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Normal Form for Local Dynamical Systems

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ABSTRACT
Normal Form is a theory that applies in the neighbourhood of an orbit of a vector field map. The theory provides an algorithmic way to generate a sequence of non-linear coordinate changes that eliminate as much non-linearity as possible at each order (where order refers to terms in Taylors series about an orbit).The normal form is intended to be the simplest form into which any system of the intended type can be transformed by changing the coordinates in a prescribed manner. Interestingly the form of non-linear that cannot be eliminated by such coordinate changes is determined by the structure of the linear part of the vector field map.

This section consists of some background knowledge, theorems and definitions necessary for understanding the concept of normal form for local dynamical systems. We briefly discuss the concept of ring of invariants and module of equivariants, and use the Groebner basis methods to compute a Groebner basis for the ideal of relations among the basic invariants.

1. INTRODUCTION

Here we collect together material needed for later chapters for easy reference.

1.1 Introduction to normal form theory

The basis for normal form theory is the observation that the vector field.

$$\dot{x} = Ax + a_2(x) + \ldots + a_j(x) + \ldots \tag{1.1.1}$$

is transformed into

$$\dot{y} = Ay + a_2(y) + \ldots + a_{j-1}(y) + b_j \ldots \tag{1.1.2}$$

By a change for co-ordinates

$$x = y + s_j(y),$$

Where s_j is homogenous of degree j, and

$$L_A S_j = a_j - b_j$$

with

$$(L_A v)x = v'(x)Ax - Av(x) \tag{1.1.3}$$

A normal form is computed by repeating such calculations for $j = 1 \ldots k$ up to some desired finite k, reverting to the original notation after each calculation. At each stage it is necessary to choose b_j so that $a_j - b_j \in im\, L_A$; then s_j exists. In order to proceed systematically, it is best to select a complement to $im\, L_A$ in each degree, and determine the b_j by projecting a_j into that complement. The problem, then, comes down to selecting a complement to $im\, L_A$. This is called the choice of a normal form style.

1.2 Literature review

The method of finding Stanley decomposition for equivariants of $N_{222....2}$ was first solved by Richard Cushman, Sanders and Neil White [1] using the method called "covariants of special equivariants." Their method involved creating a scalar problem that is larger than the vector problem. Our method begins by studying a scalar problem (of equivariants).

Mudrocks [6] used the method of S1(2) and inner product to find the Stanley decomposition of N_4.

Malonza [4] used also the method of SI(2) to find the SD for $N_{222...2}$. In our work we have used the inner product method to find the SD for N_{333} with the hope that it will generalize to $N_{33...3}$,

1.3 Invariants and Equivariants

Let $p_j\left(\Re^n,\Re^m\right)$denote the vector space for homogenous polynomials of degree j on $\Re^n$ with coefficients in $\Re^m$. Let P($\Re^n,\Re^m$ be the vector space of all such polynomials of any degree and let $p\left(\Re^n,\Re^m\right)$ be the space of normal power series. If $m = 1$, $p_*\left(\Re^n,\Re^m\right)$ becomes a ring of (scalar) formal power series on $\Re^n$, where $\Re$ denotes the set of real numbers. From the viewpoint of smooth vector fields, it is most natural to work with formal power series (Taylor series), but since in practice these must be truncated at some degree, it is sufficient to work with polynomials. Now, for any matrix A, let the Lie operator

$$L_A : P_j(\Re^n,\Re) \to P_j(\Re^n,\Re^n)$$

be as defined in equation (1.1.3) and the differential operator

$$D_{Ax} : P_j(\Re^n,\Re) \to P_j(\Re^n,\Re)$$

be defined by

$$(D_{Ax} f)(x) = f'(x)A(x) = (N(x).\nabla) f(x) \qquad (1.3.1)$$

In addition, notice that

$$L_A(fv) = (D_A f)v + fL_A v. \qquad (1.3.2)$$

Therefore, L_A is not a module homomorphism of $P(\Re^n,\Re^n)$ into itself but is a linear mapping. Recall that with every vector field $a(x) = (a_1(x), a_2(x), a_3(x)...a_n(x))$ there is an associated differential operator given by

$$D_{a(x)} = a_1(x)\frac{\partial}{\partial x_1} + ... + a_n(x)\frac{\partial}{\partial x_n} \qquad (1.3.3)$$

Acting on the space $P_j(\Re^n,\Re)$ of smooth (scalar) functions. Furthermore if v is a vector field and f is a scalar field, then $D_{v(x)} f$ is a scalar field called the derivation of f long (the flow of) $v(x)$. We will write D_A for $D_{A(x)}$, the derivation along the linear vector field Ax.

Observe that

$$D_A : P\left(\Re^n,\Re\right) \to p\left(\Re^n,\Re\right)$$

A function f is called an invariant of (the flow of) Ax if $\frac{\partial}{\partial t} f(e^{At}x)\big|_{t=0} = 0$ or equivalently $D_A f = 0$ or $f \in \ker D_A$. Since

$$D_A\left(f + g\right) = D_A f + D_A g \qquad (1.3.4)$$

$$D_A(fg) = fD_A g + gD_A f, \tag{1.3.5}$$

It follows that, if f and g are invariants, then so are $f+g$ and fg ; that is $\ker D_A$ is both a vector space over $\Re$, and also a subring of $p(\Re^n, \Re)$,known as the ring of invariants. Similarly a vector field v is called an equivariant of (the flow of) Ax, if $\frac{\partial}{\partial t} f(e^{At}x)\big|_{t=0} = 0$, that is $L_A v = 0$ or $v \in \ker L_A$ It turns out that the set of differential equations that have linear part and are in normal form to all orders possesses the structure of a module over a ring as the following lemma shows.

Lemma 1.3.1.*For any matrix A, the space of equivariants* $\ker L_A$ *is a module over the ring of invariants* $\ker D_A$.

Theorem 1.3.1.*Suppose that V is a finite dimensional vector space and $\{X, Y, Z\}$ is a triad of linear operators on V satisfying*

$[X, Y] = Z, [Z, X] = 2X, [Z, Y] = -2Y.$

Then the following properties hold:

P1. X and Y are nilpotent.

