

The Degree of Coconvex Multi Polynomial Approximation

Mayada Ali Kareem

Mathematics Department, College of Education for pure science, University of Babylon, Hilla, Babylon, Iraq
kmayada84@yahoo.com

Abstract

Let $f \in C[-1,1]^d$ change its convexity finitely many times in the interval, say s times, at $Y_s : -1 < y_{11} < y_{21} < \dots < y_{s1} < 1, \dots, -1 < y_{1d} < y_{2d} < \dots < y_{sd} < 1$. We estimate the degree of approximation of f by multi polynomials of degree n , which change convexity exactly at the points Y_s . We show that provided n is sufficiently large, depending on the location of the points Y_s , the rate of approximation is estimated by the multi third Ditzian –Totik moduli of smoothness of f multiplied by a constant $C(s)$.

Keywords: coconvex approximation, moduli of smoothness, multi approximation.

1. Introduction and Main Results .

Let $f \in C[-1,1]^d$ change its convexity finitely many times, say s , at the points $Y_s : -1 < y_{11} < y_{21} < \dots < y_{s1} < 1, \dots, -1 < y_{1d} < y_{2d} < \dots < y_{sd} < 1$ in $[-1,1]^d$ where $C[-1,1]^d$ the space of all continuous functions on $[-1,1]^d$ such that $x = (x_1, \dots, x_d), x \in [-1,1]^d$. For later reference set $y_{0j} = -1$ and $y_{sj+1} = 1$ where $j = 1, \dots, d$. Note that if $f \in C^2[-1,1]^d$, then the above is equivalent to $f''(x_1, \dots, x_d) \prod_{i=1}^s ((x_1 - y_{i1}) \dots (x_d - y_{id})) \geq 0$, in $[-1,1]^d$. We wish to approximate f by means of polynomials which are coconvex with f , that is, which change convexity exactly at the points $y_1 = (y_{11}, \dots, y_{1d}), \dots, y_s = (y_{s1}, \dots, y_{sd})$. The first Jackson estimate involving the Ditzian – Totik moduli of smoothness is due to Leviatan [3]. As was pointed out to us by the referee, the degree of coconvex polynomial approximation apparently was first discussed in the Ph. D dissertation of Diana C. Myers [4] where she obtained the Jackson estimate $\omega\left(f, \frac{1}{n}\right)$ for nearly coconvex approximation. The first estimates on the degree of coconvex approximation for a twice continuously differentiable function with an arbitrary finite number of convexity changes have recently been obtained by Kopotun [1] and the purpose of this note is to improve those result in that we do not assume the existence everywhere, and continuity of even the first derivative. We are going to make use of some special polynomials related to the function f which were constructed in that article [1] based upon the polynomials introduced by Shevchuk [5]. We remark that in the above mentioned paper, Kopotun was able to estimates on the simultaneous approximation of the function and its derivatives by the polynomials and their derivatives, thus obtaining simultaneously, coconvex approximation to f and comonotone approximation to its derivative. In order to state our main result we definition of the multi m th order Ditzian –Totik moduli of smoothness $\omega_m^\varphi(f, \delta)$. for $f \in C[-1,1]^d$, we set

$$\omega_m^\varphi(f, \delta) = \sup_{\substack{0 < h_j \leq \delta_j \\ j=1, \dots, d}} \left\| \Delta_\eta^m \left(f; ((x_1, \dots, x_d)) \right) \right\|,$$

$h = (h_1, \dots, h_d), \delta = (\delta_1, \dots, \delta_d), \eta = (\eta_1, \dots, \eta_d) = (h_1 \varphi(x_1), \dots, h_d \varphi(x_d))$, where the norm is the max-norm, $\varphi(x_j) = \sqrt{1 - x_j^2}, j = 1, \dots, d$ and

$$\Delta_\eta^m f((x_1, \dots, x_d)) = \begin{cases} \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} f\left(x_1 - \frac{m}{2} \eta_1 + i \eta_1, \dots, x_d - \frac{m}{2} \eta_d + i \eta_d\right) \\ 0, \end{cases} \quad \begin{array}{l} \text{if } x \pm \frac{m}{2} \eta \in [-1,1]^d, \\ \text{otherwise.} \end{array}$$

is the symmetric multi m th difference.

