Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.14, 2014

Stability and bifurcation of aquatic food chain model

Sara Jabbar Rashid and Raid Kamel Naji
Department of Mathematic, College of Science, University of Baghdad, Baghdad, Iraq.
email: sara.jabbar9@gmail.com , rknaji@gmail.com
The research is financed by Asian Development Bank. No. 2006-A171(Sponsoring information)

Abstract:

The objective of this paper is to study the dynamical behavior of an aquatic food chain system. A mathematical model that includes nutrients and harmful phytoplankton and zooplankton is proposed and analyzed. The phytoplankton produces a toxic substance as a strategy of defense against predation by zooplankton. Its assumed that all the feeding processes in this food chain are depending on the Holling type-II functional response. Further a portion of dead phytoplankton and zooplankton are return to the nutrient due to the decomposition process. This model is represented mathematically by the set of nonlinear differential equations. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. The occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork) a long with Hopf bifurcation near each of the equilibrium points are discussed. Finally, numerical simulation is used to study the global dynamics of this model.

Keywords: nutrient , phytoplankton, zooplankton, stability analysis, stability bifurcation.

1. Introduction:

The nutrient –phytoplankton–zooplankton (NPZ) model is a common tool in oceanographic research. The NPZ model incorporates one of the simplest set of dynamics that usefully to describe oceanic plankton dynamics. Their acceptability as a research tool is by no means universal [franks and chen, 2000]**[1].**Phytoplankton transform mineral nutrients into primitive biotic material using external energy provided by the sun. The dynamic relationship between phytoplankton and nutrients has long been of great interest in both experimental and mathematical ecology its universal existence and important **[2].** Some type of phytoplankton release toxic as a defensive strategy against the predation, these types are known as harmful phytoplankton. For example harmful algal bloom (HABs), which have adverse effects on human health, fishery, tourism, and the environment. In recent year considerable scientific attention has been given to HABs. Since the phytoplankton is a base of all the aquatic food chain systems and most of zooplankton organism depends directly on the phytoplankton in its feeding process. Therefore toxic substances released by harmful plankton play an important role in this context see for example **[3].** phytoplankton organisms are the dominant primary producers in the pelagic environment. They convert inorganic materials into new organic compounds by the process of photosynthesis, starting thereby most aquatic food webs **[4].** phytoplankton production is responsible for approximately 40% of the global primary productivity, hence the stocks of these tiny plank tonic algae play a significant role for marine reserves and fishery management. Pollution of freshwater and marine systems by anthropogenic sources has become a concern over the last decades. Organic (e.g triazine herbicides) **[5-8]**or inorganic (e.g. heavy metal) compounds **[5,6,9-12]** both may have
harmful effects.

The dynamics of most the biological systems can be represented mathematically by a set of first order nonlinear differential equations. These systems are known as a mathematical model for such biological systems and can be studied mathematically. Consequently, in this paper one kind of aquatic ecological model is proposed and analyzed .

2. Mathematical model formulation:

Consider the ecological system that consisting of nutrient, phytoplankton and zooplankton with Holling type-II functional response. According to the
specific hypotheses, mentioned below, this system can be modeled mathematically by using the following set of nonlinear equations:

$$\begin{aligned}
\frac{dN}{dt} &= D(N_0 - N) - \frac{a_1 NP}{a_2 + N} + m_1\varepsilon_1 P + m_2\varepsilon_2 Z = f_1(t) \\
\frac{dP}{dt} &= \frac{k_1 a_1 NP}{a_2 + N} - (D + \varepsilon_1)P - \frac{\beta PZ}{\beta_1 + P} = f_2(t) \\
\frac{dZ}{dt} &= \frac{k_2 \beta PZ}{\beta_1 + P} - (D + \varepsilon_2)Z - \theta\alpha PZ = f_3(t)
\end{aligned} \tag{1}$$

here $N(t)$, $P(t)$ and $Z(t)$ denoted the densities of nutrient, phytoplankton and zooplankton at time t respectively, while the parameters are assumed to be positive parameters and can be described as following: D is the dilution rate, N_0 is the concentration of the input nutrient, $m_i < 1;\ (i = 1,2)$ is the conversion factor from dead phytoplankton species and dead zooplankton species respectively to the nutrient due to the decomposition operation, a_1 and a_2 are the maximum specific ingestion rate and the half saturation constant of phytoplankton, β and β_1 are maximum attack rate and half saturation constant of the zooplankton, $k_i < 1;\ (i = 1,2)$ is the conversion factor from nutrient to phytoplankton and from phytoplankton to zooplankton respectively. θ the liberation rate of toxin substance by the harmful ,(a)the maximum zooplankton in gestation rates for the toxic substance produced by phytoplankton, finally the natural mortality rate of phytoplankton and zooplankton are denoted by ε_1 and ε_2 respectively. The above model is build depending on the following set of hypotheses:

1. There is a constant concentration of nutrient with constant rate of dilution. The nutrient up taken by the phytoplankton according to the Holling type-II
functional response. On the other hand a portion of the dead phytoplankton and zooplankton return to the nutrient due to the decomposition operation.
2. In the absence of nutrient the phytoplankton decay exponentially due to dilution and natural death. Further decay facing the phytoplankton due to the feeding process by zooplankton.
3. The zooplankton feeds on the phytoplankton according to the Holling type-II and decay exponentially due to dilution and natural death. Further it is assumed that the zooplankton affected by the toxin produced by the phytoplankton during the predation process.

Theorem (1) : All the solutions of the system **(1)** are bounded:
Proof: Let $W = N + P + Z$ then by differentiating this variable with respect to time we obtain:

$$\frac{dW}{dt} = DN_0 - DN - (1-k_1)\frac{a_1 NP}{a_2 + N} - (1-m_1)\varepsilon_1 P - (1-m_2)\varepsilon_2 Z$$

$$- DP - (1-k_2)\frac{\beta PZ}{\beta_1 + P} - DZ - \theta a PZ.$$

$$\frac{dW}{dt} = DN_0 - D(N+P+Z). \Rightarrow \frac{dW}{dt} \le DN_0 - DW.$$

Consequently, by solving this differential inequality we get that

$$W(t) \le W_0 \exp(-Dt) + \frac{m}{D}(\exp(-Dt) - 1) \qquad \text{So, as } t \to \infty \text{ we get that } W(t) \le \frac{m}{D}. \quad \blacksquare$$

3. Existence of equilibrium point:

In this section the existence of all possible equilibrium points of system **(1)** is discussed as shown below:

The nutrient equilibrium point $E_0 = (N_0, 0, 0)$ always exists.

The zooplankton free equilibrium point $E_1 = (N', P', 0)$ where

$$N' = \frac{a_2(D+\varepsilon_1)}{k_1 a_1 - (D+\varepsilon_1)},\ P' = \frac{k_1 D(N_0 - N')}{(D+\varepsilon_1) - m_1\varepsilon_1 k_1}$$

Clearly, E_1 belongs to the interior of the positive quadrant of $NP-$plane, which denoted by $Int.R_+^2 = \{(N,P) \in R^2 : N > 0, p > 0\}$, provided that one set of the following sets of conditions holds:

$$N' < N_0 \text{ and } m_1\varepsilon_1 k_1 < D + \varepsilon_1 < a_1 k_1 \qquad \textbf{(2a)}$$

Or

$$N' > N_0 \text{ and } D + \varepsilon_1 < \min.\{a_1 k_1, m_1\varepsilon_1 k_1\} \qquad \textbf{(2b)}$$

The coexistence or positive equilibrium point can be established as follows.
From the third equation of system **(1)** we obtain that:

$$\frac{K_2 \beta P}{\beta_1 + P} - (D+\varepsilon_2) - \theta a P = 0$$

Hence we obtain that

$$\theta a P^2 + (D+\varepsilon_2 + \beta_1\theta a - k_2\beta)P + \beta_1(D+\varepsilon_2) = 0$$

Clearly, due to discard rule we have either two positive real roots (given below) or else there are no positive roots depending on the following conditions whether it hold or violate respectively.

$$(D+\varepsilon_2) + \beta_1\theta a < k_2\beta \quad \textbf{(3a)}$$

$$(D+\varepsilon_2 + \beta_1\theta a - k_2\beta)^2 > 4\theta a\beta_1(D+\varepsilon_2) \quad \textbf{(3b)}$$