P2. Z is diagonalizable and has integer eigenvalues (called weights).

P3. $Ker\ X$ has basis consisting of weight vectors (eigenvectors of Z).

P4. Any basis $\{v_1, v_2, \ldots v_s\}$ of ker X consisting of weight vectors can be taken as a set of tops for Jordan chains for Y: that is, each sequence $v_j, Y v_j, Y^2 v_j, \ldots$ $v_j, Yv_j, Y^2 v_j, \ldots$ terminates with 0 and constitutes (an independent) Jordan chain for Y, so that the nonzero vectors of the form $Y^i v_j$ form a basis for V in particular, it follows that

$V = ker\ X \oplus im\ Y$

(the term chain tops suggests that Y be viewed as mapping down the chains.)

P5. The vectors $Y^i v_j$ are also weight vectors, with weights given by

$$Wt(Y^i v_j) = wt(v_j - 2i).$$

P6. The length of the chain headed by v_j is $wt(v_j) + 1$, implying that the bottom vector of each chain is $Y^{wt(v_j)} v_j$ and has weight $- wt(v_j)$.

P7. The action of X on the basis vectors is given by

$$X(Y^i v_j) pr(Y^i v_j)(Y^{i-1} v_j).$$

Where $pr(Y^i v_j)$ is the non zero constant

$$pr(Y^i v_j) = wt(v_j) + wt(Yv_j) + \ldots + wt(Y^{i-1} v_j).$$

The constant $pr(Y^i v_j)$ will be called the pressure on $(Y^i v_j)$ because it is the sum of the weights of the vectors above $(Y^i v_j)$ in its Jordan chain.

P8. The number of chain tops of weight $w \geq 0$ *equals)* $m(w)-m(w+2)$ *,where* $m(w)$ *is the multiplicity of w as an eigenvalue.*

1.4 Term orders

The set of power product is defined by $Tn = \{x^{\beta n},\ldots,x^{\beta n} | \beta_i \in N, i=1,\ldots,n\}$ $T^n = \{x_1^{\beta_n},\ldots x_n^{\beta_n} | \beta_i \in \mathrm{N}, i=1,\ldots n\}$

We denote $x_1^{\beta_n},\ldots x_n^{\beta_n}$ by x^β where $\beta = (\beta_1,\ldots\beta_n) \in \mathrm{N}^n$

Power product will refer to a product of the x_i variables and "term" will always refer to a coefficient times a power product. So every power product is a term (with coefficient 1) but a term is not necessarily a power product. We will also always assume that the different terms in a polynomial have different power products.

The ordering must extend the divisibility relations. That is if x^α divides x^β then we should have $x^\alpha \leq x^\beta$ or equivalently if $\alpha_i \leq \beta_i$ α_i for all $i=1,\ldots n$ then $x^\alpha \leq x^\beta$

The ordering of terms must be total, that is, given by $x^\alpha, x^\beta \in T^n$, exactly one of the following three relations must hold

$x^\alpha < x^\beta$, $x^\alpha = x^\beta$ *or* $x^\alpha > x^\beta$.

Term ordering must satisfy

I. The reduction $\rightarrow$ must stop after a finite number of steps.
II. Whenever $f \rightarrow +r$, the polynomial r must be such that the leading power product must be less than the leading power product of 'g'.

The following definitions will help to capture these conditions.

Definitions 1.4.1*By a term order on* T^n *we mean* $\mathcal{G}$ *total order < on* T^n *satisfying the following two conditions.*

I. $1 < x^\beta$ *for all* $x^\beta \in T^n, x^\beta \neq 1$
II. $x^\alpha < x^\beta$ *then* $x^\alpha x^\gamma < x^\beta x^\gamma$ *for all* $x^\gamma \in T^n$

Definition 1.4.2*We define the lexicographical order on* T^n *with* $x_1 > x_2,\ldots x_n$ as follows:

for $\alpha = (\alpha_1,\ldots\alpha_n), \beta = (\beta_1,\ldots\beta_n) \in \mathrm{N}^n$

We define

$x_\alpha < x_\beta$, $\Leftrightarrow$ the first coordinate α_1 and β_1 in α and β from left , which are different, satisfies $\alpha_1 < \beta_1$

So in the case of two variables x_1 and x_2, we have

$1 < x_2 < x_1 < x_2^2 < x_2^3 < \ldots < x_1 < x_2 x_1 < x_2^2 x_1 < \ldots < x_2^2 < \ldots$

If the lexicographical order with $x < y$ then *we* have

$1 < x < x^2 < x^3 < \ldots < y < xy < x^2 y < \ldots < y^2 < \ldots$

We will always denote the lexicography order by"lex"

Definitions 1.4.3*We define the degree lexicographical order on* T^n *with*

$x_1 > x_2 > \ldots > x_n$ *as follows for*

$\alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \in \mathrm{N}^n$

We define

$$x^\alpha < x^\beta \Leftrightarrow \begin{cases} \sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i \\ \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i \end{cases}$$

And $x^\alpha < x^\beta$ with respect to lex with $x_1 > x_2 > ,\ldots, > x_n$

So with this order we first order by total degree and break ties by the lex order. In the case of two variables x_1 and x_2 we have

$1 < x_2 < x_1 < x_2^2 < x_1 x_2 < x_1^2 < x_2^3 < x_1 x_2^2 < x_1^2 x_2 < x_1^3 < \ldots$

Or using the degree lexicographical ordering in $k[x, y]$ with $x < y$ we have

$1 < x < y < x^2 < x y < y^2 < x^3 < x^2 y < x y^2 < y^3 < \ldots$

We will always denote this order by “deglex”.

Definition 1.4.4 We define the degree reverse lexicographical order on T^n with

$x_1 > x_2 > \ldots > x_n$ as follows : for $\alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$

We define $x_\alpha < x_\beta \Leftrightarrow \begin{cases} \sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i \\ \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i \end{cases}$

And the first coordinates α_I and β_I in α and β from right , which are different, satisfy $\alpha_I > \beta_I$

We will denote this order by “deglex”.

We define

- $L(f) = x^{\alpha_1}$, the leading power product of *f*;
- $Lc(f) = a_1$, the leading coefficient of *f*;
- $Lt(f) = a_1 x^{\alpha_1}$,the leading termof *f*.

