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On comparison of Aunu permutation pattern and generalized permutation patterns using Wilf-equivalence

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Abstract

A large number of articles have been published in the last decade studying pattern avoidance on permutations. From the point of view of enumeration, one tries to examine permutations avoiding certain patterns according to their lengths. We tackle the problem of refining this enumeration by considering the statistics “first/last entry” in the case of generalized patterns as well as “first entry unity” in the case of Aunu patterns. We give results for classical patterns of type (123) and/or (132) (Aunu patterns) as well as for at least one of the mentioned statistics for every generalized pattern of type $(1,2)$ or $(2,1)$.

Keywords: Aunu numbers, special (restricted) 123-avoiding and/or 132-avoiding patterns of permutations.

An overview of Aunu Numbers/patterns:

Aunu numbers first emerged out of a study conducted by the founder, Aminu A. Ibrahim in an attempt to find the order of arrangement of some objects governed by some precedence relation. The term 'Aunu Patterns' referrers to a pairing scheme involving pairs of numbers associated by some precedence relation. Aunu permutation/pattern is a partial permutation in which a first entry of every permutation is unity (one) and its length is prime. For more detailed see www.algebragroup.org .

Preliminary definitions and results:

Given a string of numbers $s = s_1s_2 \ldots s_n$, the reduction of s, denoted by red(s), is the string obtained by replacing the ith smallest letter(s) of s with i. For example, red (15487864) = 13265642 because in 15487864 the smallest letter is 1, the second smallest letter is 4, the third smallest letter is 5, etc., and so in the reduction 4 is replaced with 2, 5 is replaced with 3, etc. This definition gives rise to a generalized notion of one string being contained in another string, namely: We say that the permutation $p = p_1 p_2 \; . . . p_n \in S$ *Contains a q pattern* if there is a subsequence $p_{i_1} p_{i_2} \; . . . p_{i_k}$ of *p* that reduces to *q*, that is, $\mathrm{red}(p_{i_1} p_{i_2} \; . . . p_{i_k}) = q$. Otherwise we say that *p* is *q*-avoiding. For example, 3142 contains a 132 pattern because red (142) = 132, whereas 3124 does not contain 132 because red (124)=123, which is 132-avoiding.

We shall deal with permutations avoiding some specific patterns. For us, a permutation on [1*;* *n*] will mean a list of the integers 1 up to *n* in some order. For example, here are all permutations on [1*;* 3]:

$$123; \; 132; \; 213; \; 231; \; 312; \; 321.$$

These are possible permutations of a string of numbers with length three. The corresponding (normal) permutation is obtained by letting the element *i* go into the element in position *i*. For example, in 1432, 1 goes to 1, 2 goes to 4, 3 goes to 3 and 4 goes to 2, hence the permutation is (24). Again, this is the number of permutations in a string with length four. Thus in classical patterns, a string with length n there are $n!$ number of permutations. The term *pattern* will mean for us the same thing as *permutation*, only with different semantics. Let $\pi \in S_m$ and $\alpha \in S_n$ be two strings of numbers, where S_t denotes the symmetric group on $[t] = \{1, \; 2, \; . . . , \; t\}$. We say that π contains α and denote this by $\alpha < \pi$ if there are $\pi_{i_1} \; \pi_{i_2} \; . . . \; \pi_{i_n}$ with $1 \le i_1 < i_2 < \ldots < i_n \le m$ which are order-isomorphic to α, that is having the entries π_{i_j} in the same relative order of the entries of α. Otherwise we say that π avoids α. The permutation α is called a (classical) pattern. For example, a permutation contains the pattern (123) iff it has an ascending subsequence of length three. Here, note that *members need not actually be consecutive*, merely ascending (Wilf 1997). Therefore, of the $3! = 6$

partitions of $\{1,2,3\}$, all but $\{3,2,1\}$ (i.e., $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, and $\{3,1,2\}$) contain the pattern (12) (i.e., an increasing subsequence of length two). We denote the set of all α - avoiding permutations of S_m with $S_m(\alpha)$. We say that a permutation π *avoids* a pattern α if no *subperm* of π is ordered as the pattern α . By a *subperm* we mean a certain number of elements taken *in order* but not necessarily *together*. For example, the seven non-empty subperms of 123 are

$$1;\ 2;\ 3;\ 12;\ 13;\ 23;\ 123.$$

A subperm is said to be *ordered according to* some pattern of the same length *n* if for any $1 \le k \le n$, the *k*-largest element is located in the same position in both the subperm and the pattern. For example, the subperm $\alpha\beta\gamma$ is ordered as the pattern 132 if $\alpha < \gamma < \beta$.

