

## Approximation of Entire Function of Slow Growth in Several Complex Variables

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### Abstract

In the present paper, we study the polynomial approximation of entire function in Banach space ( $B(p, q, k)$  space, Hardy space and Bergman space). The coefficient characterizations of generalized type of entire function of slow growth in several complex variables have been obtained in terms of the approximation errors.

**Keyword:** Entire function, generalized order, generalized type, approximation error.

### 1. Introduction

Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be an entire function and  $M(r, f) = \max_{|z|=r} |f(z)|$  be its maximum modulus.

The growth of  $f(z)$  is measured in terms of its order  $\rho$  and type  $\tau$  defined as under

$$\limsup_{r \rightarrow \infty} \frac{\ln \ln M(r, f)}{r^\rho} = \rho \quad (1.1)$$

$$\limsup_{r \rightarrow \infty} \frac{\ln \ln M(r, f)}{r^\rho} = \tau \quad (1.2)$$

for  $0 < \rho < \infty$ . Various workers have given different characterizations for entire function of fast growth ( $\rho = \infty$ ). M. N. Seremeta [6] defined the generalized order and generalized type with the help of general functions as follows.

Let  $L^0$  denoted the class of functions  $h$  satisfying the following conditions

- (i)  $h(x)$  is defined on  $[a, \infty)$  and is positive, strictly increasing, differentiable and tend to  $\infty$  as  $x \rightarrow \infty$ ,  
 (ii)

$$\lim_{x \rightarrow \infty} \frac{h\{(1 + 1/\psi(x))x\}}{h(x)} = 1,$$

for every function  $\psi(x)$  such that  $\psi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Let  $\Lambda$  denoted the class of function  $h$  satisfying condition (i) and

$$\lim_{x \rightarrow \infty} \frac{h(cx)}{h(x)} = 1$$

for every  $c > 0$ , that is,  $h(x)$  is slowly increasing.

For the entire function  $f(z)$  and function  $\alpha(x) \in \Lambda$ ,  $\beta(x) \in L^0$ , the generalized order of an entire function in the terms of maximum modulus is defined as

$$\rho(\alpha, \beta, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\ln M(r, f)]}{\beta(\ln r)}. \quad (1.3)$$

Further, for  $\alpha(x) \in L^0$ ,  $\beta^{-1}(x) \in L^0$ ,  $\gamma(x) \in L^0$ , generalized type of an entire function  $f$  of finite generalized order  $\rho$  is defined as

$$\tau(\alpha, \beta, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\ln M(r, f)]}{\beta[(\gamma(r))^\rho]}. \quad (1.4)$$

where  $0 < \rho < \infty$  is a fixed number.

Above relation were obtained under certain conditions which do not hold if  $\alpha = \beta$ . To overcome this difficulty, G. P. Kapoor and Nautiyal [4] defined generalized order  $\rho(\alpha; f)$  of slow growth with the help of general functions as follows

Let  $\Omega$  be the class of functions  $h(x)$  satisfying (i) and (iv) there exists a  $\delta(x) \in \Omega$  and  $x_0, K_1$  and  $K_2$  such that

$$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq K_2 < \infty \text{ for all } x > x_0.$$

Let  $\bar{\Omega}$  be the class of functions  $h(x)$  satisfying (i) and (v)

$$\lim_{x \rightarrow \infty} \frac{d(h(x))}{d(\log x)} = K, \quad 0 < K < \infty.$$

Kapoor and Nautiyal [4] showed that class  $\Omega$  and  $\bar{\Omega}$  are contained in  $\Lambda$ . Further,  $\Omega \cap \bar{\Omega} = \phi$  and they defined the generalized order  $\rho(\alpha; f)$  for entire function  $f(z)$  of slow growth as

$$\rho(\alpha; f) = \limsup_{r \rightarrow \infty} \frac{\alpha(\ln M(r, f))}{\alpha(\ln r)},$$

where  $\alpha(x)$  either belongs to  $\Omega$  or to  $\bar{\Omega}$ .