We define $lp(0) = lc(0) = 0$

Note that Lp, Lc and Lt are commutative that is $Lp(fg) = Lp(f),\ Lc(fg) = Lc(f)Lc(g)$ and $Lt(fg) = Lt(f)Lt(g)$. Also when we change the term order then $Lp(f), Lc(f)$ and $Lt(f)$ may change.

1.5 Division algorithm

In this section a division algorithm in $k(x_1,..., x_n)$ k (x_1 ,…, x_n) will be referred to as reduction process. When dividing f by f_1 ,…, f_s we want to cancel terms using the leading terms of the $f_1 s$ (so that the new terms that are introduced are smaller than the cancelled terms) and continue this process until it cannot be done any more.

Let us first look at the special case of the division of f and g, where $f \in k(x_1,..., x_n)$. We fix a term order $k(x_1,..., x_n)$.

Definition 1.5.1*If* $f \to F + r$ *and* r *is reduced with respect to F, then we call r a remainder for f with respect to F.*

Theorem 1.5.1 *Given a set of non zero polynomials* $F = \{f_1,..., f_s\}$ *and f in* $k(x_1,..., x_n)$ *the division algorithm produces polynomials* $U_1,...,U_s$, $r \in k[x_1,..., x_n]$ *such that*

$$f = U_1 f_1 + ...U_s f_s + r$$

With $r$reduced with respect to F and

$$Lp(f) = \max(Lp(U_i)Lp(f_i)), Lp(r)$$

1.6 Groebner basis

In this section we lay the theoretical foundation for computing Greobner basis.

Let $0 \neq f, g \in k[x_1,..., x_n]0$. Let $L = LCM(Lp(f), Lp(g))$ then the polynomial $S(f, g) = \frac{L}{Lt(f)} f - \frac{L}{Lt(g)} g$ is called the S-polynomial of f and g.

Example 1. Let $f = 2xy - y, g = 3y^2 - x$ with deglex term ordering with $y > x$

Then $L = y^2 x$ and $S(f, g) = \frac{y^2 x}{2yx} f - \frac{y^2 x}{3y^2} g = \frac{1}{2} yf - \frac{1}{3} xg = -\frac{1}{2} y^2 + \frac{1}{3} x^2$

If we reduce f using f_i , we get the polynomial $h_2 = f - \frac{x}{Lt(f_i)} f_i$ and if we reduce f using f_i we get

$h_1 = f - \frac{x}{lt(f_j)} f_j$. The ambiguity that is introduced is $h_2 - h_1 = f - \frac{x}{lt(f_i)} f_i - \frac{x}{lt(f_j)} = \frac{x}{L} S(f_i, f_j)$

Now that we have introduced S-Polynomial as a way to "cancel" leading terms and to account for the ambiguity in the division algorithm we can go ahead with a strategy for computing bases.

Theorem 1.6.1 *Let* $G = \{g_1,..., g_t\}$ *be a set of non zero polynomials in* $k(x_1,..., x_n)$*. Then G*

Is a Greobner basis for the ideal $I = < g_1,..., g_t >$ *if and only if for all* $i \neq j$

$$S(g_i g_j) \to G + 0$$

Let $k[x_1,..., x_n]$denote a polynomial ring over the field k.

We now state one of the main theorems of the Greobner basis method.

Theorem 1.6.2 (Buchberger's theorem)*A basis* $G = \{g_1,..., g_t\}$ *for an ideal 1 is a Groebner basis if and only if* $S(g_i, g_j) \to G + 0$ *for all* $i \neq j$ *, that is if and only if for all pairs* $i \neq j$ *the remainder on division of* $S(g_i, g_j)$ *by* **G** *is zero.*

Corollary 1.6.1 (Buchberger's first criterion)*Given a finite set* **G**$\subset k[x_1,\ldots, x_n]$, *suppose that we have* f, g $\in$ **G** *such that* GCD $Lm(f), Lm(g) = 1$ *(lm(f),* then $S(g_i, g_j) \to G + 0$.

With this criterion S-polynomials are guaranteed to reduced to zero without doing any calculations [9]

Corollary 1.6.2 (Buchberger's second criterion) *Given a finite set* $G \subset k[x_1,..., x_n]$, *suppose that we have* $f_i, f_j, f_k \in G$ *such that i < j < k. If*

- $S(f_i, f_j) = x^{\alpha} S(f_i, f_k) + x^{\beta} S(f_i, f_k)$
- $S(f_i, f_k) \to G + 0$
- $S(f_j, f_k) \to G + 0$

Then $S(f_i, f_j) \to G + 0$

Definition 1.6.1 Let $\wp : k[y_1,..., y_m] \to k[x_1,\ldots, x_n]$ be a ring homomorphism defined by

$$\wp : y_i \to f_i$$

Where $f_j \in k[x_1,\ldots, x_n], 1 \le i \le m$

Let $h \in k[y_1,..., y_m]$,say $h[y_1,..., y_m] = \sum_{\mu} C_{\mu} y_1^{\mu_1},..., y^{\mu_m}{}_m, C_{\mu} \in k, \mu = (\mu_1,..., \mu_m) \in N^m$ and only finitely many $C_{\mu}'s$ are non zero, then we have

$\wp(h) = h(f_1,... f_m) \in k[x_1,..., x_n]$.

Recall that the kernel of $\wp$ is the ideal

ker $\wp = \{h \in k[x_1,..., y_m] : \wp(h) = 0\}$that is $h \in \ker \wp h \in$ if and only if $h(f_1,... f_m) = 0$. The ker $\wp$ if often called the ideal of relations among the polynomials $f_1,\ldots, f_m$. This ideal will play an important role in later chapters.

The following theorem provides an algorithm for computing the kernel of $\wp$ or more precisely the Groebner basis for the kernel of $\wp$.