We define $\|f((x_1, \dots, x_d))\| = \sup_{\substack{x_j \in [-1,1] \\ j=1, \dots, d}} |f((x_1, \dots, x_d))|$,

we also need the following moduli of smoothness

$$\omega_m(f, \delta) = \sup_{\substack{0 < h_j \leq \delta_j \\ j=1, \dots, d}} \left\| \Delta_h^m \left(f; (x_1, \dots, x_d) \right) \right\|.$$

For $f \in C[-1, 1]^d$, we denote by

$$E_n^{(2)}(f, Y_s) = \inf \|f - p_n\| : p_n \in \Pi_n \cap \Delta^2(Y_s)$$

the degree of *coconvex approximation* of f by algebraic multi polynomials,

where p_n has the form

$$p_n(x_1, \dots, x_d) = (a_{01} + a_{02} + \dots + a_{0d}) + a_1(x_1 + \dots + x_d) \\ + a_2(x_1^2 + \dots + x_d^2) + \dots + a_n(x_1^n + \dots + x_d^n).$$

Our main result is the following :

Theorem : Let $f \in C[-1, 1]^d$ have s changes of convexity at $Y_s : -1 < y_{11} < y_{21} < \dots < y_{s1} < 1, \dots, -1 < y_{1d} < y_{2d} < \dots < y_{sd} < 1$, and denote

$$d(Y_s) = \min \left\{ \left((1 + y_{11}) \dots (1 + y_{1d}), (y_{21} - y_{11}) \dots (y_{2d} - y_{1d}), \dots \right), \right. \\ \left. \left((y_{s1} - y_{s1-1}) \dots (y_{sd} - y_{sd-1}), ((1 - y_{s1}) \dots (1 - y_{sd})) \right) \right\}.$$

Then there exists a constant $A = A(s)$ which depends only on the number of convexity changes s , such that for $n > \frac{A(s)}{d(Y_s)}$ there is a multi polynomial p_n of degree not exceeding n , which is coconvex with f and satisfies

$$\|f - p_n\| \leq C(s) \omega_m^{\phi}(f, \delta), \tag{1}$$

where $\delta = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$.

In the sequel we will denote by C an absolute constant which may vary from one occurrence to another even in the same line. Similarly $C(\cdot)$ will denote a constant which depends on a specific parameter but may change from one occurrence to another.

2. Proof of the Theorem :

If f has no change of convexity in $[-1, 1]^d$, i.e., $s = 0$ and f is convex in $[-1, 1]^d$, then the theorem is valid, thus we will assume that $s \geq 1$. We first need to construct a smoother function at $y_1 = (y_{11}, \dots, y_{1d})$. The function f is either concave or convex in $[-1, y_{11}] \times \dots \times [-1, y_{1d}]$, and each case will need a separate though similar construction. We will detail the construction for the case where f is concave in $[-1, y_{11}] \times \dots \times [-1, y_{1d}]$. For the sake of simplicity in notation in the sequel we write $\alpha_j = y_{1j}$ where $j = 1, \dots, d$. Now let

$$x_r = (x_{r1}, \dots, x_{rd}), x_{r,n} = (x_{r1,n}, \dots, x_{rd,n}) \text{ and } \cos \frac{r\pi}{n} = \left(\cos \frac{r1\pi}{n}, \dots, \cos \frac{rd\pi}{n} \right),$$

such that $x_{rj} = x_{rj,n} = \cos \frac{rj\pi}{n}$, $r = 0, \dots, n$ and $j = 1, \dots, d$, be the chebyshev nodes; and denote $I_r = [x_{r1}, x_{r1-1}] \times \dots \times [x_{rd}, x_{rd-1}]$, $h_{rj} = h_{rj,n} = x_{rj-1} - x_{rj}$ and

$$\psi_r(x_1, \dots, x_d) = \psi_{r,n}(x_1, \dots, x_d) = \frac{(h_{r1} \dots h_{rd})}{|(x_1 - x_{r1}) \dots (x_d - x_{rd})| + (h_{r1} \dots h_{rd})}, \tag{2}$$

