

Adjoint Operator in Probabilistic Hilbert Space

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Abstract: The purpose of this paper is to give a definition for the adjoint operator in Probabilistic Hilbert Space and its properties by using Riesz Representation Theorem.

Keywords: Probabilistic Hilbert Space; Adjoint Operator; properties of adjoint operator.

Introduction

The notion of probabilistic inner product spaces can be considered as the generalization of that of inner product spaces. The definition of these spaces has been introduced in [1]. Induced norms by these spaces may have very important applications in quantum particle physics particularly in connections with both string and E-infinity theories [2], [3]. The definition of probabilistic Hilbert Space also introduced in [4]. The definition of the adjoint operator in the ordinary Hilbert Space defined in many books and papers by many authors [5], [6]. Self-adjoint operators are used in functional analysis and quantum mechanics. In quantum mechanics their importance lies in the Dirac–von Neumann formulation of quantum mechanics, in which physical observables such as position, momentum, angular momentum and spin are represented by self-adjoint operators on a Hilbert space. Our main purpose of this paper is how to give the definition for the adjoint operator in Probabilistic Hilbert Space and some of its properties by using the Riesz representation theorem that has been introduced in [4]. We also published paper in the name (Some Results in Modified Probabilistic Hilbert Space) in Journal of the College Science for Women in 2014.

1. Basic definitions

Definition (1-1) [1]

Let $\mathbb{R} = (-\infty, \infty)$, define the set D to be the set of all left continuous distributions such that:

$$D = \{F: F \text{ is left continuous distribution function}\}$$

$$\text{and let } H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

be distribution function which belongs to D .

Definition (1-2) [4]

A convolution of two functions f, g is define as follows:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau ;$$

$$= \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau \quad (\text{commutative})$$

Definition (1-3) (PIP-space) [4]

Let E be a real linear space and let $F: E \times E \rightarrow D$ be a function, then the Probabilistic inner product space is the triple $(E, F, *)$ where F is assumed to satisfy the following conditions:-

$(F_{x,y}(t))$ will represent the value of $F_{x,y}$ at $t \in R$)

$$(PI - 1) \quad F_{x,x}(0) = 0$$

$$(PI - 2) \quad F_{x,y} = F_{y,x}$$

$$(PI - 3) \quad F_{x,x}(t) = H(t) \leftrightarrow x = 0$$

(PI - 4)

$$F_{\lambda x,y}(t) = \begin{cases} F_{x,y}\left(\frac{t}{\lambda}\right) & , \lambda > 0 \\ H(t) & , \lambda = 0 \\ 1 - F_{x,y}\left(\frac{t}{\lambda} +\right) & , \lambda < 0 \end{cases}$$

where h is real number, $F_{x,y}\left(\frac{t}{\lambda} +\right)$ is the right hand limit of $F_{x,y}$ at $\frac{t}{\lambda}$

(PI - 5)

if x, y are linearly independent then

$$F_{x+y,z}(t) = (F_{x,z} * F_{y,z})(t)$$

Where

$$(F_{x,z} * F_{y,z})(t) = \int_{-\infty}^{\infty} F_{x,z}(t - u)dF_{z,y}(u)$$

Note: If x, y are linearly dependent then let $y = \alpha x$, α is scalar ($\in R$) then

$$x + y = x + \alpha x = (1 + \alpha)x, \text{ now let } h = \alpha x \text{ then}$$

$$x + y = hx$$

$$F_{x+y,z}(t) = F_{\lambda x,z}(t) = \begin{cases} F_{x,z}\left(\frac{t}{\lambda}\right) & , \lambda > 0 \\ H(t) & , \lambda = 0 \\ 1 - F_{x,z}\left(\frac{t}{\lambda} +\right) & , \lambda < 0 \end{cases}$$

which is $(PI - 4)$.

Then $(E, F, *)$ is called Probabilistic Inner Product Space.