Of the six number of permutations on [1; 3], five are $(123)-\text{avoiding}$ and they include:

$$132;213;321;231;312.$$

While five are $(132)-\text{avoiding}$ and they include:

$$123;231;312;213;321$$

Hence the sets of all (132) and $(123)-\text{avoiding}$ permutations of length three are respectively

$$\text{S}_3(132)=\{123;231;312;213;321\} \text{ and}$$

$$\text{S}_3(123)=\{132;213;321;231;312\}.$$

Trees for the Pattern-Avoiding $(\text{S}_3(132) \text{ and } \text{S}_3(123))$:

i. (132)-avoiding

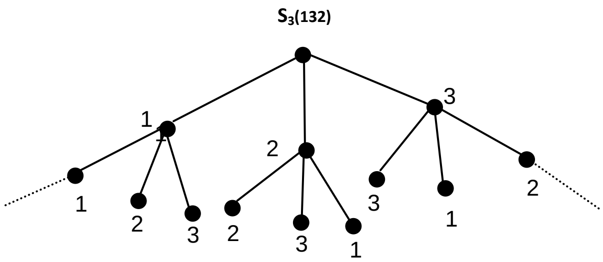

Fig 1: Tree representation for $S_3(132)$

ii. (123)-avoiding

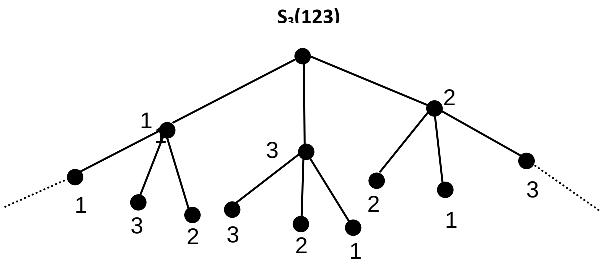

Fig 2: Tree representation for $S_3(123)$

Aunu pattern is a partial permutation patterns in the sense that it involves some restrictions which make it impossible to obey (follow) the rule of the normal permutation patterns. For instance in *Aunu permutation patterns* [1] and [2], some of the restrictions involved include: 'length of each string (sequence)' is prime and the 'first entry of each sequence is unity (one)'. As such these restrictions would make it impossible for a string with length n for instance, to have a total number of $n!$ permutations given that 'first entry of each sequence is unity (one)' as we shall see in due course. Another example in this category of permutation patterns is *generalized permutation patterns*. In [7], *generalized patterns* were introduced to study some statistics on permutations avoiding patterns. They are obtained by inserting one or more dashes among the entries of α . For instant, α=13-26-574 is a pattern of type $(2,2,3)$. Conversely, a classical pattern of length n can be seen as a pattern of type $\underbrace{(1,1,...,1)}_{n}$, assuming that a dash is inserted, but not shown between each pair of consecutive elements of the classical pattern. If $\alpha \in S_3$, then generalized pattern deriving from α are of type $(1,2)$ or $(2,1)$ according to the number of elements preceding and following the dash and they are collected in the following set

$$P = \{1-23, 12-3, 1-32, 13-2, 2-13, 21-3, 2-31, 23-1, 3-12, 31-2, 3-21, 32-1\}.$$

A permutation π (classical pattern) contains a generalized pattern $p \in P$ if adjacent elements in p are also adjacent in π . For example $\pi = 7256134$ contains the generalized $13-2$ in its subsequence $\pi_2\pi_3\pi_6 = 253$. Observe that it does not contain the pattern $1-32$, but it contains the classical pattern 132 in the subsequences $\pi_2\pi_4\pi_6 = 263$ and $\pi_2\pi_4\pi_7 = 264$. In [3] permutations avoiding two distinct patterns of length three were studied to generate trees.