Let  $f(z_1, z_2, \dots, z_n)$  be an entire function,  $z = (z_1, z_2, \dots, z_n) \in C^n$ . Let  $G$  be a full region in  $R_+^n$  (positive hyper octant). Let  $G_R \subset C^n$  denoted the region obtained from  $G$  by a similarity transformation about the origin, with ratio of similitude  $R$ . let  $d_t(G) = \sup_{z \in G} |z|^t$ , where  $|z|^t = |z_1|^{t_1} |z_2|^{t_2} \dots |z_n|^{t_n}$ , and let  $\partial G$  denoted the boundary of the region  $G$ . Let

$$f(z) = f(z_1, z_2, \dots, z_n) = \sum_{t_1, t_2, \dots, t_n=0}^{\infty} a_{t_1 \dots t_n} z_1^{t_1} \dots z_n^{t_n} = \sum_{\|t\|=0}^{\infty} a_t z^t,$$

$\|t\| = t_1 + t_2 + \dots + t_n$ , be the power series expansion of the function  $f(z)$ . Let  $M_G(R, f) = \max_{z \in G_R} |f(z)|$ . To characterize the growth of  $f$ , order ( $\rho_G$ ) and type ( $\sigma_G$ ) of  $f$  are defined as [2]

$$\rho_G = \limsup_{R \rightarrow \infty} \frac{\ln \ln M_G(R, f)}{\ln R},$$

$$\sigma_G = \limsup_{R \rightarrow \infty} \frac{\ln M_G(R, f)}{R^{\rho_G}}.$$

For the entire function,  $f(z) = \sum_{\|t\|=0}^{\infty} a_t z^t$ , A. A. Gol'dberg [3, Th .1] obtained the order and type in terms of the coefficients of its Taylor expansion as

$$\rho_G = \lim_{\|t\| \rightarrow \infty} \frac{\|t\| \ln \|t\|}{\|t\| \ln \|t\| - \ln |a_t|}. \tag{1.5}$$

$$(e \rho_G \sigma_G)^{1/\rho_G} = \limsup_{\|t\| \rightarrow \infty} \left\{ \|t\|^{1/\rho_G} [ |a_t| d_t(G) ]^{1/\|t\|} \right\}, \quad (0 < \rho_G < \infty) \tag{1.6}$$

where  $d_t(G) = \max_{r \in G} r^t$ ;  $r^t = r_1^{t_1} r_2^{t_2} \dots r_n^{t_n}$ .

for an entire function of several complex variables  $f(z) = \sum_{\|t\|=0}^{\infty} a_t z^t$ , and functions  $\alpha(x) \in \Lambda$ ,  $\beta(x) \in L^0$ , Seremeta [6, Th .1] proved that

$$\rho = \limsup_{R \rightarrow \infty} \frac{\alpha[\ln M_G(R, f)]}{\beta(\ln R)} = \limsup_{\|k\| \rightarrow \infty} \frac{\alpha(\|t\|)}{\beta[-\frac{1}{\|t\|} \ln(a_t |d_t(G))]} \quad (1.7)$$

Further, for  $\alpha(x) \in L^0$ ,  $\beta^{-1}(x) \in L^0$ ,  $\gamma(x) \in L^0$ , Seremeta [6, Th .1] proved that

$$\sigma = \limsup_{R \rightarrow \infty} \frac{\alpha[\ln M_G(R, f)]}{\beta[(\gamma(R))^\rho]} = \limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{\beta[(\gamma\{e^{1/\rho} [a_t |d_t(G)]^{-1/\|t\|}\})^\rho]} \quad (1.8)$$

where  $0 < \rho < \infty$  is a fixed number.

And let  $H'_q$  denote the Bergman space of functions  $f(z)$  satisfying the condition

$$\|f\|_{H'_q} = \left\{ \frac{1}{A} \iint_{z \in G} |f(z)|^q d\sigma_1 d\sigma_2 \dots d\sigma_n \right\}^{1/\rho} < \infty,$$

where  $z = (z_1, z_2, \dots, z_n)$ ,  $d\sigma_j = dx_j dy_j$ ,  $z_j = x_j + iy_j$ ,  $j = 1, 2, \dots, n$ . and A is the area of G.

For  $q = \infty$ , let  $\|f\|_{H'_\infty} = \|f\|_{H_\infty} = \sup \{ |f(z)|, z \in U \}$ . Then  $H_q$  and  $H'_q$  are Banach space for  $q \geq 1$ . In analogy with spaces of functions of one variable, we call  $H_q$  and  $H'_q$  the Hardy and Bergman spaces respectively.