Definition (1-4) [4]

A PIP-space $(E, F, *)$ is called with *Mathematical expectation* if:

$$\int_{-\infty}^{\infty} t dF_{x,y}(t) < \infty \quad , \forall x, y \in E$$

Theorem (1-5) [4]

Let $(E, F, *)$ be PIP – space with *Mathematical expectation* then

$$\langle x, y \rangle = \int_{-\infty}^{\infty} t dF_{x,y}(t) \quad , \forall x, y \in E$$

$(E, \langle \cdot \cdot \rangle)$ is inner product space, $(E, \|\cdot\|)$ is a normed space where $\|x\| = \sqrt{\langle x, x \rangle}$

Note $\langle x, x \rangle \geq 0 \quad \forall x \in E$.

Definition (1-6) [4]

Let $(E, F, *)$ be PIP – space, then

1. A sequence $\{x_n\}$ in E is said to be τ – converges to $x \in E$, if $\forall \epsilon > 0, \forall \lambda > 0 \exists n_0(\epsilon, \lambda)$ such that

$$F_{x_n - x, x_n - x}(\epsilon) > 1 - \lambda \quad , \forall n > n_0(\epsilon, \lambda)$$

2. A linear functional $f(x)$ defined on E is said to be continuous, if $\{x_n\}$ is τ – converges to $x \in E$ implies that $f(x_n) \rightarrow f(x), \forall \{x_n\} \in E$
3. we say that $f(x)$ is linear functional if :

$f: E \rightarrow R$ such that :

$$f(x + y) = f(x) + f(y), \forall x, y \in E$$

$$f(\alpha x) = \alpha f(x) \quad \forall \alpha \text{ scalar}, \forall x \in E$$

Definition (1-7) [4]

Let $(E, F, *)$ be PIP – space with *Mathematical expectation*. If E is complete in $\|\cdot\|$, then E is called probabilistic Hilbert space, where $\|x\| = \sqrt{\langle x, x \rangle} \quad \forall x \in E$.

Theorem (1-8) [4]

Let $(E, F, *)$ be PIP – space with Mathematical expectation. For sequence $\{x_n\}$ in E m – convergent (in norm $\|x\| = \sqrt{\langle x, x \rangle}$) implies τ – convergent.

Theorem (1-9) [4] (**Riesz Representation Theorem**)

Let $(E, F, *)$ be Probabilistic Hilbert Space, for any linear continuous functional $f(x), \exists! y \in E$ such that:

$$f(x) = \int_{-\infty}^{\infty} t dF_{x,y}(t) , \forall x \in E$$

2. Main results

In this section we will define the Adjoint operator in Probabilistic Hilbert Space by using Riesz representation theorem.

Theorem (2-1) (**Adjoint Operator in Probabilistic Hilbert Space**)

Let $(E, F, *)$ be Probabilistic Hilbert Space, Let $T \in E$ be continuous linear functional, then $\exists! T^* \in E$ such that:

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall x, y \in E$$

Proof:

Fix $y \in E$, let $\varphi_y(x) = \langle Tx, y \rangle$, $x \in E$

φ_y is linear functional on E i, e $\varphi_y: E \rightarrow R$ such that

$$\varphi_y(x + z) = \varphi_y(x) + \varphi_y(z) \quad \forall x, z \in E$$

$$\varphi_y(\alpha x) = \alpha \varphi_y(x) \quad \forall \alpha \text{ scalar}$$

Also φ_y is continuous

Then by Riesz Representation theorem $\exists! z_y \in E$ such that:

$$\varphi_y(x) = \int_{-\infty}^{\infty} t dF_{x,y}(t) \quad , \forall x \in E$$

Define $T^*: E \rightarrow R$ such that $T_y^* = z_y \quad y \in E$

So

$$\langle Tx, y \rangle = \varphi_y(x) = \int_{-\infty}^{\infty} t dF_{x, T^*y}(t) = \langle x, T^*y \rangle$$