These two positive real roots can be written as:

$$P_1, P_2 = -\frac{(D+\varepsilon_2+\beta_1\theta a - k_2\beta)}{2\theta a} \pm \frac{1}{2\theta a}\sqrt{(D+\varepsilon_2+\beta_1\theta a - k_2\beta)^2 - 4\theta a\beta_1(D+\varepsilon_2)} \quad \textbf{(3c)}$$

By substituting P_i for each $(i=1,2)$ in the second equation of system **(1)** and equating it to zero, then we get after some calculation a unique root given by that:

$$Z_i = \frac{N(\beta_1+P_i)[k_1a_1-(D+\varepsilon_1)] - a_2(\beta_1+P_i)(D+\varepsilon_1)}{\beta(a_2+N)} \quad \textbf{(3d)}$$

Now by substituting the value of P_i and Z_i for each $(i=1,2)$ in the first equation of system (2.1) and equating it to zero, we obtain after some calculation a unique positive root given by

$$N_i = \frac{\mathrm{H}_1}{2D} + \frac{\sqrt{\mathrm{H}_1^2 + 4D\mathrm{H}_2}}{2D}, \quad \textbf{(3e)}$$

here $\mathrm{H}_1 = DN_0 + m_1\varepsilon_1P_i + m_2\varepsilon_2Z_i - Da_2 - a_1P_i$

$\mathrm{H}_2 = DN_0a_2 + a_2m_1\varepsilon_1P_i + a_2m_2\varepsilon_2Z_i$

Clearly by using N_i for each $(i=1,2)$ in Z_i, then Z_i will be positive constant provided that

$$N_i > \frac{a_2(D+\varepsilon_1)}{k_1a_1-(D+\varepsilon_1)} > 0 \quad \textbf{(3f)}$$

According to the above calculation, system (2.1) have two positive equilibrium points, denoted by $E_2 = (N_1, P_1, Z_1)$ and $E_3 = (N_2, P_2, Z_2)$, belong to the positive octant of $R_+^3 = \{(N,P,Z) \in R^3 : N \geq 0, P \geq 0, Z \geq 0\}$ provided that conditions **(3a), (3b)** and **(3f)** hold.

4. Local stability analysis

In this section we will study the local stability analysis near each of the above equilibrium points with help of linearization technique as shown below.

Straightforward computation shows that the Jacobian matrix of system **(1)** can be written as

$$J = \lfloor a_{ij} \rfloor_{3\times 3} \quad \textbf{(4)}$$

where

$a_{11} = -D - \frac{a_1a_2P}{(a+N)^2} < 0$, $a_{12} = \frac{-a_1N}{a_2+N} + m_1\varepsilon_1$, $a_{13} = m_2\varepsilon_2 > 0$, $a_{21} = \frac{k_1a_1a_2P}{(a_2+N)^2} > 0$.

$a_{22} = \frac{k_1a_1N}{a_2+N} - (D+\varepsilon_1) - \frac{\beta Z}{\beta_1+P} + \frac{\beta PZ}{(\beta_1+P)^2}$, $a_{23} = -\frac{\beta P}{\beta_1+P} < 0$. $a_{31} = 0$.

$a_{32} = \frac{k_2\beta\beta_1Z}{(\beta_1+P)^2} - \theta aZ$, $a_{33} = \frac{k_2\beta P}{\beta_1+P} - (D+\varepsilon_2) - \theta aP$.

Consequently, the Jacobian matrix at each of the above equilibrium points can be written as follows

The Jacobean matrix of system **(1)** at the equilibrium point $E_0 = (N_0, 0, 0)$ can be written as:

$$J_0 = \begin{bmatrix} -D & \frac{-a_1 N_0}{a_2 + N_0} + m_1 \varepsilon_1 & m_2 \varepsilon_2 \\ 0 & \frac{[k_1 a_1 - (D+\varepsilon_1)]N_0 - a_2(D+\varepsilon_1)}{a_2 + N_0} & 0 \\ 0 & 0 & -(D+\varepsilon_2) \end{bmatrix} \quad \textbf{(5a)}$$

Clearly the eigenvalues of J_0 lie on the main diagonal and given by

$$\lambda_{0N} = -D < 0, \ \lambda_{0P} = \frac{[k_1 a_1 - (D+\varepsilon_1)]N_0 - a_2(D+\varepsilon_1)}{a_2 + N_0}, \ \lambda_{0Z} = -(D+\varepsilon_2) < 0$$

where λ_{0U} represents the eaginvalues that describe the dynamics in the U -direction. Note that the eagenvalue in the P -direction will be negative and hence the nutrient equilibrium point E_0 is locally asymptotically stable provided that

$$a_1 k_1 < D + \varepsilon_1 \quad \textbf{(5b)}$$

Or

$$k_1 a_1 > (D+\varepsilon_1) \text{ and } N_0 < \frac{a_2(D+\varepsilon_1)}{k_1 a_1 - (D+\varepsilon_1)} \quad \textbf{(5c)}$$

otherwise it will be saddle point.

The Jacobean matrix of system **(1)** at the zooplankton free equilibrium point $E_1 = (N', P', 0)$ can be written as:

$$J_1 = \begin{bmatrix} -D - \frac{a_1 a_2 P'}{(a_2 + N')^2} & \frac{-(D+\varepsilon_1) + m_1 \varepsilon_1 k_1}{k_1} & m_2 \varepsilon_2 \\ \frac{a_1 a_2 k_1 P'}{(a_2 + N')^2} & 0 & -\frac{\beta P'}{\beta_1 + P'} \\ 0 & 0 & \frac{\beta k_2 P'}{\beta_1 + P'} - (D+\varepsilon_2) - \theta a P' \end{bmatrix} \quad \textbf{(6a)}$$

Consequently the eigenvalues of J_1 satisfy the following relations

$$\lambda_{1N} + \lambda_{1P} = -D - \frac{a_1 a_2 P'}{(a_2 + N')^2} < 0$$

$$\lambda_{1N} . \lambda_{1P} = \frac{a_1 a_2 P'}{(a_2 + N')^2}[(D+\varepsilon_1) - m_1 \varepsilon_1 k_1]$$

$$\lambda_{1Z} = \frac{\beta k_2 P'}{\beta_1 + P'} - (D+\varepsilon_2) - \theta a P'$$

Straightforward computation shows that all these eigenvalues will be negative and hence $E_1 = (N', P', 0)$ is locally asymptotically stable provided that the following conditions holds, otherwise it is saddle point

$$(D+\varepsilon_1) > m_1 \varepsilon_1 k_1 \quad \textbf{(6b)}$$

$$\frac{\beta k_2 P'}{\beta_1 + P'} < (D+\varepsilon_2) + \theta a P' \quad \textbf{(6c)}$$

The Jacobean matrix of system **(1)** at the positive equilibrium point $E_2 = (N_1, P_1, Z_1)$, similarly for $E_3 = (N_2, P_2, Z_2)$, can be written as

$$J_2 = [M_{ij}]_{3\times 3} \quad \textbf{(7)}$$

where

$$M_{11} = -D - \frac{a_1 \ a_2 P_1}{(a_2 + N_1)^2}, M_{12} = -\frac{a_1 N_1}{a_2 + N_1} + m_1 \varepsilon_1, M_{13} = m_2 \varepsilon_2, M_{21} = \frac{k_1 a_1 a_2 P_1}{(a_2 + N_1)^2}.$$

$M_{22} = \frac{\beta P_1 Z_1}{(\beta_1 + P_1)^2}$, $M_{23} = -\frac{\beta P_1}{\beta_1 + P_1}$, $M_{31} = 0$, $M_{32} = \frac{k_2 \beta \beta_1 Z_1}{(\beta_1 + P_1)^2} - \theta a Z_1$, $M_{33} = 0$.

The characteristic equation of J_2 is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \quad \textbf{(7a)}$$

where

$$A_1 = -(M_{11} + M_{22}).$$
$$A_2 = M_{11}M_{22} - M_{12}M_{21} - M_{23}M_{32}.$$
$$A_3 = M_{32}[M_{11}M_{23} - M_{13}M_{21}].$$

And

$$\Delta = -(M_{11} + M_{22})[M_{11}M_{22} - M_{12}M_{21}] + M_{32}[M_{22}M_{23} + M_{13}M_{21}] \quad \textbf{(7b)}$$

Note that according to the Routh-Hurwitz criterion all the eigenvalues of J_2 have negative real parts and hence E_2 is locally asymptotically stable if and
only if $A_i > 0$ for $i = 1,3$ and $\Delta > 0$. Therefore the conditions that guarantee the local stability of E_2 are established in the following theorem.