The function  $f(z)$  analytic in U belong to the space  $B(p, q, k)$ , where  $0 < p < q \leq \infty$ , and  $0 < k \leq \infty$ , if

$$\|f\|_{p,q,k} = \left\{ \int_0^1 (1-R)^{k(1/p-1/q)-1} M_{q,G}^k(R, f) dR \right\}^{1/k} < \infty.$$

And

$$\|f\|_{p,q,\infty} = \sup \{ (1-R)^{1/p-1/q} M_{q,G}(R, f); 0 < R < 1 \} < \infty.$$

It is known [1] that  $B(p, q, k)$  is a Banach space for  $p > 0$  and  $q, k \geq 1$ , otherwise it is a Freachet space. Further, we have

$$H_q \subseteq H'_q = B\left(\frac{q}{2}, q, q\right), \quad 1 \leq q < \infty.$$

Let  $P_m = \{p : p = \sum_{\|t\| \leq m} a_t z^t\}$  be the class polynomials of degree at most m and let X denote one of the

Banach spaces defined. Then we defined error of an entire function f on the region G as

$$E_{\|t\|}(f) = E_{\|t\|}(f, G, X) = \inf \{ \|f - p\|_X : p \in P_m \}.$$

Vakarchuk and Zhir [5] obtained the characterizations of generalized order and generalized type of  $f(z)$  in terms of the errors  $E_{\|t\|}(f)$  defined above. These characterizations do not hold good when  $\alpha = \beta = \gamma$ . i.e. for entire functions of slow growth. In this paper we have tried to fill this gap. We define the generalized type  $\tau(\alpha; f)$  of an entire function  $f(z)$  having finite generalized order as

$$\tau(\alpha; f) = \limsup_{r \rightarrow \infty} \frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^\rho}.$$

Where  $\alpha(x)$  either belongs to  $\Omega$  or to  $\bar{\Omega}$ .

## 2. Main Results

**Theorem 2.1:** Let  $\alpha(x) \in \overline{\Omega}$ , then the entire function  $f(z)$  of generalized order  $\rho$ ,  $1 < \rho < \infty$ , is of generalized type  $\tau$  if and only if

$$\tau = \limsup_{R \rightarrow \infty} \frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^\rho} = \limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}}, \quad (2.1)$$

provided  $dF(x; \tau, \rho)/d \ln x = O(1)$  as  $x \rightarrow \infty$  for all  $\tau$ ,  $0 < \tau < \infty$ .

**Proof.** Let

$$\limsup_{R \rightarrow \infty} \frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^\rho} = \tau.$$

We suppose  $\tau < \infty$ . Then for every  $\varepsilon > 0$ ,  $\exists R(\varepsilon) \ni$

$$\frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^\rho} \leq \tau + \varepsilon = \bar{\tau}, \quad \forall R \geq R(\varepsilon).$$

$$\text{(or)} \quad \ln M_G(R, f) \leq (\alpha^{-1}\{\bar{\tau} [\alpha(\ln R)]^\rho\}).$$

Choose  $R = R(t)$  to be the unique root of the equation

$$t = \frac{\rho}{\ln R} F[\ln R; \bar{\tau}, \frac{1}{\rho}]. \quad (2.2)$$

Then

$$\ln R = \alpha^{-1}\left[\left(\frac{1}{\bar{\tau}} \alpha\left(\frac{\|t\|}{\rho}\right)\right)^{1/(\rho-1)}\right] = F\left[\frac{\|t\|}{\rho}; \frac{1}{\bar{\tau}}, \rho-1\right]. \quad (2.3)$$

By Cauchy's inequality,

$$\begin{aligned} |a_t| d_t(G) &\leq R^{-\|t\|} M_G(R, f) \\ &\leq \exp\{-\|t\| \ln R + (\alpha^{-1}\{\bar{\tau} [\alpha(\ln R)]^\rho\})\} \end{aligned}$$

By using (2.5) and (2.6), we get

$$|a_t| d_t(G) \leq \exp\{-\|t\| F + \frac{\|t\|}{\rho} F\}$$

or

$$\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|} \geq \alpha^{-1}\left\{\left[\frac{1}{\bar{\tau}} \alpha\left(\frac{\|t\|}{\rho}\right)\right]^{1/(\rho-1)}\right\}$$

or

$$\bar{\tau} = \tau + \varepsilon \geq \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}}.$$

Now proceeding to limits, we obtain

$$\tau \geq \limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}}. \quad (2.4)$$

Inequality (2.4) obviously holds when  $\tau = \infty$ .

