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A Comparative Study of Some Methods Based On Non – Polynomial Interpolant

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Abstract

The paper presents a comparative study of some numerical methods based on non-polynomial of great interest are the pioneer work of Lambert and Shaw (1965),Ibijola (1998),and Fatunla (1976).The class of formulae for the numerical solution of equation of the form.

$$y^1 = f(x, y),\ y(0) = y_0$$

In which the underlying interpolants which are rational functions are also considered and analyzed. Some comparative results are presented.

Keywords: Rational polynomials, Interpolants, Rational functions, Numerical Methods

1. Introduction

For the purpose of this work, we shall consider an initial value problem of the form,

$$y^1 = f(x, y),\ y(x_o) = y_0 \qquad (1)$$

Equation (1) is a common equation in science and Engineering. We shall assume that (1) has a solution that is locally represented by an interpolant which has a sample pole. Fatunla (1982) proposed a non-step rational method of the form;

$$y_{n+1} = \frac{yn^2}{yn - hy^1n} \qquad (2)$$

With the interpolating function;

$$f(x) = \frac{A}{1 + a1X} \qquad (3)$$

Throughout this paper, we shall assume that one may need to mention that various authors have suggested methods which can provide solution to equation (1) with a specific singularity. Krylov (1962) discusses the problem of singularity in relation to quadrature, and Fox (1957) refers to a similar situation in second order ordinary differential equations. Lambert and Shaw (1965) proposed a method that does not require previous knowledge of the nature and position of the singularity.

It is a well-known fact that finite difference scheme is one of the oldest and popular techniques for numerical solution of ordinary differential equations for the most equations in Mathematical Physics, Engineering and in some physical sciences, finite differences schemes have been designed and investigated both from the theoretical point of view, which is the convergence aspect and the practical point of view which is the consistency and stability point of view (Anguelov and Lubama 2003).

This paper is divided into three stages namely; stage one will revisit the theoretical aspect of Lambert and Shaw (1965). We shall examine critically, the theoretical aspect of the relationship between Fatunla (1976), Ibijola, (1998), and that of Lambert and Shaw (1965)

The last stage is the comparison of the result obtained using the above method

2. A Method for the Numerical Solution of y ¹= f(x, y) Based on a self – Adjusting Non Polynomial

In this section, we revisit the work of Lambert and Shaw (1965). We define the following

$$X_n = X_0 + nh, \; N = 0, 1, 2\ldots\ldots \quad (4)$$

We also assume that the theoretical solution of (1) is locally represented by

$$f(x) = \sum_{p=0}^{k} ap\, x^{a} + b/A + x/^{N} \quad (5)$$

Where a , b, A and N are real, K is a positive integer and N € {0, 1, 2,…, K},In the closed interval [a, b] and we define

$$h = \frac{b-a}{n} \quad (6)$$

The following Shaw (1965), Lambert and k+2 constraints, b, a_p (p=0, 1,… k) are regarded as an undetermined coefficients, while A and N are the parameters which control the nature of any positive singularity of F(X).

The following facts emerged from the work of Lambert and Shaw (1965)

S/N	Nature of N	Consequences
1	N<0(Negative)	F(X) has singularity at X= - A
2	N>1 (positive)	F(X) reverts to a polynomial

Table 1

In either interval x < - A and x > -A. F (x) will revert to a polynomial.

To obtain a numerical method from interpolating function (5), we assume that y_n is the numerical solution which approximates to y (x_n), the theoretical solution of equation (1) at x = x_n . If we now represent the solution of (1) in the closed interval [x_n, x_{n+1}] by (5) and Assume that,

(1) $F(X) = y_n$

(2) $$\frac{d^s f(x)}{dx^s} \quad nf_n^{\,s-1},\; s = 1, 2\ldots \quad (7)$$

We can obtain (7) by applying the popular chain rule to equation (1) to obtain

$$f_n(s) = \left(\left[\frac{d^s f(xnyn)}{dx^s n}\right] = \frac{df^{(s-1)}(xn,yn)}{dx} + f(Xn,Yn)^{d(s-1)} \quad \frac{(Xn,Yn)}{dyn}\right) \quad (8)$$