WILF CLASSES:

Two patterns T_1 and T_2 belong to the same Wilf class if $|S_n(T_1)| = |S_n(T_2)|$ for all n , where $S_n(T)$ denotes the set of permutations on $[n] = \{1,...,n\}$ that avoids the pattern T . Two sets having the same Wilf class are said to be Wilf equivalent. Let $S_n(\alpha)$ denote the number of permutation on [1*;* *n*] avoiding the pattern α . For example $S_3(123)$ = 5, for obvious reasons. For each pattern length *m*, we say that the patterns α and α^c are in the same Wilf class if $S_n(\alpha)$ = $S_n(\alpha^c)$ for any permutation length *n*. Wilf-equivalence defines an equivalence relation on sets of permutations, and we call the resulting equivalence classes *Wilf-classes*. The problem of counting the permutations avoiding a given permutation or set of permutations is a rich one. One of the oldest and most famous results in the area is a theorem of Erdös and Szekeres, which states that $|S_n(12\ ...\ k, (l)(l-1)\ ...\ 1)| = 0$ for $n > (k-1)(l-1)$. The field has experienced rapid growth in the last twenty years, beginning with Simion and Schmidt's proof that {123} and {132} are Wilf equivalent [9]. Since then, all permutations of length 7 and less have been Wilf-classified (see [6]), as well as all sets of two permutations both of length 4 or less (see [4], [5], and [8]). In 2004, Marcus and Tardos proved the Stanley-Wilf conjecture, which states that for any set π, $|S_n(\pi)|$ grows at most exponentially in *n* [10]. The study of permutation avoidance has also found applications to a variety of other problems in combinatorics, as well as areas of algebraic geometry and computer science. There are three simple operations on patterns which are guaranteed to preserve the Wilf class: reversing, complementing and inverting.

Reversing

The *reverse* of a pattern α, denoted α^R, is just the pattern α reversed as a text string. Thus

$$\alpha_i^R = \alpha_{n+1-i}$$

For example, $123^R = 321$. It is clear that π avoids α if, and only if, π^R avoids α^R; it is further clear that reversing is a permutation on S_n, hence α and α^R share the same Wilf class.

Complementing

The *complement* of a pattern α, denoted α^C, is obtained by subtracting each element of α from $m+1$, *m* being the length of α .

$$\alpha_i^C = n+1-\alpha_i$$

For example, to complement 1432 we subtract each element from 5 to obtain $1432^C = 4123$. It is easy to see that π avoids α if, and only if, π^C avoids α^C,
since complementing corresponds to switching < and >; it is further clear that complementing is a permutation on S_n, hence α and α^C share the same Wilf class.

Inverting

The *inverse* of a pattern α , denoted α^{-1}, is obtained by inverting α as a permutation. Alternatively, one encodes α using *positional notation*: that is, the first element is the position of 1 in α , the second is the position of 2, and so on. For example, consider inverting the permutation 1423. According to the first definition, $1423^{-1} = (243)^{-1} = (234) = 1342$. According to the second definition, the position of 1 is 1, that of 2 is 3, that of 3 is 4, that of 4 is 2. Thus we also get 1342.

Why are the two definitions equivalents? Let us look at a permutation in a third way: as a set of (*i; j*) pairs, meaning *i* goes to *j*. For example,

$$1423 = \{(1;\ 1);\ (2;\ 4);\ (3;\ 2);\ (4;\ 3)\}.$$

Inverting the permutation means reversing the pairs, which amounts to the positional notation. Now we can easily show that a permutation π avoids the pattern α if, and only if, π^{-1} avoids α^{-1}: for this amounts only to decoding the pairs in the set representation the opposite way. Hence α and α^{-1} belong to the same Wilf class.

The graphs of 123 and 132-avoiding permutations

The connection between the graphs and 132-avoiding permutations is quite straightforward. What does it mean for a permutation π to avoid 132? Suppose $|\pi| = n$ (that is, π is of size *n*).

Let us write $\pi = \alpha n \beta$. Then clearly α and β must be 132-avoiding, and furthermore, every element in α must be larger than any element in β. It is also clear that if these two conditions hold, then π avoids 132.