Conversely, let

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}} = \sigma.$$

Suppose  $\sigma < \infty$ . Then for every  $\varepsilon > 0$  and for all  $\|t\| \geq N(\varepsilon)$ , we have

$$\frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1} \ln(a_t | d_t(G))^{-1/\|t\|}]\}^{\rho-1}} \leq \sigma + \varepsilon = \bar{\sigma}$$

$$|a_t| d_t(G) \leq \frac{1}{\exp\{(\rho-1) \frac{\|t\|}{\rho} F[\frac{\|t\|}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]\}}. \quad (2.5)$$

The inequality

$$\sqrt{\|t\|} |a_t| d_t(G) R^{\|t\|} \leq \text{Re}^{-\left(\frac{\rho-1}{\rho}\right) F[\frac{\|t\|}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]} \leq \frac{1}{2} \quad (2.6)$$

is fulfilled beginning with some  $\|t\| = m(R)$ . Then

$$\sum_{\|t\|=m(R)+1}^{\infty} |a_t| d_t(G) R^{\|t\|} \leq \sum_{\|t\|=m(R)+1}^{\infty} \frac{1}{2^{\|t\|}} \leq 1. \quad (2.7)$$

We now express  $m(R)$  in terms of  $R$ . From inequality (2.6),

$$2R \leq \exp\left\{\left(\frac{\rho-1}{\rho}\right) F\left[\frac{\|t\|}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1\right]\right\},$$

we can take  $m(R) = E[\rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R + \ln 2))^{\rho-1}\}]$ . We consider the function  $\psi(x) = R^x \exp\{-\left(\frac{\rho-1}{\rho}\right) x F[\frac{x}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]\}$ . Let

$$\frac{\psi'(x)}{\psi(x)} = \ln R - \left(\frac{\rho-1}{\rho}\right) F\left[\frac{x}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1\right] - \frac{dF[\frac{x}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]}{d \ln x} = 0. \quad (2.8)$$

As  $x \rightarrow \infty$ , by the assumption of the theorem, for finite  $\sigma (0 < \sigma < \infty)$ ,

$dF[x; \bar{\sigma}, \rho-1]/d \ln x$  is bounded. So there is a  $A > 0$  such that for  $x > x_1$  we have

$$\left| \frac{dF[\frac{x}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]}{d \ln x} \right| \leq A. \quad (2.9)$$

We can take  $A > \ln 2$ . It is then obvious that inequalities (2.6) and (2.7) hold for  $\|t\| \geq m_1(R) = E[\rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R + \ln 2))^{\rho-1}\}] + 1$ . We let  $m_0$  designate the number  $\max(N(\varepsilon), E[x_1] + 1)$ . For  $R > R_1(m_0)$  we have  $\psi'(m_0)/\psi(m_0) > 0$ . From (2.9) and (2.8) it follows that  $\psi'(m_1(R))/\psi(m_1(R)) < 0$ . We hence obtain that if for  $R > R_1(m_0)$  we let  $x^*(R)$  designate the point where  $\psi(x^*(R)) = \max_{m_0 \leq x \leq m_1(R)} \psi(x)$ , then

$$m_0 < x^*(R) < m_1(R) \text{ and } x^*(R) = \rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R - a(R)))^{\rho-1}\}.$$

Where

$$-A < a(R) = \frac{dF[\frac{x}{\rho}; \frac{1}{\bar{\sigma}}, \rho-1]}{d \ln x} \Big|_{x=x^*(R)} < A.$$

Further

$$\begin{aligned} \max_{m_0 < \|t\| < m_1(R)} (|a_t| d_t(G) R^{\|t\|}) &\leq \max_{m_0 < x < m_1(R)} \psi(x) = \frac{R^{\rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R - a(R)))^{\rho-1}\}}}{e^{\rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R - a(R)))^{\rho-1}\} (\ln R - a(R))}} = \\ &= \exp\{a(R) \rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R - a(R)))^{\rho-1}\}\} \leq \\ &\leq \exp\{A \rho \alpha^{-1} \{\bar{\sigma}(\alpha(\ln R + A))^{\rho-1}\}\}. \end{aligned}$$