Equation (8) is true if all the derivatives exist

With the application of (7) and (8) to (1), then we have

$$y_{n+1} - y_n = \sum_{a=1}^{k} \frac{h^a}{a!} \; f^{(a-1)}{}_n + \frac{(A+xn)^{k+1}}{\propto^N} \; f^{(k)}{}_n[(1 + \frac{h}{A+xn})^a]^n - 1 - \sum_{a=1}^{k} \frac{\alpha^{a-1}}{a!}\left(\frac{h}{A+x_n}\right) \quad (9)$$

Where α_r^m = m (m-1) … (m-r), r is a non-negative integer.

Lambert and Shaw (1965) obtain the transaction error for the above method (a) as

$$\text{Truncation Error} = y_{n+1} - y(x_{n+1}) \tag{10}$$

$$\text{Truncation Error} = \sum_{q=1}^{\infty} Tq \frac{h^{k+q+1}}{(k+q+1)!} \tag{11}$$

On evaluation, it turns out to be
Where Tq is define as

$$Tq = -f_n^{(k+q)} + \frac{\alpha q - 1^{N-k-1}}{(A+xn)q} f_n^k \tag{12}$$

The values of the parameters A and N can also be obtained if we equate $T_1 = T_2 = 0$. This was obtained by Lambert and show as;

$$A(n) = x_n - \frac{Fn^{(k+1)} f n^k}{(f(f^{+1})^2 - f^{n(k)} f^{[k+2]}} \tag{13}$$

$$N(n) = K + 1 + \frac{Fn^{(k+1)} f n^k}{(f(f^{+1})^2 - f^{n(k)} f^{[k+2]}} \tag{14}$$

There are some advantages of the formulations of Lambert and Shaw (1965).formula (9)provide us with opportunity to determine the parameters which detect and control the nature of the solution of equation (1).The formula (13) and (14) (A(n) and N (n)) which is readily available enable us to carry out a direct computation of the parameters A and N based on derivatives of (x, y) at the points $x = x_n$, $y = y_n$. The value of k which is the degree of the polynomial of (5) can be obtained from (5).We must add here that in the practical application of (9) we must take into consideration the cost of computing higher derivatives so as to minimize the local truncation solution of equation (1) can be achieved either by increasing the order of the derivative of f(x, y) involved ,the first procedure ,with its inherent difficulties and possible instability ,can only be argued if the labour and financial cost in evaluating higher derivations of f(x,y) is unacceptable (c.f Lambert and Shaw 1965)

3. A method for the numerical solution of $y^1 = f(x, y)$ with oscillatory or exponential solution

In this section , we give a typical example of the formulation of Lambert and Shaw (1965).the method presented here is a typical one step method for the solution of equation (1).this type of construction was first reported in FATUNLA (1976) and further extended by IBIJOLA (1998).The resulting method is particularly well suited to initial value problem having oscillatory and exponential solution .For our construction, we shall assume that the theoretical solution to equation (1) can be written compactly as

$$F(X) = a_0 + a_1x + b \text{ Re } e^{\rho^{x+\mu}} \quad (15)$$

Where a_o, a_1 and b a real undetermined coefficients while P and N are complex parameters. In particular, we define

$$\rho = \rho 1 + i\, \rho 2 \quad (16)$$

And

$$\mu = i\sigma, i^2 = 1 \quad (17)$$

It is necessary at this stage to compare the interpolating function (15) with that of (5). Equation (15) can be written as

$$F(X) = \sum_{p=0}^{1} a_p x^p + b\text{Re}^{3x+u} \quad (18)$$

Here k=1, the parameters A and N are similar to ρ and μ we can say that (15) is a particular case of (5) equation (15) with definition (16) (17) transform to;

$$F(X) = a_0 + a_1 + be^{\rho ix} \cos(\rho_2 x + \sigma) \quad (19)$$

If we follow the argument of Lambert and Shaw (1965), we can obtain b and a (where a_0 drops out) as

$$\frac{f^{(1)}n}{Rn[\rho^2{}_1 - p_2)\cos\theta n - 2\rho_1\rho_2 \sin\theta n]} \quad (20)$$

$$a_1 = f_n - \frac{(\rho^1\cos\theta n - \rho^2 \sin\theta n) f^1 n}{Rn[\rho^2{}_1 - p_2)\cos\theta n - 2\rho_1\rho_2 \sin\theta n]} \quad (21)$$