It is well know that $h_{rj \pm 1} < 3h_{rj}$ and that for $x \in I_r$, $\Delta_n(x_j) \leq h_{rj} < 5\Delta_n(x_j)$, where

$\Delta_n(x) = (\Delta_n(x_1), \dots, \Delta_n(x_d))$ and $\Delta_n(x_j) = \frac{\sqrt{1-x_j^2}}{n} + \frac{1}{n^2}$. We assume that $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $\alpha \in [x_{r01}, x_{r01-1}] \times \dots \times [x_{r0d}, x_{r0d-1}]$. Then, if $n > N_\alpha = \max \left\{ \frac{50}{(y_{21}-\alpha_1) \dots (y_{2d}-\alpha_d)}, \frac{50}{(1+\alpha_1) \dots (1+\alpha_d)} \right\}$, we are

assured that $x_{r_{0j+3}} \geq -1$ and that $x_{r_{0j-4}} \leq y_{2j}$. Set $h_j = c\Delta_n(\alpha_j) < \frac{1}{6}h_{r_{0j}}$, where c is chosen sufficiently small to guarantee the right inequality. Note that this implies $x_{r_{0j+1}} < \alpha_j - 2h_j < \alpha_j + 2h_j < x_{r_{0j-2}}$.

We are going to replace f on the interval $[\alpha_1 - h_1, \alpha_1 + h_1] \times \dots \times [\alpha_d - h_d, \alpha_d + h_d]$ in a way that will keep us near the original function and at the same time the new function, will be smoother at $\alpha = (\alpha_1, \dots, \alpha_d)$. When $s = 0$, then (1) holds for all $n \geq 2$. Thus, we proceed by induction. To this end we note that either $\Delta_h^2 f((\alpha_1, \dots, \alpha_d)) \geq 0$, or $\Delta_h^2 f((\alpha_1, \dots, \alpha_d)) < 0$.

In the first case, let $L_1((x_1, \dots, x_d))$ denote the linear function interpolating f at $(\alpha_1 - h_1, \dots, \alpha_d - h_d)$ and $(\alpha_1, \dots, \alpha_d)$. Then the function $g = f - L_1$ satisfies

$$g((\alpha_1 - h_1, \dots, \alpha_d - h_d)) = g((\alpha_1, \dots, \alpha_d)) = 0,$$

$$g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) \geq 0 \text{ and } g((x_1, \dots, x_d)) \leq 0, \quad -1 \leq x_j < \alpha_j - h_j.$$

Hence, for $\bar{J}_0 = [x_{r_{01+1}}, x_{r_{01-2}}] \times \dots \times [x_{r_{0d+1}}, x_{r_{0d-2}}]$, we have,

$$\begin{aligned} 0 &\leq g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) \\ &\leq g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) - g((\alpha_1 - 2h_1, \dots, \alpha_d - 2h_d)) \\ &= \\ &g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) - 3g((\alpha_1, \dots, \alpha_d)) + 3g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) - \\ &g((\alpha_1 - 2h_1, \dots, \alpha_d - 2h_d)) \\ &\leq \omega_3(g, h, \bar{J}_0) = \omega_3(f, h, \bar{J}_0). \end{aligned}$$

