y_0,z_0)} \xi(t)d \quad +b\int_0^{\frac{d(z_0,Sz_0)[1+d(y_0,Sy_0)]}{1+d(y_0,z_0)}} \xi(t)dt \quad +c\int_0^{\frac{d(y_0,Sy_0)+d(z_0,Sz_0)}{1+d(y_0,z_0).d(z_0,Sy_0)}} \xi(t)dt$

$M(y_0, z_0) = a\int_0^{d(y_0,z_0)} \xi(t)dt$ (3.1.9)

So $\varphi\left(a\int_0^{d(y_0,z_0)} \xi(t)dt\right) = 0$, so $d(y_0, z_0) = 0$, that is $y_0 = z_0$.

Theorem 3.2: Let (X, d) be a complete metric space, let $S: X \to X$ be a mapping which satisfies the following condition:

$\psi(\int_0^{d(Sx,Sy)} \xi(t)dt) \leq \psi(M(x,y)) - \varphi(M(x,y))$ (3.2.1)

For each $x, y \in X, x \neq y, \alpha, \beta > 0, 2\alpha + 3\beta < 1$, where ψ, φ are altering distance functions, and

$M(x,y) = \alpha\int_0^{\left[\frac{d^2(x,Sx)+d^2(y,Sy)+d^2(y,Sx)}{d(x,Sx)+d(y,Sy)+d(y,Sx)}\right]} \xi(t)dt \quad +\beta\int_0^{\left[\frac{d^2(x,Sy)+d^2(y,Sx)+d^2(x,y)}{d(x,Sy)+d(y,Sx)+d(x,y)}\right]} \xi(t)dt$

(3.2.2)

Where $\xi: R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$\int_0^{\epsilon_0} \xi(t)dt > 0$

Then S has a unique fixed point $z_0 \in X$.

Proof: Can be proved easily as theorem 3.1

Theorem 3.3: Let (X, d) be a complete metric space, let $S, T: X \to X$ be a mapping which satisfies the following condition:

$\psi(\int_0^{d(Sx,Sy)} \xi(t)dt) \leq \psi(M(x,y)) - \varphi(M(x,y))$ (3.3.1)

For each $x, y \in X, x \neq y, \alpha, \beta > 0, 2\alpha + 3\beta < 1$, where ψ, φ are altering distance functions, and

$M(x,y) = \alpha\int_0^{\left[\frac{d^2(x,Sx)+d^2(y,Ty)+d^2(y,Sx)}{d(x,Sx)+d(y,Ty)+d(y,Sx)}\right]} \xi(t)dt \quad +\beta\int_0^{\left[\frac{d^2(x,Ty)+d^2(y,Sx)+d^2(x,y)}{d(x,Ty)+d(y,Sx)+d(x,y)}\right]} \xi(t)dt$ (3.3.2)

Where $\xi: R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$\int_0^{\epsilon_0} \xi(t)dt > 0$.Then S and T have a common fixed point $z_0 \in X$.

Proof: Can be proved easily as theorem 3.1 and 3.2

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