With b and a determined, and with the assumptions in section (2.0), we have equation (1) as;

$$y_{n+1} = y_n + h\left\{ f_n - \frac{(\rho^1\cos\theta n - \rho^2 \sin\theta n) f^1 n}{Rn[\rho^2{}_1 - p_2)\cos\theta n - 2\rho_1\rho_2 \sin\theta n]} + \frac{F^1 \ell^{\rho IH} [Cos\theta n Cos(\rho_2 h) - Sin\,\theta n Sin(\rho^2 h) - Sin\theta n]}{[\rho^2{}_1 - p^2{}_2)\cos\theta n - 2\rho_1\rho_2 \sin\theta n]} \right\} \quad (22)$$

The determination of the parameters ρ_1, ρ_2 and σ follow the same procedure of Lambert and Shaw (1965) with some little modification. All we need is to define the logical truncation error as $T_{n+1} = y(_{xn+1}) - y_{n+1})$ is the theoretical solution at $x = x_{n+1}$ y_{n+1} is the numerical solution generated by (22). We also apply Taylor series expansion to y (x_{n+1}), Maclaurin series expansion to cos $(\rho_2 h)$ or sin $(\rho_2 h)$ and ℓ^{pIh}.
Also we assume that the coefficients of $h^2_?$ h^3 and h^4 varnish. With all these assumptions, and following the arguments by Lambert and Shaw (1962), we obtain the following simultaneous equations in $\rho_1.\rho^2\ \Theta_n$

$[\rho^{2}- \rho_2) \cos \Theta n- 2\rho_1 \rho_2 \sin \Theta n]\, fn - [\,_1 \cos \Theta n - \rho_2 \sin \Theta n]\, fn\,(1) = 0$ (23)

$[(\rho_1^3 - 3\rho_1\rho_2^2) \cos\theta_n - 3(\rho^2_1\, \rho_2 - \rho^3_2) \sin \theta_n] f_n - [\rho_1 \cos\theta_n - \rho_2 \sin \theta_n] f^{(2)}_n \equiv 0$ (24)

$[(\rho_1^4 - 6\rho^2_1\, \rho_2^2 + \rho_2^4) \cos\theta_n - 4(\rho^3_1\, \rho_2 - \rho_1\, \rho_1^3 \sin \theta_n] f_n - [\rho_1 \cos\theta_n - \rho_2 \sin \theta_n] f^{(3)}_n = 0$ (25)

The above equations can be solved using any of Newton like interactive schemes.

Using equation (23), we can obtain the local truncation error as

$T_{n+1} = \frac{h^5}{5![\rho 1 \cos\theta n + \rho 2 \sin \theta n]}$

$\{[(\rho_1^5 \rho_2^3 + 3\rho_1\, \rho_2^4) \cos\theta_n - (\rho_1^4 - 10\rho^2_1\, \rho_2^3 + 5\rho_1^4\, \rho_2) \sin \theta_n] f_n - [\rho_1 \cos\theta_n - \rho_2 \sin \theta n]\, f4^{(4)}$

$+ O(h^6)$ (26)

Ibijola (1998) extended the above formalism by extending the order K of the polynomial in (15). That is

$F(X) = \sum_{p=0}^{2} a_p x^p + b\, \mathrm{Re} e^{\rho_2 x + \mu}$ (27)

And obtain numerical methods of

$Y_{n+1} = y_n + [f_n - \{f_n^{(1)} - \frac{[(\rho^{1^2} \rho^{2^2} \cos\theta n - 2\rho^1\rho^2 \sin \theta n]\, f^{(2)} n\, [a + ah]}{(\rho^{1^2} - 3\rho^1\rho^{2^2} \cos\theta n + (\rho^{2^3} - \rho^{12}\rho^1) \sin \theta n}$