It is obvious that (for  $R > R_1(m_0)$ )

$$M_G(R, f) \leq \sum_{\|t\|=0}^{\infty} |a_t| d_t(G) R^{\|t\|} =$$

$$\begin{aligned}
 &= \sum_{\|t\|=0}^{m_0} |a_t| d_t(G) R^{\|t\|} + \sum_{\|t\|=m_0+1}^{m_1(R)} |a_t| d_t(G) R^{\|t\|} + \sum_{\|t\|=m_1(R)+1}^{\infty} |a_t| d_t(G) R^{\|t\|} \\
 &\leq O(R^{m_0}) + m_1(R) \max_{m_0 < \|t\| < m_1(R)} (|a_t| d_t(G) R^{\|t\|}) + 1 \\
 M_G(R, f)(1 + o(1)) &\leq \exp \{ (A\rho + o(1)) \alpha^{-1} [\bar{\sigma}(\alpha(\ln R + A))^{\rho-1}] \} \\
 \alpha(\ln M_G(R, f)) &\leq \bar{\sigma}[\alpha(\ln R + A)]^{\rho-1} \leq \bar{\sigma}[\alpha(\ln R + A)]^{\rho}.
 \end{aligned}$$

We then have

$$\frac{\alpha[(A\rho + o(1))^{-1} \ln M_G(R, f)]}{[\alpha(\ln R + A)]^{\rho}} \leq \bar{\sigma} = \sigma + \varepsilon.$$

Since  $\alpha(x) \in \bar{\Omega} \subseteq \Lambda$ , now proceeding to limits we obtain

$$\limsup_{R \rightarrow \infty} \frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^{\rho}} \leq \sigma. \tag{2.10}$$

From inequality (2.4) and (2.10), we get the required the result.

Now we prove

**Theorem 2.2:** Let  $\alpha(x) \in \bar{\Omega}$ , then a necessary and sufficient condition for an entire function  $f(z) \in B(p, q, k)$  to be of generalized type  $\tau$  having finite generalized order  $\rho$ ,  $1 < \rho < \infty$  is

$$\tau = \limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|}(B(p, q, k)) d_t(G))^{-1/\|t\|})\}]^{\rho-1}}. \tag{2.11}$$

**Proof.** First we consider the space  $B(p, q, k)$ ,  $q = 2$ ,  $0 < \rho < \infty$  and  $k \geq 1$ . Let  $f(z) \in B(p, q, k)$  be of generalized type  $\tau$  with generalized order  $\rho$ . Then from the Theorem 2.1, we have

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}} = \tau. \tag{2.12}$$

For a given  $\varepsilon > 0$ , and all  $\|t\| > m = m(\varepsilon)$ , we have

$$|a_t| d_t(G) \leq \frac{1}{\exp\{(\rho-1) \frac{\|t\|}{\rho} F[\frac{\|t\|}{\rho}; \frac{1}{\tau}, \rho-1]\}}. \tag{2.13}$$

Let  $g_t(f, z) = \sum_{j=0}^{\|t\|} a_j z^j$  be the  $t^{th}$  partial sum of the Taylor series of the function  $f(z)$ . Following [5, p.1396], we get

$$E_{\|t\|}(B(p, 2, k); f) \leq B^{1/k}[(\|t\| + 1)k + 1; k(1/p - 1/2)] \{ \sum_{\|j\|=\|t\|+1}^{\infty} (|a_j| d_j(G))^2 \}^{1/2} \tag{2.14}$$

where  $B(a, b)$  ( $a, b > 0$ ) denotes the beta function. By using (2.13), we have

$$E_{\|t\|}(B(p, 2, k); f) \leq \frac{B^{1/k}[(\|t\| + 1)k + 1; k(1/p - 1/2)]}{\exp\{(\rho-1) \frac{\|t\|+1}{\rho} F[\frac{\|t\|+1}{\rho}; \frac{1}{\tau}, \rho-1]\}} \{ \sum_{\|j\|=\|t\|+1}^{\infty} \psi_j^2(\alpha) \}^{1/2}, \tag{2.15}$$

where

$$\psi_j(\alpha) \cong \frac{\exp\{\frac{\|t\|+1}{\rho} (\rho-1) [\alpha^{-1} \{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}\}}{\exp\{\frac{\|j\|}{\rho} (\rho-1) [\alpha^{-1} \{(\frac{\alpha(\frac{\|j\|}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}\}}.$$

Set

$$\psi(\alpha) \cong \exp\left\{-\frac{(\rho-1)}{\rho} \left[\alpha^{-1} \left\{\left(\frac{\alpha(\frac{1}{\rho})}{\tau+\varepsilon}\right)^{1/(\rho-1)}\right\}\right]\right\}.$$