We can now outline a conversion procedure from 132-avoiding permutations into graphs:

For example, we may graph the permutation 1423 as in Figure 1.1

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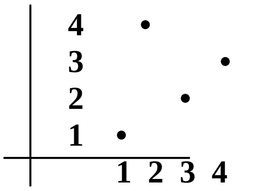


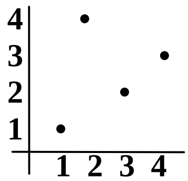
Figure 1.1: The graph of the permutation 1423

Since the graph of a permutation is necessarily on an $n \times n$ square we may use the natural symmetries of the square to determine some useful relationships between various sets $S_n(Q)$.

We consider three symmetries that are natural both for the square and in the language of permutations.
We note that:

- Reversal corresponds to flipping the graph of p over the vertical line of symmetry:

The graph of $p = 1423$ *The graph of* $p^R = 3241$



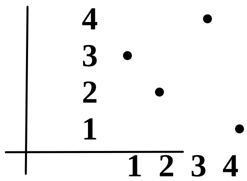
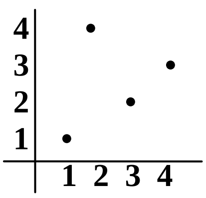


Figure 1.1: The graph of the permutation $p = 1423$

Figure 1.2: The graph of the permutation $p^r = 3241$

- Complement corresponds to flipping the graph of p over the horizontal line of symmetry:

The graph of $p = 1423$ *The graph of* $p^c = 4132$



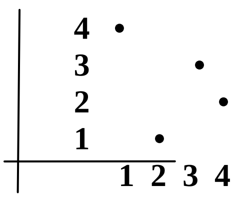


Figure 1.1: The graph of the permutation $p = 1423$

Figure 1.3: The graph of the permutation $p^c = 4132$

- Inverse corresponds to flipping the graph of p over the main diagonal line of symmetry:

The graph of $p = 1423$

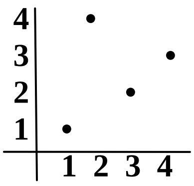


Figure 1.1:

The graph of the permutation $p = 1423$

The graph of $p^{-1} = 1342$

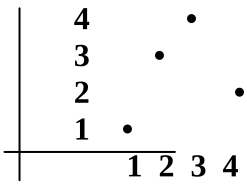


Figure 1.4:

The graph of the permutation $p^{-1} = 1342$

These lead to the natural symmetries:

Let $\mathrm{p} = \mathrm{p}_1 p_2 ... \mathrm{p}_n \in \mathrm{S}_n$. Then: $\forall p, q \in \mathrm{S}_n$

p avoids q $\Leftrightarrow p^R$ avoids q^R,

p avoids q $\Leftrightarrow p^C$ avoids q^C,

p avoids q $\Leftrightarrow p^{-1}$ avoids q^{-1};

And moreover $|\mathrm{S}_n(\mathrm{Q})| = |\mathrm{S}_n(\mathrm{Q}^R)| = |\mathrm{S}_n(\mathrm{Q}^C)| = |\mathrm{S}_n(\mathrm{Q}^{-1})|$, where Q^* is the set obtained by applying the operation * to all patterns in the set Q.

By repeatedly applying the operations of reverse, complement, and inverse, which generate the symmetries of the square, we see that we can partition sets of patterns into equivalence classes up to size 8 that will necessarily have the same enumeration. Two pattern sets Q and Q^C that yield the same sequence $\{|\mathrm{S}_n(\mathrm{Q})|\} \geq 0$ are said to be Wilf-equivalent.

There are other relations besides those given by the symmetries of the square that give Wilf-equivalent classes of patterns. For example:

Theorem 1.

Let $\mathrm{I}_t = 12...\mathrm{t}$, $\mathrm{J}_t = \mathrm{t}(\mathrm{t}-1)...1$, and let $\mathrm{A} = \mathrm{a}_{t+1}$ ... a_l be any permutation of $(t+1)$ ... l. Then $|\mathrm{S}_n(\{\mathrm{I}_t\mathrm{A}\})| = |\mathrm{S}_n(\{\mathrm{J}_t\mathrm{A}\})|$ for all $n \geq 0$, i.e. $\{I_t A\}$ and $\{J_t A\}$ are Wilf-equivalent.

Proof of Theorem 1

The symmetries and equivalences of the previous section will guide us as we seek to comprehensively enumerate Q-avoiding permutations for various Q.