T^* is linear map since

let $y, z \in E$, α, β are scalars than $\forall x \in E$

$$\begin{aligned} \langle x, T^*(\alpha y + \beta z) \rangle &= \langle Tx, \alpha y + \beta z \rangle \\ &= \int_{-\infty}^{\infty} t dF_{Tx, \alpha y + \beta z}(t) = \int_{-\infty}^{\infty} t dF_{Tx, \alpha y}(t) + \\ &\int_{-\infty}^{\infty} t dF_{Tx, \beta z}(t) \\ &= \alpha \int_{-\infty}^{\infty} t dF_{Tx, y}(t) \\ &+ \beta \int_{-\infty}^{\infty} t dF_{Tx, z}(t) = \alpha \langle Tx, y \rangle + \beta \langle Tx, z \rangle \\ &= \alpha \langle x, T^*y \rangle + \beta \langle x, T^*z \rangle \end{aligned}$$

Uniqueness of the adjoint operator

let T_1^*, T_2^* be two adjoint operators for $T \in E$ i, e

$$\langle Tx, y \rangle = \langle x, T_1^*y \rangle$$

$$, \langle Tx, y \rangle = \langle x, T_2^*y \rangle$$

Then

$$\langle x, T_1^*y \rangle = \langle x, T_2^*y \rangle$$

$$\int_{-\infty}^{\infty} t dF_{x, T_1^*y}(t) = \int_{-\infty}^{\infty} t dF_{x, T_2^*y}(t) = 0$$

$$\int_{-\infty}^{\infty} t dF_{x, T_1^*y - T_2^*y}(t) = 0$$

$$F_{x, T_1^*y - T_2^*y} = 0 \text{ iff } , (T_1^* - T_2^*)y = 0 \quad \forall x \in E$$

$$T_1^* = T_2^*$$

So T^* is unique.

Theorem (2-2)

Let $(E, F, *)$ be Probabilistic Hilbert Space, let T^* be the adjoint operator of $T \in E$, then T^* has the following properties:

- $(T^*)^* = T$

Proof:

$$\begin{aligned} \langle Tx, y \rangle &= \int_{-\infty}^{\infty} tdF_{x, T^*y}(t) = \langle x, T^*y \rangle \\ &= \int_{-\infty}^{\infty} tdF_{T^*y, x}(t) = \langle T^*y, x \rangle = \int_{-\infty}^{\infty} tdF_{y, T^{**}x}(t) = \langle y, T^{**}x \rangle \\ &= \int_{-\infty}^{\infty} tdF_{T^{**}x, y}(t) = \langle T^{**}x, y \rangle \end{aligned}$$

$$\langle Tx, y \rangle - \langle T^{**}x, y \rangle = 0$$

$$\langle (T - T^{**})(x), y \rangle = 0 \quad \forall x, y \in E$$

$$(T - T^{**})(x) = 0 \quad \forall x \in E$$

So

$$(T^*)^* = T$$

- $(\alpha T)^* = \alpha T^*$

Proof:

$$\langle \alpha Tx, y \rangle = \int_{-\infty}^{\infty} tdF_{x, (\alpha T)^*y}(t) = \langle x, (\alpha T)^*y \rangle \quad (1)$$

$$\begin{aligned} \langle \alpha Tx, y \rangle &= \int_{-\infty}^{\infty} tdF_{\alpha Tx, y}(t) \\ &= \alpha \int_{-\infty}^{\infty} tdF_{x, y}(t) = \alpha \langle Tx, y \rangle = \alpha \int_{-\infty}^{\infty} tdF_{x, T^*y}(t) = \alpha \langle x, T^*y \rangle \\ &> \quad (2) \end{aligned}$$