Since  $\alpha(x)$  is increasing and  $\|j\| \geq \|t\| + 1$ , we get

$$\psi_j(\alpha) \leq \exp\left\{\frac{(\|t\|+1)-\|j\|}{\rho}(\rho-1)\left[\alpha^{-1}\left\{\left(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon}\right)^{1/(\rho-1)}\right\}\right]\right\} \leq \psi^{\|j\|-(\|t\|+1)}(\alpha). \quad (2.16)$$

Since  $\psi(\alpha) < 1$ , we get from (2.15) and (2.16),

$$E_{\|t\|}(B(p,2,k);f) \leq \frac{B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}[\exp\{(\rho-1)\frac{\|t\|+1}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}]} \quad (2.17)$$

For  $\|t\| > m$ , (2.17) yields

$$\tau + \varepsilon \geq \frac{\alpha(\frac{\|t\|+1}{\rho})}{\left\{\alpha\left(\frac{\rho}{(1+\frac{1}{\|t\|})^{\rho-1}}\right)\left\{\ln((E_{\|t\|} d_t)^{-1/\|t\|}) + \ln\left(\frac{B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)]^{1/\|t\|}}{(1-\psi^2(\alpha))^{1/2}}\right)\right\}\right\}^{(\rho-1)}}$$

Now

$$B[(\|t\|+1)k+1;k(1/p-1/2)] = \frac{\Gamma((\|t\|+1)k+1)\Gamma(k(1/p-1/2))}{\Gamma((\|t\|+1/2+1/p)k+1)}.$$

Hence

$$B[(\|t\|+1)k+1;k(1/p-1/2)] \cong \frac{e^{-(\|t\|+1)k+1}[(\|t\|+1)k+1]^{(\|t\|+1)k+3/2}\Gamma(1/p-1/2)}{e^{[(\|t\|+1/2+1/p)k+1]}[(\|t\|+1/2+1/p)k+1]^{(\|t\|+1/2+1/p)k+3/2}}.$$

Thus

$$\{B[(\|t\|+1)k+1;k(1/p-1/2)]\}^{1/(\|t\|+1)} \cong 1. \quad (2.18)$$

Now proceeding to limits, we obtain

$$\tau \geq \limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \quad (2.19)$$

For reverse inequality, by [5, p.1398], we have

$$|a_{t+1}| B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)] \leq E_{\|t\|}(B(p,2,k);f). \quad (2.20)$$

Then for sufficiently large  $\|t\|$ , we have

$$\begin{aligned} & \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \\ & \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(|a_{t+1}| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;(1/p-1/2)])\}\}]^{(\rho-1)}} \\ & \geq \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(|a_t| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;(1/p-1/2)])\}\}]^{(\rho-1)}}. \end{aligned}$$

By applying limits and from (2.12), we obtain

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \tau. \quad (2.21)$$

From (2.19) and (2.21), we obtain the required relation

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} = \tau. \quad (2.22)$$

In the second step, we consider the spaces  $B(p, q, k)$  for  $0 < p < q, q \neq 2$ , and  $q, k \geq 1$ . Gvaradze [1] showed that, for  $p \geq p_1, q \geq q_1$  and  $k \leq k_1$ , if at least one of the inequalities is strict, then the strict inclusion  $B(p, q, k) \subset B(p_1, q_1, k_1)$  holds and the following relation is true:

$$\|f\|_{p_1, q_1, k_1} \leq 2^{1/q-1/q_1} [k(1/p-1/q)]^{1/k-1/k_1} \|f\|_{p, q, k}.$$

For any function  $f(z) \in B(p, q, k)$ , the last relation yields

$$E_{\|t\|}(B(p_1, q_1, k_1); f) \leq 2^{1/q-1/q_1} [k(1/p-1/q)]^{1/k-1/k_1} E_{\|t\|}(B(p, q, k); f). \quad (2.23)$$

For the general case  $B(p, q, k), q \neq 2$ , we prove the necessity of condition (2.13).