$-\{ \frac{(\rho_2 \cos\theta n - \rho_2 \sin \theta n]\, f^2 n\,]h}{(p_1^2 - 3\rho^1 p_2^{\,2} \cos \theta n + (p_2^3 - 3\rho_1^{\,2} p_2) \sin\theta n}$

$+ [f^{(1)}_n - [[(\rho^2_1 - \rho^2_2 \cos\theta_n - 2\rho_1\rho_2 \sin \theta_n] f^{(2)}_n]\, ah$

$+ [1/2 \{f \frac{[(p^2_1 p^2_2 \cos\theta n - 2p_1p_2 \sin \theta n]\, f^{(2)} n\, [2n+1]}{(p^3_1 - 3p_1p^2_2 \cos \theta n + (p^3_2 - p^2_1 p_2) \sin \theta n}$ (28)

$+ \frac{[e^{\,\rho\ldots}(\cos\theta n \cos(p^2 h) - \sin\theta n \sin\theta(p^2 h) - \cos\theta n\,]\, f^{(2)} n}{(p^3_1 - 3p_1p^2_2 \cos \theta n + (p^{32} - p^{21}p^2) \sin \theta n}$

(Which can be used to determine $\rho_1\ \rho_2$) and σ of the form

(1) $[((p_1^2 - p^2_2 \cos\theta n - 2p^1p^2 \sin \theta n]\, f^{(2)} n -$

$(p^{31} - 3p^1p^2_2) \cos\theta n + (3\, p_2^3 p^1 - p_2^3) \sin \theta n] f^{(1)} n$ (29)

(2) $(p^4_1 p^4_2 + 6p^2_1 p^2_2) \cos\theta n - (4\, p_1 p^3_2 - 4p^3_1 p_2) \sin \theta n$

$$p^{31} - 3p^1p^{22}\cos\theta n + (p^{32} - p^{21}p^2)\sin\theta n \quad (30)$$

(3) $$[(p^5{}_5 + p_1p^4{}_2 - 10p^3{}_1p^2{}_2)\cos\theta - (10p^2{}_{11}p^3{}_2 - 5p^4{}_1p^5{}_2)\sin\theta n$$

$$(p^3{}_1 - 3p_1p^2{}_2\cos\theta n + (p_2^3 - p_1^2p_2)\sin\theta n]f^{(4)}{}_n \quad (31)$$

And truncation error,

$$p^6{}_2 - 15p^2{}_1p^4{}_2 + 15p^4{}_1p^2{}_2 - p^6{}_1\cos\theta n + [(6p_1p^5{}_2 - 12p^3{}_1p^3{}_2 + 6p^2{}_1p_2$$

$$(p^3{}_1 - 3p_1p^2{}_2\cos\theta n + (p^3{}_2 - p^2{}_1p_2)\sin\theta n]]f^{(*)}n + 0(h^7)$$

Advantages of Ibijola (1998) over that of Fatunla (1976) is that

(a) More derivatives are involved in the evaluation and computation using Ibijola (1998) there by introducing more analytic functions in the course of computation

(b) The order of polynomial increased from one (Fatunla 1976) to two (Ibijola 1998).this eventually leads to increase in order of the method whole Fatunla (1976) is of order four ,Ibijola (1998) is of other five .

It must be stated here that with the present computer age, it is very easy to solve equation (29) to (31) with some existing computer packages in the numerical computations; a program was written in FORTAN 77 to solve equation (29) to (31). This produced the values of p_1 ,p_2 and θ_n as shown in the tables below

4. Comparison of Lambert and Shaw (1960), Fatunla (1976) and Ibijola (1998)

In this section, we present comparison of the methods far discussed. We present some result as applied to problem with singularity or points of discontinuities. The text problem is

$$y^1 = 1 + y^2,\ y(0) = 1 \quad (32)$$

This an initial value problem with theoretical solution of the form

$$y(x) = \tan\left(x + \frac{\pi}{4}\right), \quad (33)$$

Which has the point of singularity at x = $\pi/4$

In Lambert and Shaw (1965), the method was able to detect the point of singularity and also calculate or evaluate the parameters analytically. All that is needed in the derivatives of f(x,y), while in Fatunla (1976) and Ibijola (1998), more parameters are introduced which gave birth to solving simultaneous equations by Newton alternative method .This is an improvement over Lambert and Shaw (1965). More orders of the methods can be obtained using polynomial of higher order in (15). It is very clear that Fatunla (1979) and Ibijola (1988) can be consider as a family of Lambert and Shaw (1965)

The below table shows the results obtained by solving equation (32) using the method stipulated in section 2 and 3.