Theorem (2): The positive equilibrium point E_2 is locally asymptotically stable provided that the following conditions are satisfied.

$$m_1 \varepsilon_1 R_1 < a_1 N_1 \quad \textbf{(8a)}$$

$$\max\left\{\frac{\beta R_1^2 Z_1}{a_1 a_2}, \frac{\beta Z_1 R_1 (D R_1^2 + a_1 a_2 P_1)}{k_1 a_1 a_2 (a_1 N_1 - m_1 \varepsilon_1 R_1)}\right\} < R_2^2 < \min.\left\{\frac{k_2 \beta \beta}{\theta a}, \left(\frac{\beta P_1}{k_1 m_2 \varepsilon_2}\right)^2\right\} \quad \textbf{(8b)}$$

$$R_1^2 < \frac{k_1 a_1 a_2 m_2 \varepsilon_2 R_2^3}{\beta^2 P_1 Z_1} \quad \textbf{(8c)}$$

here $R_1 = a_2 + N_1$ and $R_2 = \beta_1 + P_1$.

Proof: By using the elements of J_2, it is easy to verify that condition **(8b)** guarantees that $A_1 > 0$ and $A_3 > 0$. While conditions **(8a)** and **(8b)** ensures that the first term of Δ will be positive, however conditions **(8b)** and **(8c)** guarantees the positivity of the second term of Δ. Hence all conditions of Routh-Hurwitz criterion are satisfied with the help of the above conditions. Therefore all the eigenvalues of J_2 have negative real parts and hence E_2 is locally asymptotically stable. ■

5. Global stability analysis:

In this section the region of the global stability of each equilibrium points of system **(1)** is established with the help of Lyapunov function as shown in the following theorems.

Theorem (3): Let the nutrient equilibrium point $E_0 = (N_0, 0, 0)$ is locally asymptotically stable, then it is globally asymptotically stable in the sub region G_1 of the domain R_+^3 that satisfy the following condition.

$$N_0 + k_1 < N < \min.\left\{N_0 + \frac{D + \varepsilon_1}{m_1 \varepsilon_1}, N_0 + \frac{D + \varepsilon_2}{m_1 \varepsilon_2}\right\} \quad \textbf{(9)}$$

Proof. Consider the following real valued function

$$V_0 = \frac{(N - N_0)^2}{2} + P + Z$$

Clearly $V_0 : R_+^3 \to R$ be a continuously differentiable function that satisfy that

$$V_0(N, 0, 0) > 0 \text{ and } V_0(N_0, 0, 0) = 0$$

Further,

$$\frac{dV_0}{dt}=(N-N_0)\frac{dN}{dt}+\frac{dP}{dt}+\frac{dZ}{dt}$$
$$\le -D(N-N_0)^2-\frac{a_1NP}{a_2+N}(N-N_0-k_1)-\frac{\beta PZ}{\beta_1+P}(1-k_2)$$
$$-P(m_1\varepsilon_1N_0+D+\varepsilon_1-m_1\varepsilon_1N)$$
$$-Z(m_2\varepsilon_2N_0+D+\varepsilon_2-m_2\varepsilon_2N)$$

Consequently, by using the given condition we get that

$$\frac{dV_0}{dt}\le -D(N-N_0)^2$$

Thus $\frac{dV_0}{dt}<0$ for any initial condition in the interior of G_1 and hence V_0 is a Lyapunov function. So, $E_0=(N_0,0,0)$ is globally asymptotically stable in the sub region G_1. ■

Obviously, G_1 represent the basin of attraction of the nutrient equilibrium point $E_0=(N_0,0,0)$ of system (1).

Theorem (4): Let the zooplankton free equilibrium point $E_1=(N',P',0)$ is locally asymptotically stable, then it is globally asymptotically stable in the sub region G_2 of the domain R^3_+ that satisfy the following conditions.

$$D+\varepsilon_1>\frac{a_1k_1N'}{a_2+N'} \quad \textbf{(10a)}$$

$$P>P'+k_2 \quad \textbf{(10b)}$$

$$N<N'+\frac{D+\varepsilon_2}{m_2\varepsilon_2} \quad \textbf{(10c)}$$

$${\gamma_{12}}^2<4\gamma_{11}\gamma_{22} \quad \textbf{(10 d)}$$

here $\gamma_{12}=m_1\varepsilon_1+\frac{k_1a_1a_2P}{R}+\frac{a_1N'}{a_2+N'}$,

$\gamma_{11}=D+\frac{a_1a_2P}{R}$, and $\gamma_{22}=(D+\varepsilon_1)-\frac{k_1a_1N'}{a_2+N'}$.

Proof. Consider the following real valued function

$$V_1=\frac{(N-N')^2}{2}+\frac{(P-P')^2}{2}+Z$$

Clearly $V_1:R^3_+\to R$ be a continuously differentiable function that satisfy that

$V_1(N,P,0)>0$ and $V_1(N',P',0)=0$

Further

$$\frac{dV_1}{dt}=(N-N')\frac{dN}{dt}+(P-P')\frac{dP}{dt}+\frac{dZ}{dt}$$
$$\le -\left[D+\frac{a_1a_2P}{R}\right](N-N')^2+\left[m_1\varepsilon_1+\frac{k_1a_1a_2P}{R}+\frac{a_1N'}{a_2+N'}\right](N-N')(P-P')$$
$$-\left[(D+\varepsilon_1)-\frac{k_1a_1N'}{a_2+N'}\right](P-P')^2-\left[D+\varepsilon_2+m_2\varepsilon_2N'-m_2\varepsilon_2N\right]Z$$
$$-\left[P-(P'+k_2)\right]\frac{\beta PZ}{\beta_1+P}$$

here $R=(a_2+N)(a_2+N')$. Consequently, by using the given conditions we get that

$$\frac{dV_1}{dt} \le -\left[\sqrt{\gamma_{11}}(N - N') - \sqrt{\gamma_{22}}(P - P')\right]^2$$

Thus $\frac{dV_{0\backslash 1}}{dt} < 0$ for any initial condition in the interior of G_2 and hence V_1 is a Lyapunov function. So, $E_1 = (N', P', 0)$ is globally asymptotically stable in the sub region G_2. ■

Again, G_2 represent the basin of attraction of the zooplankton free equilibrium point $E_1 = (N', P', 0)$ of system **(1).**

Further more since there are two positive equilibrium points in the interior of R_+^3 having the same local stability conditions but with different neighborhood of starting points then its not possible to study the globule stability of them using
Lyapunov function. Therefore we will study it numerically instead of analytically as shown in last chapter.

6. The local Bifurcation

In this section an investigation for the dynamical behavior of system **(1)** under the effect of varying one parameter of each time is carried out. The
occurrence of local bifurcation in the neighborhood of the equilibrium point are studied in the below theorems.
Consider the Jacobean matrix of system **(1)** given by equation **(4).** It is easy to verify that straight forward computation gives that:

$$D^2F(X,\alpha)(V,V) = \begin{pmatrix} \frac{2a_1a_2v_1}{(a_2+N_0)^2}\left(\frac{v_1P}{(a_2+N_0)} - v_2\right) \\ \frac{2k_1a_1a_2v_1}{(a_2+N_0)^2}\left(v_2 - \frac{v_1P}{(a_2+N_0)}\right) + \frac{2\beta\beta_1v_2}{(\beta_1+P)^2}\left(\frac{Zv_2}{(\beta_1+P)} - v_3\right) \\ \frac{2k_2\beta\beta_1v_2}{(\beta_1+P)^2}\left(v_3 - \frac{Zv_2}{(\beta_1+P)}\right) - 2\theta a v_2 v_3 \end{pmatrix} \quad \textbf{(11)}$$

where $V = (v_1, v_2, v_3)^T$. Further

$$D^3F(X,\alpha)(V,V,V) = \begin{pmatrix} \frac{6a_1a_2v_1^2}{(a_2+N_0)^3}\left(v_2 - \frac{v_1P}{(a_2+N_0)}\right) \\ \frac{6k_1a_1a_2v_1^2}{(a_2+N_0)^3}\left(\frac{v_1P}{(a_2+N_0)} - v_2\right) + \frac{6\beta\beta_1v_2^2}{(\beta_1+P)^3}\left(v_3 - \frac{Zv_2}{(\beta_1+P)}\right) \\ \frac{6k_2\beta\beta_1v_2^2}{(\beta_1+P)^3}\left(\frac{Zv_2}{\beta_1+P} - v_3\right) \end{pmatrix} \quad \textbf{(12)}$$

where $X = (N, P, Z)^T$, and α be any parameter .