Theorem 1.6.3 Let $K = \langle y_1 - f_1,..., y_m - f_m \rangle \subseteq k[y_1,... y_m, x_1,... x_m]$ x_n]then ker $\wp$ = $K \cap k[y_1,... y_m]$

1.7 The full ring of invariants – an example

Let $\Re \subset \Re[x_1,\ldots, x_n]$ be a subring fo the ring of polynomials. Let $R_1,..., R_2$ be subrings of R and let $f_1,... f_s \in \Re[x_1,..., x_n]$ If

$\Re = R_1 f_1 \oplus R_2 f_2 \oplus ... \oplus R_s f_s$ 1.7.1

Then (1.7.1) is called a Stanley decomposition of R and every element of R can be written as

$$\sum_{i=I}^{s} g_i, f_i \quad 1.7.2$$

for $g_i \in R_i, i = 1,..., s$. One major application of theorem (1.3.1) is the calculation of $\ker x$, the ring of invariants. Four steps are required to complete the calculation in any example.

- Compute a finite set of invariants $I_1,..., I_s$ called the basic invariants, which suffice to generate all invariants up to some given degree j.
- Compute a Groebner basis for the ideal of relations among the basic invariants.
- From the Groebner basis, determine a Stanley decomposition for the ring R of polynomials in the basic invariants.
- From the Stanley decompositions, set up a two variable generation function called the table function (Hilbert function) , and use it to test that R is in fact all of ker X. if it is not, then not all of the basic invariants have been found. In that case, return back to the first step and increase the value of j.

We observe that the operators $\{X, Y, Z\}$ map each $P_j(\Re^n, \Re)$ for $i = 1, \dots, j$ to itself. So that $P_j(\Re^n, \Re)$ can be taken to be the vector space in theorem (1.3.1). Since Z is diagonal and $Z = D_z$, the monomials in (x_1 ,.., x_n) are in eigenvectors of Z, that is the weight vectors. As an examples we will find the ring of invariants ker X for the triad $[X, Y, Z\}$ with X = N_4. The associated differential operators are as defined in equation (1.3.4) for step one, the basic invariants can be shown to be

$$\alpha = x_1$$

$$\beta = 2x_1^2 - 3x_1 x_3$$

$$y = 4x_2^3 - 9x_1 x_2 x_3 + 9x_1^2 x_4$$

$$\delta = 9x_1^2 x_4^2 - 3x_2^2 x_3^2 - 18 x_1 x_2 x_3 x_4 + 6 x_1 x_3^3 + 8x_2^2 x_4$$

For step two, the relation satisfied by this invariant is

$$y^2 = 2\beta^3 + 9\alpha^2\delta$$

and there are no other relations as demonstrated by the table of functions. Thus, $y^2 - 2\beta^3 + 9\alpha^2\delta$ is the Groebner basis for the ideal of relations. Now consider the ring $\Re$ =R[α, β, y, δ] $\subset P_j(\Re^n, \Re^n)$ of polynomials in the known basic invariants. The representation of an element of $\Re$ as a polynomial is not unique because of the relation above, but this equation itself can be used to restore the uniqueness by excluding y^2 (or any high power of y). Thus, a Stanley decomposition of $\Re$ is:

$$\Re = R[\alpha, \beta, \delta] \oplus R[\alpha, \beta, \delta]\, y. \quad 1.73$$

Another way to say this is that any polynomial in $\Re$ can be written uniquely as

$$f(\alpha, \beta, \delta) + g(\alpha, \beta, \delta)\, y \quad 1.7.4$$

Where f and g are polynomials in three variables α , β and δ . The Stanley decomposition (1.7.4) can be abbreviated as $f.I + g.y$, f ad g will be referred to as coefficient functions, and I and y as Stanley basis elements.

To generate the table function of the Stanley decomposition, we replace each term in (1.7.3) by a rational function P/Q in d and w (for "d= degree in x " and "w= weight ") constructed as follows: for each basic invariant (α , β and δ) appearing in a coefficient function (f and g), the denominator will contain a factor $1-d^p w^q$, where p and q are the degree and weight of the invariant; the numerator will $d^p w^q$, where p and q are the degree and weight of the Stanley basis element of that term. When the rational functions P/Q from each term of the Stanley decomposition are summed up we obtain the table function T given by $T=\sum_i P/Q$. Thus, for this example, the table function is:

$$T=\frac{1+d^3w^3}{(1-dw^3)(1-d^2w^2)(1-d^4)}$$

1.7.5

The following lemma gives a method to check that enough basic invariants have been found.

Lemma 1.7.1*Let { X, Y, Z} be a triad of* $n\times n$ *matrices, let { X, Y, Z} be the induced triad, and suppose that* I_l *,…,*I_t*is a finite set of polynomials in ker X, let R be a subring of* $R[I_1,\ldots I_t]$*; suppose that the relations among the* $I_1,\ldots I_t$ *have been found, and that the Stanley decomposition and its associated table function* $T(d,w)$ *have been determined.*

Then $R=\ker X\subset P_j(\Re^n,\Re^n)$ if and only if

$$\frac{\partial}{\partial w}wT\bigg|_{w=1}=\frac{1}{(1-d)_n}$$

1.7.6

In the above example of N_4, $\Re$ = ker X, since

$$\frac{\partial}{\partial w}wT\bigg|_{w=1}=\frac{1}{(1-d)_4}$$

1.7.7

1.8 The basic isomorphism and Stanley decomposition of ker *X*

The goal of this is to describe a procedure for obtaining a Stanley decomposition for ker X given a Stanley decomposition for ker x where X and x are defined as in equations 1.3.3.

Let $N_{r_1,r_2,\ldots r_k}$ be an $n\times n$ block diagonal nilpotent matrix with upper Jordan blocks of sizes $r_1,r_2,\ldots r_k$,with $r_1,r_2,\ldots r_k$ =n.Let $R_i=r_1+r_2+\ldots+r_i$, i = 1,2, ... , k, so that $R_1,R_2,\ldots,R_k$ R_1, R_2 ,…, R_k are the row numbers of the bottom of the Jordan blocks. Define a map

$$\overline{\wp}:P(\Re^n,\Re^n)\to P(\Re^n,\Re^n)$$

by

$$\overline{\wp}(v_1,\ldots v_n)=(v_{R_1},\ldots,v_{R_k})$$

Clearly $\overline{\wp}$ is a homomorphism of modules over $P(\Re^n, \Re)$. Let $\wp$ be the restriction of $\overline{\wp}$ to ker , hence we have the following theorem.