By (1), (2)

$$\langle x, (\alpha T)^*y \rangle = \alpha \langle x, T^*y \rangle$$

$$\langle x, (\alpha T)^*y \rangle - \alpha \langle x, T^*y \rangle = 0$$

$$\langle x, (\alpha T)^* y - \alpha T^* y \rangle = 0 \quad \forall x, y \in E$$

$$\langle x, ((\alpha T)^* - \alpha T^*)(y) \rangle \geq 0$$

$$((\alpha T)^* - \alpha T^*)(y) = 0 \quad \forall x \in E$$

$$(\alpha T)^* = \alpha T^*$$

$$3. (T_1^* + T_2^*) = T_1^* + T_2^*$$

Proof:

$$\langle (T_1 + T_2)x, y \rangle = \int_{-\infty}^{\infty} t dF_{x, (T_1+T_2)^* y}(t) = \langle x, (T_1 + T_2)^* y \rangle \quad (1)$$

$$\begin{aligned} \langle (T_1 + T_2)x, y \rangle &= \int_{-\infty}^{\infty} t dF_{(T_1+T_2)x, y} \\ &= \int_{-\infty}^{\infty} t dF_{T_1 x, y} + \int_{-\infty}^{\infty} t dF_{T_2 x, y} = \langle T_1 x, y \rangle + \langle T_2 x, y \rangle \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} t dF_{x, T_1^* y} + \int_{-\infty}^{\infty} t dF_{x, T_2^* y} = \int_{-\infty}^{\infty} t dF_{x, (T_1^* + T_2^*) y}(t) = \langle x, (T_1^* + T_2^*) y \rangle \\ &> \quad \quad \quad (2) \end{aligned}$$

By (1), (2)

$$\langle x, (T_1 + T_2)^* y \rangle = \langle x, (T_1^* + T_2^*) y \rangle$$

$$\langle x, (T_1 + T_2)^* y \rangle - \langle x, (T_1^* + T_2^*) y \rangle = 0$$

$$\langle x, (T_1 + T_2)^* y - (T_1^* + T_2^*) y \rangle = 0$$

$$\langle x, ((T_1 + T_2)^* - (T_1^* + T_2^*)) y \rangle = 0 \quad \forall x, y \in E$$

$$((T_1 + T_2)^* - (T_1^* + T_2^*)) y = 0 \quad \forall y \in E$$

$$(T_1^* + T_2^*) = T_1^* + T_2^*$$

$$4. (T_1 T_2)^* = T_2^* T_1^*$$

$$\langle T_1 T_2 x, y \rangle = \int_{-\infty}^{\infty} t dF_{(T_1 T_2)^* x, y} = \langle x, (T_1 T_2)^* y \rangle \quad (1)$$

$$\begin{aligned} \langle T_1 T_2 x, y \rangle &= \langle T_2 x, T_1^* y \rangle = \int_{-\infty}^{\infty} t dF_{T_2 x, T_1^* y} = \int_{-\infty}^{\infty} t dF_{x, T_2^* T_1^* y} = \langle x, T_2^* T_1^* y \rangle \\ &> \quad \quad \quad (2) \end{aligned}$$

By (1), (2)

$$\begin{aligned} \langle x, (T_1 T_2)^* y \rangle &= \langle x, T_2^* T_1^* y \rangle \\ \langle x, (T_1 T_2)^* y \rangle - \langle x, T_2^* T_1^* y \rangle &= 0 \\ \langle x, (T_1 T_2)^* y - T_2^* T_1^* y \rangle &= 0 \\ \langle x, ((T_1 T_2)^* - T_2^* T_1^*)(y) \rangle &= 0 \quad \forall x, y \in E \\ ((T_1 T_2)^* - T_2^* T_1^*)(y) &= 0 \quad \forall y \in E \\ (T_1 T_2)^* &= T_2^* T_1^* \end{aligned}$$

Conclusion

In this paper we tried to introduce a reasonable definition of the adjoint operator in probabilistic Hilbert Space. This definition can be extend as the extension of probabilistic Hilbert space. Also we proved some properties of the adjoint operator.

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