Similarly, in the latter case, let $L_1((x_1, \dots, x_d))$ denote the linear function interpolating f at $(\alpha_1 + h_1, \dots, \alpha_d + h_d)$ and $(\alpha_1, \dots, \alpha_d)$. Then the function $g = f - L_1$ satisfies

$$g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) = g((\alpha_1, \dots, \alpha_d)) = 0,$$

$$g((\alpha_1 - h_1, \dots, \alpha_d - h_d)) < 0, \text{ and } g((x_1, \dots, x_d)) \geq 0, \quad \alpha_j + h_j \leq x_j < y_{2j}. \text{ Hence,}$$

$$\begin{aligned} 0 &< -g((\alpha_1 - h_1, \dots, \alpha_d - h_d)) \\ &\leq g((\alpha_1 + 2h_1, \dots, \alpha_d + 2h_d)) - g((\alpha_1 - h_1, \dots, \alpha_d - h_d)) \\ &= g((\alpha_1 + 2h_1, \dots, \alpha_d + 2h_d)) - 3g((\alpha_1 + h_1, \dots, \alpha_d + h_d)) \\ &\quad + 3g((\alpha_1, \dots, \alpha_d)) - g((\alpha_1 - h_1, \dots, \alpha_d - h_d)) \end{aligned}$$

$$\leq \omega_3(g, h, \bar{J}_0) = \omega_3(f, h, \bar{J}_0).$$

Thus, in both case we have,

$$\max\{|g((\alpha_1 - h_1, \dots, \alpha_d - h_d))|, |g((\alpha_1, \dots, \alpha_d))|, |g((\alpha_1 + h_1, \dots, \alpha_d + h_d))|\} \leq C\omega_3(f, h, \bar{J}_0),$$

which in turn implies that the quadratic polynomial $L_2((x_1, \dots, x_d))$, interpolating g at $(\alpha_1 - h_1, \dots, \alpha_d - h_d)$, $(\alpha_1, \dots, \alpha_d)$ and $(\alpha_1 + h_1, \dots, \alpha_d + h_d)$ is bounded by the same on $[\alpha_1 - h_1, \alpha_1 + h_1] \times \dots \times [\alpha_1 - h_1, \alpha_1 + h_1]$. Thus

$$|L_2(x)| \leq C \left(\frac{|(x_1 - \alpha_1) \dots (x_d - \alpha_d)| + (h_{r_{01}} \dots h_{r_{0d}})}{(h_{r_{01}} \dots h_{r_{0d}})} \right)^2 \omega_3(f, h, \bar{J}_0), \quad (3)$$

where $x \in [-1, 1]^d$.

At the same time applying Whitney's theorem for the multi approximation [2] we conclude that

$$|g(x) - L_2(x)| \leq C(d)\omega_3(f, h, \bar{J}_0),$$

where $x \in [\alpha_1 - h_1, \alpha_1 + h_1] \times \dots \times [\alpha_1 - h_1, \alpha_1 + h_1]$.
 Since

$$|g(x) - L_2(x)| \leq C \left(\frac{|x - \alpha| + h_{r_0}}{h_{r_0}} \right)^3 \omega_3(f, h, J_0), \quad [6]$$

where $x \in [-1, 1]$ and $J_0 = [x_{r_0+1}, x_{r_0-2}]$.

Thus

$$|g(x) - L_2(x)| \leq C \left(\frac{|(x_1 - \alpha_1) \dots (x_d - \alpha_d)| + (h_{r_01} \dots h_{r_0d})}{(h_{r_01} \dots h_{r_0d})} \right)^3 \omega_3(f, h, \bar{J}_0), \quad (4)$$

where $x \in [-1, 1]^d$ and $\bar{J}_0 = [x_{r_01+1}, x_{r_01-2}] \times \dots \times [x_{r_0d+1}, x_{r_0d-2}]$.