Theorem 1.8.1 *The image of* $\wp$ is ker $x^{r_1}\oplus,\ldots,\oplus\ker x^{r_k}$ and the mapping $\wp$: ker X → $\ker X\to\ker x^{r_1}\oplus\ldots\oplus\ker x^{r_k}$ is an isomorphism of modules over the ring ker X.

Proof. Observe that if $f \in \ker X$ and g $\in P(\Re^n, \Re)$, then

$x(fg) = fxg$.

It follows that if g $\in \ker x^r$ (for any r) then, $fg \in \ker x^r$:that is ker x^r is a module over ker x. The rest of the proof will be cleared after considering the example below.

$$N_{222} = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 0 & 1 & & \\ & & 0 & 0 & & \\ & & & & 0 & 1 \\ & & & & 0 & 0 \end{bmatrix}$$

In this case if follows that $\overline{\wp}(v_1,\dots,v_6) = (v_1, v_2, v_6)$, and if v $\in \ker x$ then $x^{v_1} = 0$, $x^{v_2} = v_1$, $x^{v_3} = 0$, $x^{v_4} = 0$, $x^{v_5} = 0$, $x^{v_6} = v_5$. These conditions imply that; $x^{2v_2} = 0$, $x^{2v_4} = 0$, $x^{2v_6} = 0$, so that, $\overline{\wp}(v) = (v_2, v_4, v_6) \in \ker x^2 \oplus \ker x^4 \oplus \ker x^6$

and shows that $v \in \ker x$ can be constructed from v_2, v_4, v_6 by the reconstruction.

$$map\, \wp^{-1}(v_2, v_4, v_6) = \begin{bmatrix} x^{v_2} \\ v_2 \\ x^{v_4} \\ v_4 \\ x^{v_6} \\ v_6 \end{bmatrix}$$

Thus $\wp$ is invertible. Since it is a module homomorphism, it is an isomorphism.

Lemma 1.8.1 If $h \in p(\Re^n, \Re)$ *belong to the ker* $D_N^{\,r_s}$ *, then the vector polynomial* $v_{\{s,h\}}$ *defined by* $v_{\{s,h\}} = \sum_{i=0}^{r_s-1} (D^i\, Nh) e_{R_s - 1}$

Belong to ker L_N. For instance if

$$N_{2,3} = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{bmatrix}$$ Then $s \in \{1,2\}, r_1 = 2, R_1 = 2, r_2 = 3, R_2 = 5$

$$v\{1,h\} = \begin{bmatrix} D_n h \\ h \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ for } h \in \ker D_N^2 \text{ And } v_{\{2,h\}} = \begin{bmatrix} 0 \\ 0 \\ D_N^2 h \\ D_N h \\ h \end{bmatrix}, \quad \text{for } h \in \ker D_N^2$$

Definition 1.8.1*If J is a monomial ideal, the monomials belonging to J are called nonstandard monomials. The standard monomials with respect to this ideal are the monomials that do not belong to it.*

The following Lemma forms the basis for obtaining the Stanley decomposition for ker X.

Lemma 1.8.2 Let R be any subring fo ker X generated by homogenous polynomials I_1 ,… I_s in $X = x_1, x_2, ...x_m$ which are weight vectors for the triad {X, Y, Z},and let $\Re_{ik}$ be the vector subspace of $\Re$ consisting of polynomials homogenous in x degree *I* and weight *k*. Let a Groebner basis for the relations of I_I ,…, I_s , be selected. Then:

1. *The standard monomials in I_I ,…, I_s (with respect to the given Groebner basis) having degree i($in\, x$) and weight k form a basis for* $\Re_{ik}$
2. *If* $\Re = \ker X$ *, the standard monomials of degree i form a set of chain tops for the chains in* $p_i = (\Re^n, \Re).$

According to this lemma , the chain tops of $P(\Re^n, \Re)$ under the triad {X, Y, Z} may be taken to be the standard monomials in the basic invariants I_I ,…, I_s , with respect to the given Stanley decomposition of ker X. The chains under the chain tops can be obtained by repeated applications of Y, and a vector space basis for ker X^r can be obtained by computing the iterates down to depth *r*.

Let *f* be a standard monomial of degree *j* (in x) and let $\gamma^i f$ be a non zero entry in the chain under *f*, we define g $\in P_i(\Re^n, \Re)$ to be a replacement of $\gamma^i f$ if $x^i g^i$ is an non zero multiple of *f*.

Lemma 1.8.3 *If a vector subspace* $V \subset \ker X^r$ *contains a replacement for every chain element to depth r, then* $V = \ker X^r$

Lemma 1.8.4 L*et f be a standard monomial. A replacement for* $y^r f$ *can be found by placing r copies of y arbitrary in front of the various factors of f as long as the result is not zero*.

Recall that the maximum power of y that can be applied to an invariant equals the weight (length-1) of the invariant. By the above lemma, think of each standard monomial as being written without powers, so that $I_2^{\,3} I_3^{\,2}$ appears as I_2, I_2, I_2, I_3, I_3 . Apply y to the last factor until the power of y equals its weight , then to the factor before that, and so on, stopping when the total number of factors of y reaches $r-1$ (for the construction of replacement for the chain elements under a standard monomial to depth r.) Each replacement constructed in this manner contain two pats , a prefix which is itself a standard monomial and contains no y and a suffix, which begins with the first occurrence of y.It is clear that no basic invariant of weight zero(length one) can appear in a suffix ; we call such invariant trivial.

The next step is to describe the set of prefixes that can occur with any given suffix. Let S be a suffix and let g be the standard monomial that results from deleting all occurrences of y in S; we call g a stripped suffix. Let f be any other standard monomial. Then $f\,S$ occurs as a replacement (that is, f is a prefix for S) precisely when the following two conditions are satisfied:

1. *f*, g is a standard monomial (so that *f*, g occurs as a chain top);

2. The factors *f*, g are correctly ordered, equivalently, the final factor of *f* either precedes or equals the first factor of g.