Theorem(5) : Assume that the parameter k_1 passes through the value $k_1^* = \frac{(D+\varepsilon_1)(a_2+N)}{a_1N_0}$**, then** system **(1)** near the equilibrium point E_0 has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur
   provided that the following condition holds :

$$L = \left(\frac{k_1a_1a_2v_1}{(a_2+N_0)^2}\left(1 - \frac{T_1P}{(a_2+N_0)}\right) + \frac{2\beta Z}{\beta_1^2}\right) \ne 0 \quad \textbf{(13)}$$

3. A pitch-fork bifurcation provided that:

$$L = 0 \quad \textbf{(14a)}$$

$$L_1 = \left(\frac{k_1 a_1 a_2 T_1^2}{(a_2 + N_0)^2} \left(\frac{T_1 P}{(a_2 + N_0)} - 1 \right) - \frac{\beta Z}{\beta_1^3} \right) \neq 0 \qquad \textbf{(14b)}$$

Where $T_1 = \left(-\frac{a_1 N_0}{(a_2 + N_0)} + m_1 \varepsilon_1 \right) \frac{1}{D}$.

Proof : According to the Jacobian matrix of system **(1)** at E_0 that is given by equation **(5a),** it is easy to verify that as $k_1 = k_1^*$, then the $J_0(E_0, k_1^*)$ has the following eigenvalues : $\lambda_{0N} = -D, \lambda_{0P} = 0, \lambda_{0Z} = -(D + \varepsilon_1)$

Let $\overline{V} = (\overline{v}_1, \overline{v}_2, \overline{v}_3)^T$ be the eigenvector of $J_0(E_0, k_1^*)$ corresponding to the eigenvalue $\lambda_{0P} = 0$. Then it is easy to check that $\overline{V} = \left((T_1 \overline{v}_2, \overline{v}_2, 0\right)^T$, where $\overline{v}_2$ represents any nonzero real value

Also, let $\overline{C} = (\overline{c}_1, \overline{c}_2, \overline{c}_3)^T$ represents the eigenvector of $J_0^T(E_0, k_1^*)$ that corresponding to the eigenvalue $\lambda_{0P} = 0$. Straight forward calculation shows that $\overline{C} = (0, \overline{c}_2, 0)^T$ where $\overline{c}_2$ is any nonzero real number.

Now, since $\frac{\partial F}{\partial k_1} \equiv F_{k_1}(X, k_1) = \left(0, \frac{a_1 NP}{a_2 + N}, 0 \right)^T$, where $X = (N, P, Z)^T$ and $F = (f_1, f_2, f_3)^T$ with $f_i; i = 1,2,3$ represent the functions in the right hand side of system (1) then we get $F_{k_1}(E_0, k_1^*) = (0,0,0)^T$ and hence the following is obtained:

$$\overline{C}^T [F_{k_1}(E_0, k_1^*)] = (0, \overline{c}_2, 0)(0,0,0)^T = 0$$

Thus the system **(1)** at E_0 does not experience any saddle-node bifurcation in view of sotomayor theorem **[13]**. While the first condition of the transcritical and pitch-fork bifurcation is satisfied .
Also, since

$$\overline{C}^T [DF_{k_1}(E_0, k_1^*) \overline{V}] = (0, \overline{c}_2, 0)(0, \frac{a_1 N_0}{a_2 + N_0} \overline{v}_2, 0) = \overline{c}_2 \overline{v}_2 (\frac{a_1 N_0}{a_2 + N_0}) \neq 0$$

where $DF_{k_1}(E_0, k_1^*) = \frac{\partial}{\partial X} F_k(X, k_1)$ and then substitute $X = E_0, k_1 = k_1^*$. Moreover, according to equation **(11)** we have

$$\overline{C}^T [D^2 F(E_0, k_1^*)(\overline{V}, \overline{V})] = 2 \overline{c}_2 \overline{v}_2^2 (\frac{k_1 a_1 a_2 v_1}{(a_2 + N_0)^2} (1 - \frac{T_1 P}{(a_2 + N_0)}) + \frac{\beta Z}{\beta_1^2}) \neq 0$$

where $D^2 F(E_0, k_1^*) = DJ_0(X, k_1)$ with $X = E_0$, $k_1 = k_1^*$.

Clearly, $\overline{C}^T [D^2 F(E_0, k_1^*)(\overline{V}, \overline{V})] \neq 0$ provided that condition **(13)** holds, and then by sotomayor theorem **[13]** the system **(1)** possesses a transcritical bifurcation but not pitch-fork bifurcation near E_0. However, if the condition **(14a)** holds we get that $\overline{C}^T [D^2 F_{k_1}(E_0, k_1^*)(\overline{V}, \overline{V}] = 0$, and hence further computation shows

$$\overline{C}^T [D^3 F(E_0, k_1^*)(\overline{V}, \overline{V}, \overline{V})] = 6 \overline{c}_3 \overline{v}_3^3 (\frac{k_1 a_1 a_2 T_1^2}{(a_2 + N_0)^2} (\frac{T_1 P}{(a_2 + N_0)} - 1) - \frac{\beta Z}{\beta_1^3}) \neq 0$$

Clearly if the condition **(14b)** holds then system **(1)** possesses a Pitch-fork bifurcation near E_0 at $k_1 = k_1^*$, and hence the proof is complete. ■

Theorem(6): Assume that the parameter ε_2 passes through the value $\varepsilon_2^* = \frac{\beta k_2 P'}{\beta_1 + P'} - D - \theta a P'$, where ε_2 is positive parameter under the condition

$$\frac{\beta k_2 P'}{\beta_1 + P'} > D + \theta a P' \quad \textbf{(15a)}$$

then system **(1)** near the equilibrium point E_1 has:
1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur
provided that the following condition holds :

$$L_2 = \left(\frac{k_2 \beta \beta_1}{(\beta_1 + P')^2} \left(1 - \frac{ZT_2}{(\beta_1 + P')} - \theta a \right) \right) \neq 0 \quad \textbf{(15b)}$$

3. A pitch-fork bifurcation provided that .

$$L_2 = 0 \quad \textbf{(16a)}$$

$$L_3 = \left(\frac{ZT_2}{(\beta_1 + P')} - 1 \right) \neq 0 \quad \textbf{(16b)}$$

where $T_2 = \dfrac{k_1 \hat{v}_3}{D + \varepsilon_1(1 - m_1 k_1)} \left(m_2 \varepsilon_2 - \dfrac{\beta D (a_2 + N')^2}{k_1 a_1 a_2 (\beta_1 + P')} - \dfrac{\beta P'}{k_1(\beta_1 + P')} \right)$

Proof : According to the Jacobian matrix of system **(1)** at E_1 that is given by equation **(6a)** , it is easy to verify that if $\varepsilon_2 = \varepsilon_2^*$ the eigenvalues of equation **(6a)** satisfy the following relation:

$$\lambda_{1N} + \lambda_{1p} = -D - \frac{a_1 a_2 P'}{(a_2 + N')^2},$$

$$\lambda_{1N} . \lambda_{1P} = \frac{a_1 a_1 P'}{(a_2 + N')} [(D + \varepsilon_1) - m_1 \varepsilon_1 k_1,$$

$$\lambda_{1Z} = 0$$

Let $\hat{V} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T$ be the eigenvector of $J_1(E_1, \varepsilon_2^*)$ corresponding to the eigenvalue $\lambda_{1Z} = 0$, then it is easy to check that $\hat{V} = (\dfrac{\beta (a_2 + N')^2 \hat{v}_3}{a_1 a_2 k_1 (\beta_1 + p')}, T_2 \hat{v}_3, \hat{v}_3)$, where $\hat{v}_3$ represents any nonzero real value.

Also, let $\hat{C} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)^T$ represents the eigenvector of $J_1^T(E_1, \varepsilon_2^*)$ that corresponding to the eigenvalue $\lambda_{1Z} = 0$ .Straightforward calculation shows that $\hat{C} = (0, 0, \hat{c}_3)^T$ where $\hat{c}_3$ is any nonzero real number.