Since $|\Delta_h^3 f((x_1, \dots, x_d))| \leq |\Delta_{\eta}^3 f((x_1, \dots, x_d))|$,

$$\text{then } \omega_3(f, h, \bar{J}_0) \leq C\omega_3^\varphi(f, \delta), \quad \delta = \left(\frac{1}{n}, \dots, \frac{1}{n} \right)$$

we obtain by (3) and (4)

$$|g(x)| \leq C\psi_{r_0}^{-3} \omega_3^\varphi(f, \delta), \quad x \in [-1, 1]^d, \quad (5)$$

Now let

$$\tilde{g}(x) = \begin{cases} -g(x), & x \in [-1, \alpha_1] \times \dots \times [-1, \alpha_d] \\ g(x), & \text{otherwise,} \end{cases}$$

and finally,

$$\hat{g}(x) = \begin{cases} \tilde{g}(x), & x \notin [\alpha_1 - h_1, \alpha_1 + h_1] \times \dots \times [\alpha_1 - h_1, \alpha_1 + h_1], \\ \max\{\tilde{g}(x), 0\}, & x \in [\alpha_1 - h_1, \alpha_1 + h_1] \times \dots \times [\alpha_1 - h_1, \alpha_1 + h_1]. \end{cases}$$

By virtue of (5) we immediately have

$$\|\tilde{g} - \hat{g}\| \leq C \omega_3^\varphi(f, \delta), \quad (6)$$

Also

$$\omega_3^\varphi(\hat{g}, \delta) \leq \omega_3^\varphi(\tilde{g}, \delta) + C\omega_3^\varphi(f, \delta) \leq C\omega_3^\varphi(f, \delta), \quad (7)$$

where the first inequality follows from (6), while the second is due to the fact that for $x \notin \bar{J}_0$,

$$\left| \Delta_{\tilde{\eta}}^2(\tilde{g}((x_1, \dots, x_d))) \right| = \left| \Delta_{\tilde{\eta}}^2(g((x_1, \dots, x_d))) \right| = \left| \Delta_{\tilde{\eta}}^2(f((x_1, \dots, x_d))) \right|,$$

where $\tilde{\eta} = \left(\frac{\varphi(x_1)}{n}, \dots, \frac{\varphi(x_d)}{n} \right)$ and for $x \in \bar{J}_0$ we apply (5).

It is readily seen that $\hat{g} \in C[-1, 1]^d$, that it is convex in $[-1, y_{21}] \times \dots \times [-1, y_{2d}]$ and that it changes convexity at $Y'_{s-1} = Y_s \setminus \{y_{11}\}, \dots, \{y_{1d}\}$. If on the other hand, f was convex in $[-1, y_{11}] \times \dots \times [-1, y_{1d}]$, then \hat{g} would be concave in $[-1, y_{21}] \times \dots \times [-1, y_{2d}]$ and change convexity at Y'_{s-1} . Thus in any case \hat{g} has fewer convexity changes, so by induction, we may assume that for $n > \frac{A(s-1)}{d(Y_{s-1})}$, there exists an n th degree multi polynomial q_n which is coconvex with \hat{g} and which satisfies the analogue of (1). Namely by (7),

$$\|\hat{g} - q_n\| \leq C(s-1)\omega_3^\varphi(\hat{g}, \delta) \leq C(s)\omega_3^\varphi(f, \delta), \quad (8)$$

Thus, fixing $n > \max\left\{\frac{A(s-1)}{d(y_{s-1}^*)}, N_\alpha\right\}$, readily leads to the definition of $A(s)$. Note that since $\hat{g}(\alpha_1, \dots, \alpha_d) = 0$, we may assume that $q_n(\alpha_1, \dots, \alpha_d) = 0$ constant in (8).

Kopotun [1] has constructed for α and q_n two polynomial v_n and w_n of degree at most $20n(s+1) = 4n\mu$, with the properties that for all $x \in [-1, 1]$,

$$v_n(x) \operatorname{sgn}(x - \alpha) \geq 0.$$

Thus let

$$V_n((x_1, \dots, x_d)) = v_n(x_1) + \dots + v_n(x_d)$$

and

$$W_n((x_1, \dots, x_d)) = w_n(x_1) + \dots + w_n(x_d)$$

where

$$\begin{aligned} v_n(x_1) &= a_{01} + a_1x_1 + a_2x_1^2 + \dots + a_nx_1^n \\ &\vdots \\ v_n(x_d) &= a_{0d} + a_1x_d + a_2x_d^2 + \dots + a_nx_d^n \end{aligned}$$