Let $m_1, m_2, ..., m_p$ be the leading monomials of the Greobner basis for the basic invariants $I_1, ..., I_s$ given g, the condition (1) for fg to be standard is that f not be divisible by any of the monomials $m_i = m_i / \gcd(m_i, g)$. Let the first basic invariant appearing in g be $I_{i(g)}$. Then the condition (2) of f g to be correctly ordered, is that f not divisible by $I_{i(g)-1}, \dots I_1$ (ordering the basic invariants by($I_i < I_j$ if $j < i$). Therefore the prefix monomials f associated with the given stripped suffix g are the standard monomials with respect to the (new) ideal $\langle m'_1, ..., m'_p, I_{i(g)+1}, ... I_1 \rangle$. Now let f be the prefix monomial associated with a given suffix S, then the collection of polynomials which are linear combination of such prefix monomials for a given suffix S is a ring, called the prefix ring for S, which has a Stanley decomposition (defined by its standard prefix monomials). This Stanley decomposition will ba denoted by $P(S)$, the Stanley decomposition of the prefix ring for the suffix S. We conclude this section by the following theorem.

Theorem 1.8.5 A Stanley decomposition for ker X^r is given by

$$\ker X^r = SD(\ker X) \oplus (\bigoplus_s P(S)S),$$

Where:

1. SD (ker X) is the Stanley decomposition of the invariant ring by a particular Groebner basis for the relations among the invariants;
2. The sum ranges over all suffices S of depth $\leq r$, suffices being defined as in 1.8.4 using a selected ordering of the basic invariants; and *P(S)* is the Stanley decomposition of the prefix ring for S defined above, using as standard monomials those determined by the same Groebner basis used to obtain $SD\ (ker\ X)$.

2.RING OF INVARIANTS

2.1 Introduction

A single Takens- Bogdanov system has the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ + quadratic terms + cubic terms + } \quad \text{.Let } N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Our goal is to describe the equivariants (normal form) for the system

$$x = N_{33}\, x + h.o.t$$

Where $x \in \Re^6$ and $N = \begin{bmatrix} N_3 & \\ & N_3 \end{bmatrix}$

2.2 Creating the triads of operators

Given the nilpotent matrix N in the upper Jordan form, the first step is to create M and H, such that M is a nilpotent matrix with the same block structure as N but is modified into a lower Jordan form, H is diagonal and

$[N, M] = H, [H, N] = 2N, [H, M] = -2N$, where $[A, B]$ is a lie bracket of matrices (and of linear operators in genera) should agree with the usual commutator brackets so that $[A, B] = AB - BA$ rather than the negative of this.

“Modified lower Jordan form “means that the only non zero entries of M lie in the subdiagonal(just as for a nilpotent matrix in lower Jordan form, but the entries are not necessarily equal to one. Here we shall see N_3 and N_{33}.

$$N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

In order to give the procedure for obtaining M and H, it is only necessary to tell how to obtain numbers in the diagonal of H and sub diagonal of M.

The construction is done block wise and the entries of H are the built first.

The procedure is as follows [3]:

1. For a block of size r in N, the diagonal entries in the corresponding block of H begin with $r-1$ and decrease by 2 at each step until $1-r$ is reached at the bottom of the block.
2. The entries in the sub diagonal of the corresponding block of M is partial sums of entries in H the first entry in M is the first entry in H, the second is the sum of the first two entries in H and so forth until the block is completed. Having obtained the triad $\{N\ M\ H\}$ in this way, we create two additional triads $\{X, Y, Z\}$ and $\{x, y, z\}$ as follows
   $X = M^* = M^T, Y = N^* = N^T, Z = H^T = H, x = D_y,\ y = D_x$ *and* $z = D_z$

For N_3

$$X = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad Z = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$[X,Y] = XY - YX = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = Z$$

$$[Z,X]=ZX-XZ= \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = 2X$$

$$[Z,Y]=ZY-YZ= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} = -2Y$$

The second is a triad of differential operators, which also satisfy

$[X, Y] = Z$, $[Z, X] = 2X$, $[Z, Y] = -2Y$

For N_3

$$x = D_Y = x_1 \frac{\partial}{\partial X_2} + x_2 \frac{\partial}{\partial X_3}$$

$y = D_X = 2x_2 \frac{\partial}{\partial X_1} + 2x_3 \frac{\partial}{\partial X_2}$

$z = D_z = 2x_1 \frac{\partial}{\partial X_1} + 2x_3 \frac{\partial}{\partial X_3}$

The different operator $\{x, y, z\}$ map each vector space of homogenous scalar polynomials F_{j+1}^{n} into itself with x and y being nilpotent and z semi simple, the eigen vectors of Z (called weight vectors, but are actually scalar functions) are monomials x^m and the associated eigenvalues (called weights) are $\langle m, \mu \rangle$, where $\mu = (\mu_1, \mu_2 \ldots \mu_n)$ are the eigenvalues (diagonal elements) of Z, that is Z (x^m) = $\langle m, \mu \rangle$ x^m

For the case of N_3

$$Z(x_1^{m_1} x_2^{m_2} x_3^{m_3}) = (2m_1 - 2m_3)(x_1^{m_1} x_2^{m_2} x_3^{m_3})$$

2.3 Generating Jordan chain of y

The procedure for generating the Jordan chains of y on any given vector space F_{j+1}^{n}., under the assumption that $A = N$, breaks into the following steps

1. Construct a weight table for z on F_{j+1}^{n}.
2. Construct the top weight list derived from the weight table.
3. Determine the weight vector that fills each position on the top weight list.
4. The vectors found in the previous steps will be the tops of a set of Jordan chains for the nilpotent operator y **.**Apply y to these top weight vectors to generate the Jordan chains of y . Make a table of these chains, in which y is represented as mapping downwards.
5. The vectors in the table just describes will be notified Jordan chains for nilpotent operator x = $D_Y = DN^3$, regarded as a mapping downward.

The steps described above will be discussed at greater length below using the example of N_3, later on N_{33} will be discussed.

Step 1 is to construct a weight table of Z on F_{j+1}^{n}. This is done by making a list of all the multi indices '*m*' with $|m| = j+1$. (so that $x^m \in F_{j+1}^{n}$) computing $\langle m, \mu \rangle$ for each m and the corresponding multiplicity with which each eigen value of z occurs. It is only necessary to record only the positive weight because the set of weights is symmetrical around zero.