Now, since $\dfrac{\partial F}{\partial \varepsilon_2} \equiv F_{\varepsilon_2}(X, \varepsilon_2) = (m_2 Z, 0, -Z)^T$, *where* $X = (N, P, Z)^T$ and $F = (f_1, f_2, f_3)^T$ with $f_i; i = 1,2,3$ represent the right hand side of system **(1)** then we get $F_{\varepsilon_2}(E_1, \varepsilon_2^*) = (0,0,0)^T$ and the following is obtained:

$$\hat{C}^T [F_{\varepsilon_2}(E_1, \varepsilon_2^*)] = (0, 0, \hat{c}_3)(0,0,0)^T = 0$$

Thus the system **(1)** at E_0 does not experience any saddle-node bifurcation in view of sotomayor theorem**[13]** .
While the first condition of the Transcritical and Pitch-fork bifurcation is satisfied. Also, since

$$\hat{C}^T [DF_{\varepsilon_2}(E_1, \varepsilon_2^*) \hat{V}] = (0, 0, \hat{c}_3)(m_2 \hat{v}_3, 0, -\hat{v}_3)^T = -\hat{c}_3 \hat{v}_3 \neq 0$$

where $DF_{\varepsilon_2}(E_1, \varepsilon_2^*) = \dfrac{\partial}{\partial X} F_{\varepsilon_2}(X, \varepsilon_2)$ and then substitute $X = E_1$, $\varepsilon_2 = \varepsilon_2^*$. Moreover, According to equation **(11)** we have

$$\hat{C}^T[D^2F(E_1,\varepsilon_2^*)(\hat{V},\hat{V})] = 2\hat{c}_3\hat{v}_3^2T_2\left(\frac{k_2\beta\beta_1}{(\beta_1+P')^2}(1-\frac{ZT_2}{(\beta_1+P')}-\theta a)\right)\neq 0$$

$$where \quad D^2F(E_1,\varepsilon_2^*) = DJ_1(X,\varepsilon_2) \quad with \quad X = E_1\ ,\varepsilon_2 = \varepsilon_2^*$$

Clearly , $\hat{C}^T[D^2F(E_1,\varepsilon_2^*)(\hat{V},\hat{V}]\neq 0$ provided that condition **(15b)** holds, and then by sotomayor theorems**[13]**, the system **(1)** possesses a transcritical bifurcation but not pitch-fork bifurcation near E_0.

However , if the condition **(16a)** holds we get that $\hat{C}^T[D^2F(E_1,\varepsilon_2^*)(\hat{V},\hat{V})] = 0$.

Also, According to equation **(12)** we have $\hat{C}^T[D^3F(E_1,\varepsilon_2^*)(\hat{V},\hat{V},\hat{V})]$

$$= \left(\frac{6k_2\beta\beta_1T_2^2\hat{v}_3^3\hat{c}_3}{(\beta_1+P')^3}(\frac{ZT_2}{(\beta_1+P')}-1)\right)\neq 0$$

Clearly if the condition **(16b)** holds then system **(1)** possesses a Pitch-fork bifurcation near E_0 at $\varepsilon_2 = \varepsilon*$, and hence the proof is complete. ■

Theorem(7): Assume that the parameter θ passes through the value $\theta^* = \dfrac{k_2\beta\beta_1}{(\beta_1+P_1)^2a}$**,** then system **(1)** near the equilibrium point E_2 has:

1. Saddle-node bifurcation provided that :

$$\left(M_{22}-\frac{M_{12}M_{21}}{M_{11}}\right)\neq 0$$
$$\left(M_{23}-\frac{M_{13}M_{21}}{M_{11}}\right)\neq 0 \qquad \textbf{(17)}$$

2. No transcritical bifurcation .
3. No pitch-fork bifurcation .

Proof : According to the Jacobian matrix of system **(2.1)** at E_2 that is given by equation **(7)** , It is easy to verify that as $\theta = \theta^*$. The characteristic equation of J_2 that given by Eq**. (7a).**

Since $A_3 = M_{32}[M_{11}M_{23}-M_{13}M_{21}]$, where M_{ij} represents the elements of the Jacobian matrix J_2. We obtain that $A_3 = 0$ for $\theta = \theta^* = \dfrac{k_2\beta\beta_1}{(\beta_1+P_1)^2a}$ and hence .the eigenvalues of J_2 satisfy the following relation $\lambda(\lambda^2+A_1\lambda+A_2)=0,$ so either $\lambda = 0$ or $(\lambda^2+A_1\lambda+A_2)=0$. Now if we assume that $\lambda = 0$. Then it $\tilde{V} = (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^T$ be the eigenvector of $J_2(E_2,\theta^*)$ corresponding to the eigenvalue $\lambda = 0$. Then it is easy to check that $\tilde{V} = \left(\left(\left(\frac{M_{12}M_{11}M_{23}-M_{12}M_{21}M_{13}}{M_{22}M_{11}-M_{12}M_{21}}-M_{13}\right)\frac{\tilde{v}_3}{M_{11}}\right),\left(\frac{M_{21}M_{13}-M_{11}M_{23}}{M_{22}M_{11}-M_{12}M_{21}}\right)\tilde{v}_3,\tilde{v}_3\right)^T$

where $\tilde{v}_3$ represents any nonzero real value.

Also, let $\tilde{C} = (\tilde{c}_1,\tilde{c}_2,\tilde{c}_3)^T$ represents the eigenvector of $J_2^T(E_2,\theta^*)$ that corresponding to the eigenvalue $\lambda_1 = 0$. Straightforward competition and under the condition **(17)** shows that $\tilde{C} = (0,0,\tilde{c}_3)^T$ where $\tilde{c}_3$ is any nonzero real number.

Now, Since $\dfrac{\partial F}{\partial\theta} \equiv F\theta(X,\theta) = (0,0,-aPZ)^T$, $where \quad X = (N,P,Z)^T$ and $F = (f_1, f_2, f_3)$ with $F_i; i = 1,2,3$ represent the right hand side of system (1) then we get $F_\theta(E_2,\theta) = (0,0,-aP_1Z_1)^T$ and the following is obtained:

$$\tilde{C}^{T}[F_{\theta}(E_2,\theta^{*})] = -c_3 a P_1 Z_1 \neq 0$$

Then by sotomayor theorems **[13]**, the system (1) possesses a saddle-node bifurcation but not transcritical and pitch fork bifurcation. ■

7. The Hopf bifurcation:

In this section, the occurrence of Hopf-bifurcation near the equilibrium point is studied below .

The Hopf bifurcation analysis near E_0 and E_1:

According to the Jacobean matrix of system **(1)** at E_0 and E_1 which are given by equation **(5a)** and **(6a)** respectively, it is easy to verify that J_0 and J_1 has three real eigenvalues. So, the necessary and sufficient conditions for a Hop bifurcation to occur are not satisfies.

Theorem(8): Assume that the conditions **(8a)** and **(8c)** with the following condition are hold:

$$\max\left\{\frac{\beta R_1^2 Z_1}{a_1 a_2}, \frac{\beta Z_1 R_1 (D R_1{}^2 + a_1 a_2 P_1)}{k_1 a_1 a_2 (a_1 N_1 - m_1 \varepsilon_1 R_1)}\right\} < R_2{}^2 < \left\{\frac{k_2 \beta\beta}{\theta a}\right\} \qquad \textbf{(18a)}$$

Then system **(1)** possesses a Hopf bifurcation around the equilibrium point E_2 when the parameter θ passes through $\theta = \theta^{*}$, where

$$\theta^{*} = \frac{-(M_{11}+M_{22})[M_{11}M_{22}-M_{12}M_{21}]}{aZ_1(M_{22}M_{23}+M_{13}M_{21}} + \frac{k_2\beta\beta_1}{a(\beta_1+P_1)^2}$$

and M_{ij} represent the elements of the Jacobian matrix given by **(7).**

Proof: Consider The characteristic equation of J_2 that given by Eq. **(7a)** and Δ that given in Eq. **(7b).**

Now, suppose that $\Delta = A_1 A_2 - A_3 = 0$ then according to **[14]** there is possibility to occurrence of Hopf-bifurcation if and only if the Jacobian matrix of system **(1)** near E_2 has two complex conjugate eigenvalues, say $\lambda_i = \omega_1 \pm i\omega_2$ with the third eigenvalue is real and negative, in addition to the following two conditions:

$$\omega_1(\theta^{*}) = 0 \qquad \textbf{(18b)}$$

$$\left.\frac{d\omega_1}{dl}\right|_{\theta=\theta^{*}} \neq 0 \qquad \textbf{(18c)}$$

where $\theta = \theta^{*}$ represents the bifurcation parameter.

Now, to verify the necessary and sufficient conditions for a Hopf bifurcation to occur we need to find a parameter satisfy $\Delta(\theta^{*}) = 0,\ A_1(\theta^{*}) > 0 \ and\ A_2(\theta^{*}) > 0$.