Let  $f(z) \in B(p, q, k)$  be an entire transcendental function having finite generalized order  $\rho(\alpha; f)$  whose generalized type is defined by (2.12). Using the relation (2.13), for  $\|t\| > m$  we estimate the value of the best polynomial approximation as follows

$$E_{\|t\|}(B(p, q, k); f) = \|f - g_t(f)\|_{p, q, k} \leq (\int_0^1 (1-R)^{(k(1/p-1/q)-1)} M_{q, G}^k dR)^{1/k}.$$

Now

$$|f|^q = |\sum a_t z^t|^q \leq (\sum |a_t r^t|)^q \leq (r^{\|t\|+1} \sum_{\|k\|=\|t\|+1}^{\infty} |a_k|)^q.$$

Hence

$$\begin{aligned} E_{\|t\|}(B(p, q, k); f) d_t(G) &\leq B^{1/k} [(\|t\| + 1)k + 1; k(1/p - 1/2)] \sum_{\|k\|=\|t\|+1}^{\infty} |a_k| \\ &\leq \frac{B^{1/k} [(\|t\| + 1)k + 1; k(1/p - 1/2)]}{(1 - \psi(\alpha)) [\exp\{\frac{\|t\|+1}{\rho} (\rho - 1) [\alpha^{-1} \{(\frac{\|t\|+1}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]} \end{aligned} \quad (2.24)$$

For  $\|t\| < m$ , (2.24) yields

$$\tau + \varepsilon \geq \frac{\alpha(\frac{\|t\|+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{\|t\|})^{(\rho-1)}}) \{\ln((E_{\|t\|} d_t(G))^{-1/\|t\|}) + \ln(\frac{B^{1/k} [(\|t\|+1)k+1; k(1/p-1/2)]^{1/\|t\|}}{(1-\psi(\alpha))})\}\}^{(\rho-1)}}$$

Since  $\psi(\alpha) < 1$ , and  $\alpha \in \bar{\Omega}$ , proceeding to limits and using (2.18), we obtain

$$\tau \geq \limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}.$$

For the reverse inequality, let  $0 < p < q < 2$  and  $k, q \geq 1$ . By (2.23), where  $p_1 = p, q_1 = 2$ , and  $k_1 = k$ , and the condition (2.13) is already proved for the space  $B(p, 2, k)$ , we get

$$\begin{aligned} &\limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|}(B(p, q, k); f) d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \\ &\geq \limsup_{\|t\| \rightarrow \infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|}(B(p, 2, k); f) d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} = \tau. \end{aligned}$$

Now let  $0 < p \leq 2 < q$ . Since we have



$$M_{2,G}(R, f) \leq M_{q,G}(R, f), \quad 0 < R < 1,$$

therefore

$$E_{\|t\|}(B(p, q, k); f) \geq \left| a_{\|t\|+1} \right| B^{1/k} [(\|t\| + 1)k + 1; k(1/p - 1/q)]. \quad (2.25)$$

Then for sufficiently large  $\|t\|$ , we have

$$\begin{aligned} & \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \\ & \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(\left| a_{\|t\|+1} \right| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|^k} [(\|t\| + 1)k + 1; k(1/p - 1/q)])\})\}]^{(\rho-1)}} \\ & \geq \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(\left| a_{\|t\|} \right| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|^k} [(\|t\| + 1)k + 1; k(1/p - 1/q)])\})\}]^{(\rho-1)}}. \end{aligned}$$

By applying limits and from (2.12), we obtain

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln(\left| a_{\|t\|} \right| d_t(G))^{-1/\|t\|}\}]^{(\rho-1)}} = \tau.$$

Now we assume that  $2 \leq p < q$ . Set  $q_1 = q, k_1 = k$ , and  $0 < p_1 < 2$  in the inequality (2.23), where  $p_1$  is an arbitrary fixed number. Substituting  $p_1$  for  $p$  in (2.25), we get

$$E_{\|t\|}(B(p, q, k); f) \geq \left| a_{\|t\|+1} \right| B^{1/k} [(\|t\| + 1)k + 1; k(1/p_1 - 1/q)]. \quad (2.26)$$

Using (2.26) and applying the same analogy as in the previous case  $0 < p \leq 2 < q$ , for sufficiently large  $\|t\|$ , we have

$$\begin{aligned} & \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \\ & \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(\left| a_{\|t\|+1} \right| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|^k} [(\|t\| + 1)k + 1; k(1/p_1 - 1/q)])\})\}]^{(\rho-1)}} \\ & \geq \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \{\ln(\left| a_{\|t\|} \right| d_t(G))^{-1/\|t\|} + \ln(B^{-\rho/\|t\|^k} [(\|t\| + 1)k + 1; k(1/p - 1/q)])\})\}]^{(\rho-1)}} \end{aligned}$$

By applying limits and using (2.12), we obtain

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|} d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \tau.$$

From relation (2.19) and (2.21), and the above inequality, we obtain the required relation (2.22).