For the case of N_3, with j = 0, we have $m_1 + m_2 + m_3 = 3, \quad 2m_1 - 2m_3 = w.$

m_1	**m_2**	**m_3**	**M_4**
3	0	0	6
2	1	0	4
2	0	1	2
1	1	1	0
0	3	0	0
0	2	1	-2
1	2	0	2
1	0	2	-2
0	0	3	-6

Weight	6	4	2	0
Multiplication	1	1	2	2

For j = 1

For j = 1$m_1 + m_2 + m_3 = 2$

$2m_1 - 2m_3 = w$

m_1	**m_2**	**m_3**	**m_4**
0	2	0	0
2	0	0	4
0	0	2	-4
1	1	0	2
1	0	1	0
0	1	1	-2

Weight	4	2	0
Multiplication	1	1	2

For j = 2

$m_1 + m_2 + m_3 = 1$

$2m_1 - 2m_3 = w$

m_1	m_2	m_3	w
1	0	0	2
0	1	0	0
0	0	1	-2

Weight	2	0
Multiplication	1	1

Step 2 is to construct top weights list from the weight table as follows.

For each non negative weight 'w' occurring in the weight table, compute the multiplicity of w minus multiplicity of $w = 2$ (if $w = 2$ does not occur in the weight table its multiplicity is zero). The result is the number of times w occurs in the top weight list.

The complete result for N_3 with $j = 0 \ldots 3$ degree 1 through 3 with vertical line separating the degrees is found to be 2| 4 0| 6 2|

Step 3 is to find weight vector (or weight polynomial, that is a scalar polynomial that is an eigen vector of Z) that fills each position in the top weight list, meaning that it has the required degree and eigen value (weight).

There are several techniques available to find polynomials that fill the required positions. Two of these techniques which we call kernel principle and the multiplication principle are especially important and are sufficient to handle every problem. So we limit ourselves to these methods. Other methods are cross – section method and the method of tran-vectants.

2.4 The kernel principle
To find the weight vectors of a given weight and degree. It suffices to take any basis for the kernel of operator X , regarded as an operator on the space spanned by the monomial of the specified weight and degree.

2.5 Multiplication principle
Any product of weight polynomial is a weight polynomial; the degree and weight of the product is the sum of degree and weight of the factors.

So we have

$$\begin{array}{c|cc|cc} 2 & 4 & 0 & 6 & 2 \\ \alpha & \alpha^2 & \beta & \alpha^3 & \alpha\beta \end{array}$$

Where $\alpha = x_1,\ \beta = x^2{}_2 - 2x_1x_2$

α and β are referred to as basic invarints for N_3 obtained by the kernel principle .

Using the same procedure we can now find the basic invariants for N_{33}.

$$N = N_{33}\begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \end{bmatrix} M = M_{33} = \begin{bmatrix} 0 & 0 & 0 & & & \\ 2 & 0 & 0 & & & \\ 0 & 2 & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & 0 & 2 & 0 \end{bmatrix}$$

$$H = H_{33}\begin{bmatrix} 2 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & -2 & & & \\ & & & 2 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & -2 \end{bmatrix} \quad X = M^{T}\begin{bmatrix} 0 & 2 & 0 & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 2 & 0 \\ & & & 0 & 0 & 2 \\ & & & 0 & 0 & 0 \end{bmatrix}$$

$$Y = N^{T=}\begin{bmatrix} 0 & 0 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix} \quad Z = H^{*} = H^{T}\begin{bmatrix} 2 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & -2 & & & \\ & & & 2 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & -2 \end{bmatrix}$$

$$Z(x_1^{m_1} y_1^{m_1} z_1^{m_3} x_2^{m_4} y_2^{m_5} z_2^{m_6}) = 2m_1 - 2m_3 + 2m_4 - 2m_6(x_1^{m_1} y_1^{m_1} z_1^{m_3} x_2^{m_4} y_2^{m_5} z_2^{m_6})$$

$$x = D_Y = x_1 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial z_1} + x_2 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial z_2}$$

$$y = D_x = 2y_1 \frac{\partial}{\partial x_1} + 2z_1 \frac{\partial}{\partial y_2} + 2y_2 \frac{\partial}{\partial x_2} + 2z_2 \frac{\partial}{\partial y_2}$$

$$z = D_z = 2x_1 \frac{\partial}{\partial x_1} - 2z_2 \frac{\partial}{\partial z_2} + 2x_2 \frac{\partial}{\partial x_2} - 2z_2 \frac{\partial}{\partial z_2}$$

For j=0, $h \in f_1^{6}$

$$m_1 + m_2 + m_3 + m_4 + m_5 + m_6 = 1$$

$$2m_1 - 2m_3 + 2m_4 - 2m_6 = w$$

m_1	m_2	m_3	m_4	m_5	m_6	w
1	0	0	0	0	0	2
0	1	0	0	0	0	0
0	0	1	0	0	0	-2
0	0	0	1	0	0	2
0	0	0	0	1	0	0
0	0	0	0	0	1	-2

Weight	2	0
Multiplication	2	2
Top weight list	2	2

For j = 1

$$m_1 + m_2 + m_3 + m_4 + m_5 + m_6 = 2$$

$$2m_1 - 2m_3 + 2m_4 - 2m_6 = w$$

m_1	m_2	m_3	m_4	m_5	m_6	w
2	0	0	0	0	0	4
0	2	0	0	0	0	0
0	0	2	0	0	0	-4
0	0	0	2	0	0	4
0	0	0	0	2	0	0
0	0	0	0	0	2	-4
1	1	0	0	0	0	2
1	0	1	0	0	0	0
1	0	0	1	0	0	4
1	0	0	0	1	0	2
1	0	0	0	0	1	0

Weight	4	2	0
Multiplicity	3	4	7
Top weight list	4	2	0
Multiplicity	3	1	3

The following lemma helps us to know that no term has been omitted.