and

$$\begin{aligned} w_n(x_1) &= c_{01} + c_1x_1 + c_2x_1^2 + \dots + c_nx_1^n \\ &\vdots \\ w_n(x_d) &= c_{0d} + c_1x_d + c_2x_d^2 + \dots + c_nx_d^n \end{aligned}$$

two multi polynomials of degree at most $20n(s+1) = 4n\mu$ with the properties that for all $x \in [-1, 1]^d$,

$$\begin{aligned} &(v_n(x_1) + \dots + v_n(x_d))(\operatorname{sgn}(x_1 - \alpha_1) + \dots + \operatorname{sgn}(x_d - \alpha_d)) \\ &\geq v_n(x_1) \operatorname{sgn}(x_1 - \alpha_1) + \dots + v_n(x_d) \operatorname{sgn}(x_d - \alpha_d) \geq 0 \end{aligned}$$

where $\operatorname{sgn}(x_j - \alpha_j) = \begin{cases} 1 & x_j - \alpha_j > 0 \\ -1 & x_j - \alpha_j < 0 \end{cases}$

and $\operatorname{sgn}((x_1 - \alpha_1, \dots, x_d - \alpha_d)) = \operatorname{sgn}(x_1 - \alpha_1) + \dots + \operatorname{sgn}(x_d - \alpha_d)$,

$$V_n'((x_1, \dots, x_d))q_n''((x_1, \dots, x_d))\left(q_n'((x_1, \dots, x_d)) - q_n'((\alpha_1, \dots, \alpha_d))\right) \operatorname{sgn}((x_1 - \alpha_1, \dots, x_d - \alpha_d)) \geq 0$$

$$\begin{aligned} &|(v_n(x_1) + \dots + v_n(x_d)) - (\operatorname{sgn}(x_1 - \alpha_1) + \dots + \operatorname{sgn}(x_d - \alpha_d))| \\ &\leq |v_n(x_1) - \operatorname{sgn}(x_1 - \alpha_1)| + \dots + |v_n(x_d) - \operatorname{sgn}(x_d - \alpha_d)| \\ &\leq C(s) \psi_{r_0}^\mu((x_1, \dots, x_d)), \end{aligned} \quad (9)$$

$$\begin{aligned} &|(w_n(x_1) + \dots + w_n(x_d)) - (\operatorname{sgn}(x_1 - \alpha_1) + \dots + \operatorname{sgn}(x_d - \alpha_d))| \\ &\leq |w_n(x_1) - \operatorname{sgn}(x_1 - \alpha_1)| + \dots + |w_n(x_d) - \operatorname{sgn}(x_d - \alpha_d)| \\ &\leq C(s) \psi_{r_0}^\mu((x_1, \dots, x_d)), \end{aligned} \quad (10)$$

$$|V_n'((x_1, \dots, x_d))| = |v_n'(x_1) + \dots + v_n'(x_d)|$$

$$\begin{aligned} &\leq |v'_n(x_1)| + \dots + |v'_n(x_d)| \\ &\leq C(s) \psi_{r_0}^\mu(x_1, \dots, x_d)(h_{r_0 1}^{-1} \dots h_{r_0 d}^{-1}), \end{aligned} \tag{11}$$

and finally

$$W'_n(x_1, \dots, x_d) \operatorname{sgn} q'_n(\alpha_1, \dots, \alpha_d) \leq 0,$$

$x \in [y_{2r1}, y_{2r1+1}] \times \dots \times [y_{2rd}, y_{2rd+1}]$, $r = 0, \dots, [s/2]$, and

$$W'_n(x_1, \dots, x_d) \operatorname{sgn} q'_n(\alpha_1, \dots, \alpha_d) \geq 0,$$

$x \in [y_{2r1+1}, y_{2r1+2}] \times \dots \times [y_{2rd+1}, y_{2rd+2}]$, $r = 0, \dots, [(s-1)/2]$.