**Theorem 2.3:** Assuming that the condition of Theorem 2.2 are satisfied and  $\xi(\alpha)$  is a positive number, a necessary and sufficient condition for a function  $f(z) \in H_q$  to be an entire function of generalized type  $\xi(\alpha)$  having finite generalized order  $\rho$  is that

$$\limsup_{\|t\| \rightarrow \infty} \frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|}(H_q; f) d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} = \xi(\alpha). \quad (2.27)$$

**Proof.** Let  $f(z) = \sum_{|t|=0}^{\infty} a_t z^t$  be an entire transcendental function having finite generalized order  $\rho$  and generalized type  $\tau$ . Since

$$\lim_{|t| \rightarrow \infty} \sqrt[|t|]{|a_t|} = 0 \tag{2.28}$$

$f(z) \in B(p, q, k)$ , where  $0 < p < q \leq \infty$  and  $q, k \geq 1$ . From relation (1.8), we get

$$E_{|t|}(B(q/2, q, q); f) \leq \zeta_q E_{|t|}(H_q; f), \quad 1 \leq q < \infty. \tag{2.29}$$

where  $\zeta_q$  is a constant independent of  $|t|$  and  $f$ . In the case of Hardy space  $H_{\infty}$ ,

$$E_{|t|}(B(p, \infty, \infty); f) \leq E_{|t|}(H_{\infty}; f), \quad 1 < p < \infty. \tag{2.30}$$

Since

$$\begin{aligned} \xi(\alpha; f) &= \limsup_{|t| \rightarrow \infty} \frac{\alpha(\frac{|t|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{|t|}(H_q; f) d_t(G))^{-1/|t|})\}]^{(\rho-1)}} \\ &\geq \limsup_{|t| \rightarrow \infty} \frac{\alpha(\frac{|t|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{|t|}(B(q/2, q, q); f) d_t(G))^{-1/|t|})\}]^{(\rho-1)}} \\ &\geq \tau, \quad 1 \leq q < \infty. \end{aligned} \tag{2.31}$$

Using estimate (2.30) we prove inequality (2.31) in the case  $q = \infty$ .

For the reverse inequality

$$\xi(\alpha; f) \leq \tau, \tag{2.32}$$

we use the relation (2.13), which is valid for  $|t| > m$ , and estimate from above, the generalized type  $\tau$  of an entire transcendental function  $f(z)$  having finite generalized order  $\rho$ , as follows. We have

$$\begin{aligned} E_{|t|}(H_q; f) &\leq \|f - g_t\|_{H_q} \\ &\leq \sum_{|j|=|t|+1}^{\infty} |a_j| \\ &\leq \frac{1}{[\exp\{(\rho-1) \frac{|t|+1}{\rho} [\alpha^{-1}\{(\frac{\alpha(\frac{|t|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}]} \sum_{|j|=|t|+1}^{\infty} \psi_j(\alpha) \end{aligned}$$

Using (2.16),

$$\begin{aligned} E_{|t|}(H_q; f) &\leq \|f - g_t\|_{H_q} \\ &\leq \frac{1}{d_t(G) (1 - \psi(\alpha)) [\exp\{(\rho-1) \frac{|t|+1}{\rho} [\alpha^{-1}\{(\frac{\alpha(\frac{|t|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}]} \\ \frac{1}{E_{|t|}(H_q; f) d_t(G)} &\geq (1 - \psi(\alpha)) [\exp\{(\rho-1) \frac{|t|+1}{\rho} [\alpha^{-1}\{(\frac{\alpha(\frac{|t|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}]\}] \end{aligned}$$

This yields

$$\tau + \varepsilon \geq \frac{\alpha(\frac{|t|+1}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} [\ln((E_{|t|}(H_q; f) d_t(G))^{-1/|t|+1}) + \ln((1 - \psi(\alpha))^{-1/|t|+1})\}]^{(\rho-1)}}. \tag{2.33}$$

Since  $\psi(\alpha) < 1$  and by applying the properties of the function  $\alpha$ , passing to the limit as  $|t| \rightarrow \infty$  in (2.33), we obtain inequality (2.32). thus we have finally

$$\xi(\alpha) = \tau. \tag{2.34}$$

This prove Theorem 2.2.

**Remark :** An analog of Theorem 2.3 for the Bergman space follows from (1.8) for  $1 \leq q < \infty$  and from Theorem 2.2 for  $q = \infty$ .

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