Consequently, if

$$0 = -(M_{11}+M_{22})\big[M_{11}M_{22}-M_{12}M_{21}\big] + M_{32}[M_{22}M_{23}+M_{13}M_{21}]$$

it is easy to verify that the parameter's value that satisfy the above equation is:

$$\theta^{*} = \frac{-(M_{11}+M_{22})[M_{11}M_{22}-M_{12}M_{21}]}{aZ_1(M_{22}M_{23}+M_{13}M_{21}} + \frac{k_2\beta\beta_1}{a(\beta_1+P_1)^2}$$

where θ^{*} is a positive parameter under the condition **(8a), (8c)** and **(18a).** Hence the coefficients of the characteristic equation can be written as:

$$A_1(\theta^{*}) = -(M_{11}+M_{22}) > 0 .$$

$$A_2(\theta^{*}) = M_{11}M_{22} - M_{12}M_{21} - M_{23}\left(\frac{k_2\beta\beta_1 Z_1}{(\beta_1+P_1)^2} - \theta^{*} a Z_1\right) > 0$$

$$A_3(\theta^{*}) = \left(\frac{k_2\beta\beta_1 Z_1}{(\beta_1+P_1)^2} - \theta^{*} a Z_1\right)[M_{11}M_{23} - M_{13}M_{21}] .$$

So for $\theta = \theta^{*}$ we have $A_1 A_2 = A_3$ and then the characteristic equation becomes:

$$P_3(\lambda)=(\lambda+A_1)(\lambda^2+A_2)=0$$

Thus the roots become $\lambda_1=-A_1$ *and* $\lambda_{2,3}=\pm i\sqrt{A_2}$.

Clearly, at $\theta=\theta^*$ there are two pure imaginary eigenvalues $\lambda_{2,3}=\pm i\sqrt{A_2}$ with the other one of eigenvalues real and negative, that is mean, $\mathrm{Re}(\lambda_i(\theta))|_{\theta=\theta^*}=\omega_1(\theta^*)=0,\ i=2,3$. Thus the first condition **(18b)** of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $\theta=\theta^*$.

Let as now check the second condition **(18c).** Since, in general, the complex eigenvalues for any value of θ can be written as: $\lambda_{2,3}=\omega_1\pm i\omega_2$

So to verify the transversality condition we apply the bifurcation criterion given by **[23]** as stated in chapter one . Note that for $\theta=\theta^*$ we have $\omega_1=0$ and $\omega_2=\pm i\sqrt{A_2}$, substitution into Eq**. (1.25)** gives that

$$\Psi(\theta^*)=-2(M_{11}M_{22}-M_{12}M_{21}-M_{23}(\frac{k_2\beta\beta_1Z_1}{(\beta_1+P_1)^2}-\theta^*aZ_1).$$

$$\Phi(\theta^*)=-2(M_{11}+M_{22})\left(\sqrt{M_{11}M_{22}-M_{12}M_{21}-M_{23}(\frac{k_2\beta\beta_1Z_1}{(\beta_1+P_1)^2}-\theta^*aZ_1)}\right)$$

$$\Theta(\theta^*)=-aZ_1(M_{11}M_{23}-M_{13}M_{21}).$$

$$\Gamma(\theta^*)=(aZ_1M_{23})\left(\sqrt{M_{11}M_{22}-M_{12}M_{21}-M_{23}(\frac{k_2\beta\beta_1Z_1}{(\beta_1+P_1)^2}-\theta^*aZ_1)}\right)$$

Further by substitution into Eq. $\Theta(\mu)\Psi(\mu)+\Gamma(\mu)\Phi(\mu)$ we obtain

$$\Theta(\mu)\Psi(\mu)+\Gamma(\mu)\Phi(\mu)=$$

$$2aZ_1\left(M_{11}M_{22}-M_{12}M_{21}-M_{23}\left[\frac{k_2\beta\beta_1Z_1}{(\beta_1+P_1)^2}-\theta^*aZ_1\right]\right)\left((M_{11}M_{23}-M_{13}M_{21})-M_{23}(M_{11}+M_{22})\right)$$

Now, according to **[14],** the necessary and sufficient conditions for a Hop bifurcation to occur are satisfies if in addition to the above the following condition should be satisfied $\Theta(\mu)\Psi(\mu)+\Gamma(\mu)\Phi(\mu)\neq 0$. Clearly this is will be hold under the given conditions and hence the proof is complete. ■

8. Numerical simulation

In this section the dynamical behavior of system **(1)** is studied numerically for different sets of parameters. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system **(1)** and second confirm our obtained analytical results. It is observed that, for the following hypothetical values of the parameters with initial point (14.46,6.47,1.69); system **(1)** has asymptotically stable positive equilibrium point, see Fig. **(1).**

$$\begin{aligned}&D=0.2,\ N_0=40,\ a_1=1.25,\ a_2=8,\ m_1=0.5,\\&m_2=0.8,\ k_1=0.4,\ k_2=0.6,\ \beta=0.6,\ \beta_1=4,\\&\theta=0.02,\ a=0.02.,\ \varepsilon_1=0.025,\ \varepsilon_2=0.02.\end{aligned} \quad \textbf{(19)}$$

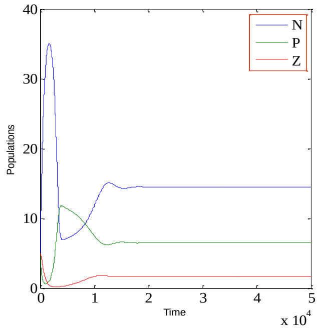

Fig. (1): The solution of system (1) approaches asymptotically to stable positive point (14.46,6.47,1.69)

Now in order to discuss the effect of varying the parameters values of system **(1)** on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. **(19)** with varying one parameter each time. It is observed that, for the data given in Eq. **(19)** with varying one of the parameters values m_2 or ε_1, there is no change in the dynamical behavior of system **(1)** and the system still approaches to positive equilibrium point.

For the data given in Eq. **(19)** with $25 \le N_0 \le 49$, the solution of system **(1)** approaches to the positive equilibrium point as shown in Fig. **(2a)**, however for $N_0 > 49$ the solution of system **(2.1)** approaches to periodic dynamic in $Int.R_+^3$ as shown in Fig**.(2b),** while for $7 \le N_0 \le 24$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the interior of positive quadrant of NP-plane as shown in Fig**.(2c)**. Further for $N_0 \le 6$ the solution of system **(2.1)** approaches asymptotically to $E_0 = (N_0, 0, 0)$ on the boundary $N-$axis as shown in Fig**.(2d).**

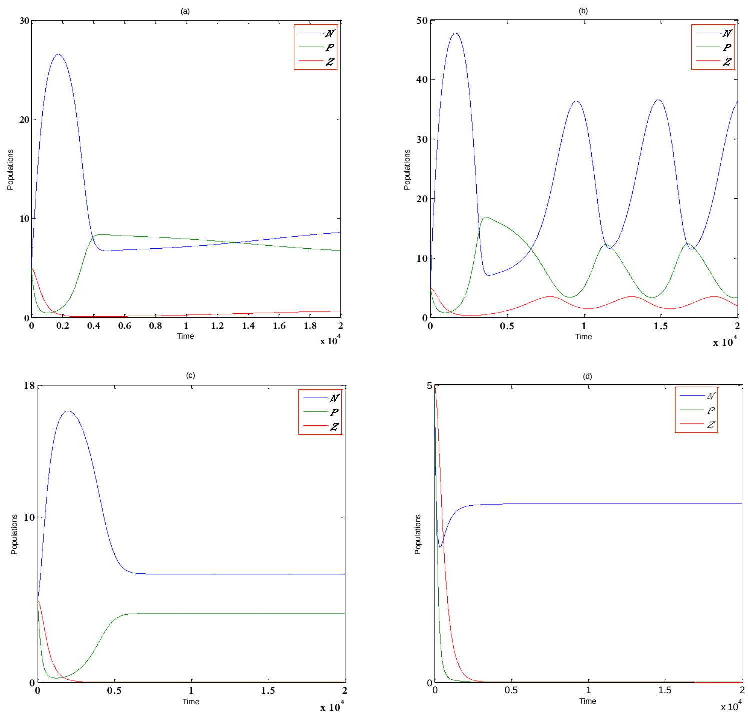

Fig.(4.2): Time series of the solution of system (1) for the data given by Eq (19) with different values of N_0. (a) System (1) approaches to (9.0096,6.47,0.69) in $Int.R_+^3$ for $N_0 = 30$. (b) System(2.1) approaches to periodic dynamic in $Int.R_+^3$ for $N_0 = 55$ (c) System (1) approaches to (6.54,4.16,0) in $Int.R_+^2$ of NP-plane for $N_0 = 18$. (d) System (1) approaches to (3,0,0) on the N- axis for $N_0 = 3$.