Lambert and Shaw (1965)

x	**y (x_n)**	y_n	**N (n)**	-A (n)
0.00	1.000000000	1.000000000	-2.000000000	+1.000000000
0.05	1.105355590	1.105355493	-1.675437652	+1.92080447
0.10	1.223048880	1.223048668	-1.459538749	1.871052433
0.15	1.356087851	1.356087497	-1.311939388	1.839170053
0.20	1.508497647	1.508497114	-1.209581045	1.818606761
0.25	1.685796417	1.685795650	-1.138345499	1.818606761
0.30	1.895765122	1.895764043	-1.089014193	1.797042894
0.35	2.149747640	2.149746124	-1.032812028	1.788793751
0.40	2.464962756	2.464960611	-1.032812028	1.788793751
0.45	2.868884028	2.868880924	-1.018291295	1.787043022
0.50	3.408223442	3.4082018788	-1.009367173	1.786114151
0.55	4.169364045	4.169356669	-1.004253830	1.785666280
0.60	5.331855223	5.331842457	-1.001612640	1.785478455
0.65	7.340436575	7.340410941	-1.000453697	1.785415020
0.70	11.681373800	11.681304752	-1.000071263	1.785400289
0.75	28.238252850	28.237817988	-1.000002095	1.785398727

TABLE 1: Take $y' = 1 + y^2$, y (0) = 1; $y(x) = \tan(x + \frac{\pi}{4})$

x	**Theoretical solution y (x_n)**	**Numerical solution y_n**	**Truncation error**
0.00	.1000000D+01	.1000000D+01	.4371139D-07
0.01	.1020203D+01	.1020202D+01	.9025532D-07
0.02	.1040822D+01	.1040822D+01	.1407313D-06
0.03	.1061875D+01	.1061875D+01	.1962785D-06
0.04	.10833880D+01	.10833879D+01	.2554902D-06
0.05	.1105356D+01	.1105355D+01	.3193564D-06
0.06	.1127823D+01	.1127822D+01	.389289D-06
0.07	.1150803D+01	.1150802D+01	.4657207D-06
0.08	.1174317D+01	.1174317D+01	.5478229D-06
0.09	.1198391D+01	.1198391D+01	.6373651D-06
0.10	.122304D+01	.1223048D+01	.7354548D-06

TABLE 2: Take $y' = 1 + y^2$, y (0) = 1; $y(x) = \tan(x + \frac{\pi}{4})$, ρ_1 = -10.6774200, ρ_2 =

8.5483870, θ = 1.2015280, h = 0.01

5. Observation Comments and Conclusion

We observe that the three methods namely Lambert and Shaw (1965), Fatunla (1976) and Ibijola (1998) can be classified as special case of the general one – step methods. in fact Lambert Shaw (1965) is a special case of Taylor series method .two parameters , namely N(n) and A(n) are needed in the computation procedure of Lambert and Shaw but in Fatunla (1976) we need the values of three parameters which are computed using three simultaneously equations. This involves more derivatives and there by introducing analytic function into the calculations. Ibijola (1998) used smaller mesh – size (h) to improve the accuracy of the mesh-(h) to improve the accuracy of the method.

Ibijola (1998) and Fatunla (1976) developed a method that is A-stable and L-stable, consistent having oscillatory or exponential solution .These methods can also solve problems in which there are singularities in the solution .one of the advantages of Lambert and Shaw (1965) over Fatunla (1965) and Ibijola (1998) if that the method can detect and manage the point of singularities .we must point out is that Fatunla (1976) has few derivatives to be evaluated. We have carried out details study of three families of methods that are capable of solving special initial value problems with oscillatory or exponential solutions. These methods can also detect points of singularities and solve those problems in which the solutions contains such points

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