We first consider results for when Q contains exactly one pattern:

. Length 1:

$|\mathrm{S}_n(\{1\})| = 0$ for $\mathrm{n} \geq 1$ since any permutation with at least one letter contains a 1 pattern.

. Length 2:

$|\mathrm{S}_n(\{12\})| = 1$ for $\mathrm{n} \geq 1$, which counts the strictly decreasing permutations. Also, since $21 = 12^R$, we have $|\mathrm{S}_n(\{21\})| = 1$.

. Length 3:

Recall that from the symmetries of the square, we have $|\mathrm{S}_n(\{123\})| = |\mathrm{S}_n(\{321\})|$ and $|\mathrm{S}_n(\{132\})| = |\mathrm{S}_n(\{231\})| = |\mathrm{S}_n(\{213\})| = |\mathrm{S}_n(\{312\})|$. Simion and Schmidt in Pudwell Provided a bijection between (123)-avoiding permutations and (132)-avoiding Permutations, and moreover showed that $|\mathrm{S}_n(\{\pi_3\})| = \frac{1}{n+1}\binom{2n}{n} = C_{n,}$ where π_3 is any permutation of length 3, and $C_{n,}$ denotes the nth Catalan number.

From what has been enumerated above, it suffices to deduce that:

$$|S_3(\{123\})| = |S_3(\{132\})| = 5$$

More interestingly for Aunu patterns (where the first element of each permutation is 1 and n is a prime number), we deduse from the above that:

$$|A_3(123)| = |A_3(132)| = 1,$$

$$\vdots \quad \vdots$$

$$|A_n(123)| = |A_n(132)|.$$

Hence $A_n(123)$ and $A_n(132)$ are Wilf-equivalent.

The next theorem shows that if m is the order of an element α in a cyclic group $(G,.)$ generated by α (for which $\alpha^m = e$), then $(G,.)$ contains m elements.

A group $(G,.)$, or just G for short, is called a cyclic if there is an element $\alpha \in G$ for which $G = \{\alpha^i \mid i \in Z\}$. In this case, α is called a cyclic generator for G. More generally, suppose α is an element of a group G, and let $H = \{\alpha^i \mid i \in Z\}$. Then H is a subgroup of G called the cyclic group generated by α [11].

Definitions . A group $(G,\bullet)$, or just G for short, is called *cyclic* if there is an element $\alpha \in G$ for which $G = \{\alpha^i \mid i \in Z\}$. In this case, α is called a *cyclic generator* for G.

Theorem 2

Suppose α *is an element in a group* G. If m is the smallest positive integer for which $\alpha^m = e$, where e is the identity element in G, then the cyclic group generated by α contains m elements.

Proof of Theorem 2

Let $G = \{\alpha^i \mid i \in Z\}$ (i.e. $\alpha \in G$ and α generates G). Suppose also $\alpha^i = \alpha^j$ … (i) for some $0 < i < j$. Then since G is a group, it follows from equation (i) that $\alpha^{j-i} = \alpha^j \alpha^{-i} = e$, where e is the identity element in G. Thus there is a smallest positive integer m for which $\alpha^m = e$ … (ii). Now let $\alpha^t = e$ … (iii) where t is an integer. Then since $t = mq + r$ for some $0 \le r < m$, and $\alpha^t = \alpha^{mq+r} = (\alpha^m)^q \alpha^r = (e)^q \alpha^r = \alpha^r$ i.e. $\alpha^t = \alpha^r$ …(iv), it follows from (iii) and (iv) that $r = 0$

[i.e. $(iv) \div (iii) \Rightarrow \alpha^{t-t} = \alpha^r e^{-1} \Rightarrow \alpha^0 = \alpha^r \Rightarrow r = 0$]

Hence m divides t. Since $\alpha^i = \alpha^j$ for $i < j$ forces $\alpha^{j-i} = e$, a contradiction if $0 < j - i < m$, the set $\{\alpha^i \mid 0 \le i < m\}$ consists of m distinct elements. Furthermore, for any integer k we can write $k = mq + r$ for some $0 \le r < m$ with $\alpha^k = \alpha^r$. Therefore, $H = \{\alpha^i \mid 0 \le i < m\}$ and H contains m elements.

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