Similarly by varying the parameter a_2 keeping the rest of parameters values as in Eq. **(19)**, It is observed that for $26 \le a_2 \le 49$ the system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the interior of positive quadrant of $NP-$plane, however for $0.05 \le a_2 \le 25$ the system approaches to the positive equilibrium point, while for $a_2 \le 0.04$ the system**(1)**

approaches to periodic dynamic in $Int.R_+^3$, but for $a_2 \ge 50$ the solution of system **(1)** approaches asymptotically to $E_0 = (N_0, 0, 0)$ on the boundary $N-$axis.

For the parameters values given in Eq. **(19)** with varying D in the range $0.18 \le D \le 0.25$ the system approaches to the positive equilibrium point as shown in Fig. **(3a),** however for $D \le 0.17$ the system **(1)** approaches to periodic dynamic in $Int.R_+^3$ as shown in Fig. **(3b).** Moreover for $D \ge 0.26$ the system (1) approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane as shown in Fig. **(3c).**

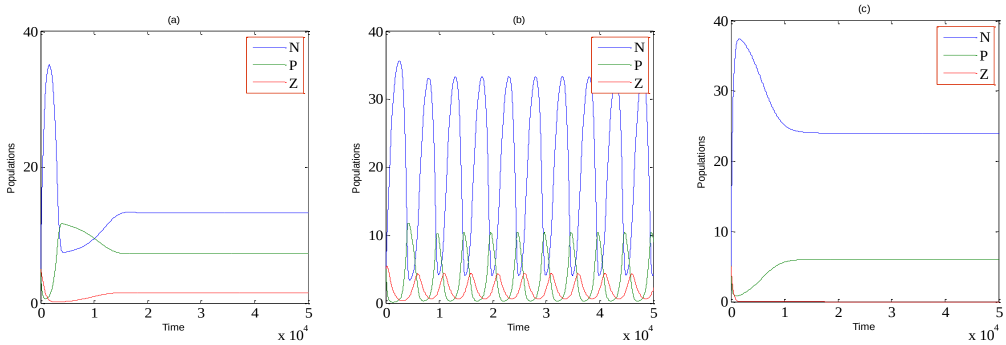

Fig.(3): Time series of the solution of system (2.1) for the data given by Eq (19) with different values of D. (a) System (1) approaches to (13.29,7.33,1.45) in $Int.R_+^3$ for D=0.21. (b) System(2.1) approaches to periodic dynamic in $Int.R_+^3$ for D=0.11. (c) System (2.1) approaches to (24,6.05,0) in $Int.R_+^2$ of NP-plane for D=0.35.

Now by varying the parameter k_2 keeping the rest of parameters values as in Eq. **(19),** it is observed that for $k_2 \ge 0.7$ the system**(1)** approaches to periodic dynamic in $Int.R_+^3$, while for $k_2 \le 0.5$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the interior of positive quadrant of $NP-$plane.

Also varying the parameter β keeping the rest of parameter values as in Eq. **(19),** it is observed that for $\beta = 0.5$, $\beta = 0.6$, the solution of system **(2.1)** approaches to the positive equilibrium point. However for $\beta \ge 0.7$ the system **(1)** approaches to periodic dynamic in $Int.R_+^3$, while for $\beta \le 0.4$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane.

For the parameters values given in Eq.**(19)** with varying β_1 in the range $\beta_1 < 0.1$ *and* $4 \le \beta_1 \le 7$ the system approaches to the positive equilibrium point. However for $0.2 \le \beta_1 \le 3$ the system **(1)** approaches to periodic dynamic in $Int.R_+^3$, while for $\beta_1 \ge 8$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane.

For the parameters values given in Eq.**(19)** with varying a_1, it is observed that for $a_1 \ge 0.8$, the solution of system **(1)** approaches to the positive equilibrium point as shown in Fig. **(4a),** however for $a_1 = 0.7$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane as shown in Fig. **(4b).** While for $a_1 \le 0.6$ the solution of system **(1)** approaches asymptotically to $E_0 = (N_0, 0, 0)$ on the boundary $N-$axis as shown in Fig**. (4c).**

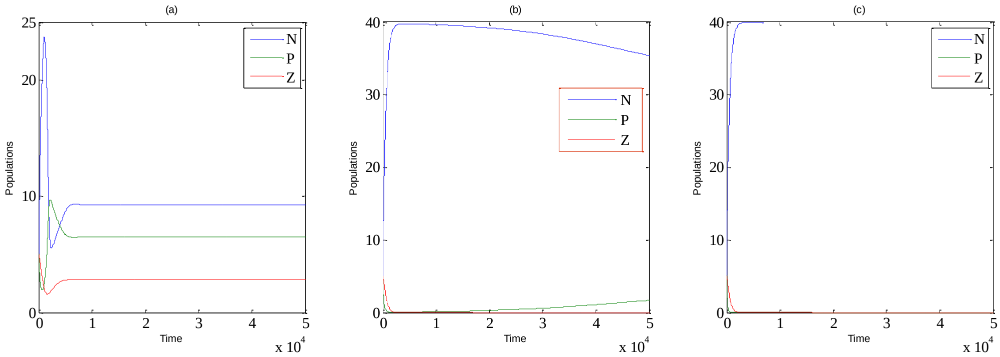

Fig.(4): Time series of the solution of system (1) for the data given by Eq (19) with different values of a_1. (a) System (1) approaches to (9.29,6.47,2.82) in $Int.R_+^3$ for $a_1 = 1.8$. (b) System (1) approaches to (32,7.2.64,0) in $Int.R_+^2$ of NP-plane for $a_1 = 0.7$. (c) System (2.1) approaches to (40,0,0) on the N- axis for $a_1 = 0.1$.

For the parameters values given in Eq.**(19)** with varying k_1, it is observed that for $0.3 \le k_1 \le 0.5$ the system approaches to the positive equilibrium point as shown in Fig. **(5a).** However for $0.6 \le k_1 \le 7$ the system **(2.1)** approaches to periodic dynamic in $Int.R_+^3$ as shown in Fig. **(5b),** while for $k_1 \le 0.2$ the solution of system **(1)** approaches asymptotically to $E_0 = (N_0,0,0)$ on the boundary $N-$axis as shown in Fig. **(5c).**

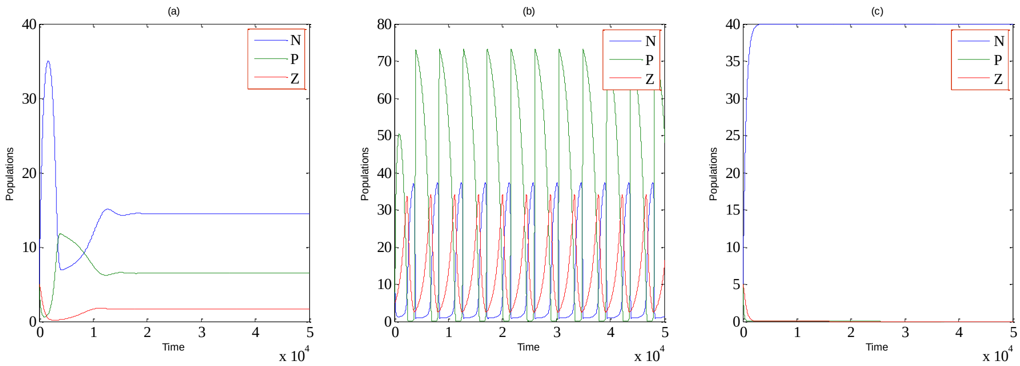

Fig.(5): Time series of the solution of system (1) for the data given by Eq (19) with different values of k_1. (a) System (2.1) approaches to (14.46,6.47,1.69) in $Int.R_+^3$ for $k_1 = 0.4$. (b) System(2.1) approaches to periodic

dynamic in $Int.R_+^3$ for $k_1 = 2$. (c) System (2.1) approaches to (40,0,0) on the N- axis for $k_1 = 0.1$.