Lemma 2.1 The dimesions of vector spaces $f_j^{\,n}$ and $V_j^{\,n}$ are given by the binomial coefficients:

dim $f_j^{\,n} = \binom{n+j-1}{j}$ and dim $V_j^{\,n} = \binom{n(n+j)}{j+1}$

Using the kernel and multiplication principle we have the following basic invariants.

$$f_1 = x_1,\ f_2 = x_2,\ f_4 = y_1^{\,2} - 2x_1z_1,\ f_5 = y_1^{\,2} - 2x_2z_2,\ f_6 = y_1y_2 + x_1z_2 - x_2z_1$$

2.6 Groebner basis for the invariants

In this sections, we find the Groebner basis for the basic invariants, associated with the ring of invariants ker x for the inner product normal form for the system.

$$x = N_{33} + h.o.t$$

Elimination

Consider two sets of variables($x_1...,x_n$) and ($y_1,\dots, y_n$). Assume that the power product in the x variable and power product in the y variables are ordered by term order Lx, Ly respectively. We define a term order L on the power products in the x, y variables as follows.

Definition 2.1 Eor x_1 and x_2 power products in x variables and y_1 and y_2 power products in the y variables we define

$$X_1Y_1 < X_2Y_2 \Leftrightarrow \begin{cases} X_1 <_x X_2 \\ or \\ X_1 = X_2 \, and \ \ Y_1 <_Y Y_2 \end{cases}$$

This term order is called an elimination order with the x variables larger than the y variables.

We will use the theory of elimination to determine.

I. The kernel φ or more precisely basis for the kernel of φ.

II. The image of φ or more precisely an algorithm to decide whether a polynomial f is in the image of φ and an algorithm to decide whether φ is onto.

We now have an algorithm for computing a Groebner basis for the Kernel of φ. We first compute a Greobner basis **G.**

Let $\phi : Q(a,b,c,d,e,f) \to Q[x,y,z]$ be the map defined by

$$a \to x_1,\ b \to x_2,\ c \to x_1y_2 - y_1x_2,\ d \to y_1^2\text{-}2x_1z_1,\ e \to y_2^2\text{-}2x_2z_2,\ f \to y_1y_2 + x_1z_2 - x_2z_1$$

We compute the Groebner basis **G** for the ideal

$$k = \left\langle a - x_1, b - x_2, c - x_1y_2 - y_1x_2, d - y_1^2 + 2x_1z_1, e - y_2^2 + 2x_2z_2, f - y_1y_2 + x_1z_2 - x_2z_1 \right\rangle$$

with respect to deglex term ordering on x, y, z with $x_1 < y_1 < z_1 < x_2 < y_2 < z_2$ and the deglex term ordering on a,b,c,d,e,f variables with $a < b < c < d < e < f$. with an elimination order between them with x,y,z variables larger than the a,b,c,d,e,f variables.

$f_1 = x_1 - a$, $f_2 = x_2 - b$, $f_3 = x_1y_2 - y_1x_2 - c$, $f_4 = y_1^2\text{-}2x_1z_1\text{-}d$, $f_5 = y_2^2\ \text{-}2x_2z_2\text{-}e$,
$f_6 = y_1y_2 + x_1z_2 - x_2z_1 - f$

By collorary 1.6.2 (Buchbergers' first criterion)

$$S(f_1,f_2)\xrightarrow{G}0, S(f_1,f_4)\xrightarrow{G}0, S(f_1,f_5)\xrightarrow{G}0, S(f_1,f_6)\xrightarrow{G}0, S(f_2,f_3)\xrightarrow{G}0$$
$$S(f_2,f_5)\xrightarrow{G}0, S(f_3,f_4)\xrightarrow{G}0$$

By theorem 1.6.2(Buchbergers theorem)

$$S(f_5,f_6) = \frac{y_1y_2^2}{y_2^2}\left(y_2^2 - 2x_2z_2 - e\right) - \frac{y_1y_2^2}{y_2^2}\left(y_1y_2 + x_1z_2 - x_2z_1 - f\right)$$

$$= y_1y_2^2 - 2x_2z_2y_1 - ey_1 - y_1y_2^2 - x_1z_2y_2 + x_2z_1y_2 - fy_2$$

	$z_1 + z_1y_2$
f_3	$x_1z_2y_2 - 2x_2z_2y_2 + x_2z_1y_2 + fy_2 - ey_1$ $-$ $x_1z_2y_2 - 2x_2z_2y_1 - cz_2$
f_2	$x_2z_1y_2 + cz_2 + fy_2 - ey_1$ $-$ $x_2z_1y_2 - bz_1y_2$
	$bz_1y_2 + cz_2 + fy_2 - ey_1$

Or

$$S(f_5, f_6) = x_1z_2y_2 - 2x_2z_2y_2 + x_2z_1y_2 + fy_2 - ey_1 \xrightarrow{f3} x_2z_1y_2 + cz_2 + fy_2 - ey_1 \xrightarrow{f2} bz_1y_2 + cz_2 + fy_2 - ey_1$$

Note now that in the last polynomial namely $bz_1y_2 + cz_2 + fy_2 - ey_1$ no term is divisible by the leading power of the given polynomials and so this procedure cannot continue.

In the same way we obtain the following remainders.

$$f_7 = abz_2 - cy_2 + b^2z_1 + ea - bf,\ f_8 = a^2z_2 - abz_1 + cy_1 + af - d^2b$$
$$f_9 = ay_1z_2 + by_1 + dy_2 + 2cz_1 - fy,\ f_{10} = by_2z_1 + cz_2 + fy_2 - ey_1$$
$$f_{11} = a^3e + b^2d - c^2$$

By Theorem 1.5.1

We get **G**= $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\}$

Is a Groebner basis. Therefore the Grobner basis for the ker φ is

$$\mathbf{G} \cap k[a,b,c,d,e,f] = \{a^2e + b^2d - c^2\}$$

3.0 RECOMMENDATIONS

It is recommended that more research work be directed to finding the Stanley decomposition for the normal form module $\ker x$ and the Stanley decomposition of $\ker x^r$ given $\ker x$.

4.0CONCLUSION

The Groebner basis has been found and we observe that:

1. The method is Local in the sense that the coordinate transformations are generated in the neighbourhood of a known solution.
2. In general the coordinate transformations will be on non linear functions of the dependent variable.However the important point is that these coordinate transformation are found by solving a sequence of linear problems.

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