Now for the parameters values given in Eq.**(19)** with varying ε_2 in the range $\varepsilon_2 \ge 0.07$ the system approaches to the positive equilibrium point as shown in Fig. **(6a).** However for $\varepsilon_2 < 0.07$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane as shown in Fig. **(6b).**

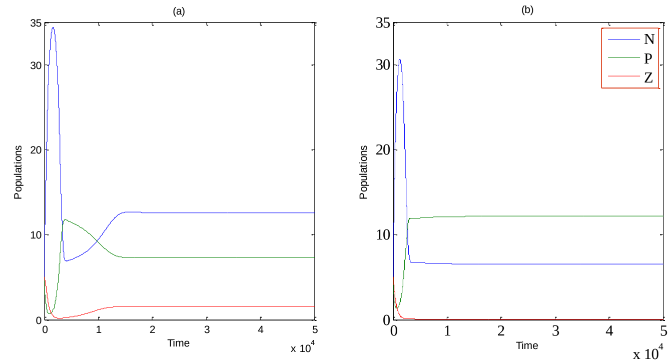

Fig.(6): Time series of the solution of system (1) for the data given by Eq (19) with different values of ε_2. (a) System (2.1) approaches to (12.6,7.33,1.52) in $Int.R_+^3$ for $\varepsilon_2 = 0.09$. (b) System (2.1) approaches to (6.54,12.16,0) in $Int.R_+^2$ of NP-plane for $\varepsilon_2 = 0.03$.

Similarly varying the parameter θ keeping the rest of parameters values as in Eq. **(19)** shows that for $\theta \geq 0.3$, the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane, while for $\theta \leq 0.2$ the system approaches to the positive equilibrium point.

Also varying the parameter (a) keeping the rest of parameters values as in Eq. **(19)** gives that for $a \leq 0.2$ the system approaches to the positive equilibrium point. However for $a \geq 0.3$ the solution of system **(1)** approaches asymptotically to $E_1 = (N', P', 0)$ in the $Int.R_+^2$ of NP-plane.

Finally for the parameters values given in Eq.**(19)** with varying m_1, it is observed that when $m_1 \leq 6.9$, the solution of system **(1)** approaches to the positive equilibrium point as shown in Fig. **(7a)**, however for $m_1 \geq 7$ the system **(1)** approaches to periodic dynamic in $Int.R_+^3$ as shown in Fig. **(7b).**

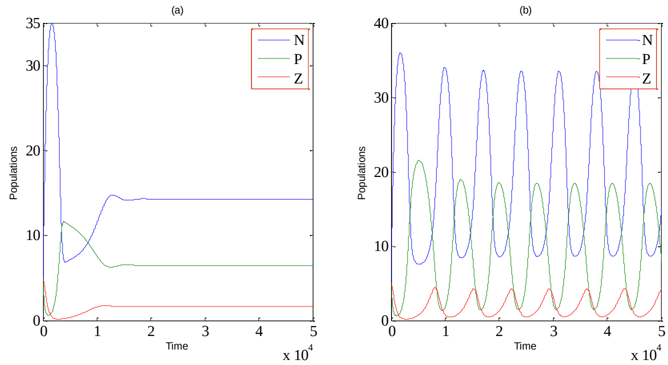

Fig.(7): Time series of the solution of system (1) for the data given by Eq (19) with different values of m_1. (a) System (1) approaches to (14.266,6.47,1.66) in $Int.R_+^3$ for $m_1 = 0.1$. (b) System(2.1) approaches to periodic dynamic in $Int.R_+^3$ for $m_1 = 12.2$.

9. conclusions and Discussions:

In this chapter, a mathematical model consisting of nutrients, harmful phytoplankton and zooplankton species has been proposed and analyzed. it is assumed that the phytoplankton producing a toxin substance as a defensive strategy against the predation by zooplankton. However the zooplankton consumes the phytoplankton according to Holling type-II functional response. Finally a portion of the dead species of phytoplankton and zooplankton is returned to nutrient due to the decomposition operation. The boundedness of the proposed system **(1)** has been discussed. The dynamical behavior of system **(1)** has been investigated locally as well as globally. Further, it is observed that the nutrient equilibrium point (E_0) always exist and it is locally asymptotically stable point if and only if the conditions **(5b)** or **(5c)** hold otherwise it will be saddle point. The zooplankton free equilibrium point (E_1) exists if the conditions **(2a)** and **(2b)** hold, and it is locally asymptotically stable point if and only if the conditions **(6b)** and **(6c)** hold. Further it is observed that there are two positive equilibrium point (E_2) and (E_3) provided that the conditions **(3a), (3b)** and **(3f)** are hold, and they are locally asymptotically stable points if and only if the conditions **(8a), (8b)** and **(8c)** are hold.

To understand the effect of varying each parameter on the dynamics of system **(1)** and to confirm our obtained analytical results, system **(1)** has been solved numerically using the hypothetical parameters values given by Eq. **(19)** and the following results are obtained:

1. It is observed that varying the parameters values m_2 *and* ε_1 do not have any effect on the dynamical behavior of system (1) and the system still approaches to positive equilibrium point.
2. Gradually decreasing the parameter D, which stand for dilution rate, causes destabilizing of the system (2.1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. However increasing the value of D, causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Thus there are two bifurcation points of this parameter.
3. Gradually decreasing the parameter N_0, which stand for the concentration of the input nutrient, causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Moreover further decreasing of N_0 leads to extinction of phytoplankton species too and the system approaches to E_0 on the boundary of N-axis. However increasing the value of N_0, causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. Thus there are three bifurcation points of this parameter.
4. Gradually decreasing the parameter a_1, which stand for the maximum specific ingestion rate, causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Moreover further decreasing of a_1 leads to extinction of phytoplankton species too and the system approaches to E_0 on the boundary of N-axis. However as increasing the value of a_1 the system (1) still approaches asymptotically to the positive point in the $Int.R_+^3$ that means increasing a_1 has a stabilizing effect on the system. Thus there are two bifurcation points of this parameter.
5. Gradually decreasing the parameter a_2, which stand for the half saturation constant of phytoplankton, causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. However increasing the value of a_2 causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Moreover further increasing of a_2 leads to extinction of phytoplankton species too and the system approaches to E_0 on the boundary of N-axis. Thus there are three bifurcation points of this parameter.
6. Gradually decreasing the parameter m_1, which stand for the conversion factor from dead phytoplankton species to nutrient, has no effect on the system and the solution still approaches to the positive equilibrium point. However increasing the value of m_1 causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. Thus there are two bifurcation points for this parameter.

7. Gradually decreasing the parameter k_1, which stand for the conversion rate from nutrient to phytoplankton , causes extinction in the phytoplankton and zooplankton populations and the system approaches to E_0 on the boundary of N-axis. However increasing the value of k_1 causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. Thus there are two bifurcation points for this parameter.

8. Gradually decreasing the parameter k_2, which stand for the conversion rate from phytoplankton to zooplankton, causes extinction in the zooplankton species and the system approaches to E_1 in the $Int.R_+^2$ of NP-plane. However increasing the value of k_2 causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. Thus there are two bifurcation points for this parameter.

9. Similarly the maximum attack rate β of zooplankton has the same behavior as that of the conversion rate from phytoplankton to zooplankton k_2.

10. Gradually decreasing the parameter β_1, which stand for the half saturation constant of zooplankton, causes destabilizing of the system (1) and the solution approaches asymptotically to periodic dynamic in the $Int.R_+^3$ that means the system still persist. Moreover further decreasing of β_1 leads to return back to stable case and again the system (1) approaches asymptotically to the positive point in the $Int.R_+^3$ that means decreasing β_1 less than a specific value has a stabilizing effect on the system. However increasing the value of β_1 causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Thus there are three bifurcation points for this parameter.

11. Gradually decreasing the parameter θ, which stand for the dilution rate, has no effect on the system and the solution still approaches to the positive equilibrium point. However increasing the value of θ causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Thus there are two bifurcation points for this parameter.

12. Similarly the liberation rate of toxin substance by the harmful a has the same behavior as that of the dilution rate θ.

13. Finally decreasing the parameter ε_2, which stand for the natural mortality rate of zooplankton, has no effect on the system and the solution still approaches to the positive equilibrium point. However increasing the value of ε_2 causes extinction of zooplankton species and the system (1) approaches to equilibrium point E_1 in the $Int.R_+^2$ of NP-plane. Moreover further increasing of ε_2 leads to return back to stable case and again the system (1) approaches asymptotically to the positive point in the $Int.R_+^3$ that means increasing ε_2 more than a specific value has a stabilizing effect on the system. Thus there are three bifurcation points of this parameter.

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