We are ready to define the multi polynomial

$$\begin{aligned} p_n(x_1, \dots, x_d) &= \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} \left[(q'_n(u_1, \dots, u_d) - q'_n(\alpha_1, \dots, \alpha_d)) V_n(u_1, \dots, u_d) \right. \\ &\quad \left. + q'_n(\alpha_1, \dots, \alpha_d) W_n(u_1, \dots, u_d) \right] du_1 \dots du_d, \end{aligned}$$

of degree at most $5nu$, which evidently is coconvex with f . Note that $P_n(x_1, \dots, x_d) = L_1(x_1, \dots, x_d) + p_n(x_1, \dots, x_d)$ is of the same degree and it too is coconvex with f . Hence, we conclude the induction step by proving (1) for P_n .

We begin with

$$\begin{aligned} &|f(x_1, \dots, x_d) - P_n(x_1, \dots, x_d)| = |g(x_1, \dots, x_d) - p_n(x_1, \dots, x_d)| \\ &= |\tilde{g}(x_1, \dots, x_d) \operatorname{sgn}(x_1 - \alpha_1, \dots, x_d - \alpha_d) - p_n(x_1, \dots, x_d)| \\ &\leq \|\tilde{g} - \hat{g}\| + |\hat{g}(x_1, \dots, x_d) \operatorname{sgn}(x_1 - \alpha_1, \dots, x_d - \alpha_d) - p_n(x_1, \dots, x_d)| \\ &\leq C\omega_3^\varphi(f, \delta) + \left| \left(\hat{g}(x_1, \dots, x_d) - q_n(x_1, \dots, x_d) \right) \operatorname{sgn}(x_1 - \alpha_1, \dots, x_d - \alpha_d) \right| \\ &\quad + \left| q_n(x_1, \dots, x_d) \operatorname{sgn}(x_1 - \alpha_1, \dots, x_d - \alpha_d) \right. \\ &\quad \left. - \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} q'_n(u_1, \dots, u_d) V_n(u_1, \dots, u_d) du_1 \dots du_d \right| \\ &\quad + \left| q'_n(\alpha_1, \dots, \alpha_d) \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} (V_n(u_1, \dots, u_d) - W_n(u_1, \dots, u_d)) du_1 \dots du_d \right| \end{aligned}$$

$= E_1 + E_2 + E_3 + E_4$, say.

By virtue of (8),

$$E_2 \leq C(s)\omega_3^\varphi(f, \delta), \tag{12}$$

Recalling that $q_n(\alpha_1, \dots, \alpha_d) = 0$, integration by parts and (11) yield,

$$\begin{aligned}
 E_3 &= \left| q_n((x_1, \dots, x_d)) \left(\text{sgn}((x_1 - \alpha_1, \dots, x_d - \alpha_d)) - V_n((x_1, \dots, x_d)) \right) \right. \\
 &\quad \left. + \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} q_n((u_1, \dots, u_d)) V_n'((u_1, \dots, u_d)) du_1 \dots du_d \right| \\
 &\leq C(s) \left(|q_n((x_1, \dots, x_d))| \psi_{r_0}^\mu((x_1, \dots, x_d)) \right. \\
 &\quad \left. + \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} |q_n((u_1, \dots, u_d))| \psi_{r_0}^\mu((u_1, \dots, u_d)) (h_{r_{01}}^{-1} \dots h_{r_{0d}}^{-1}) du_1 \dots du_d \right) \\
 &\leq C(s) \omega_3^\varphi(f, \delta), \tag{13}
 \end{aligned}$$

where the last inequality in (13) follows from (5), (6) and (7),

$$\begin{aligned}
 |q_n((x_1, \dots, x_d))| &\leq |\tilde{g}((x_1, \dots, x_d))| + |\tilde{g}((x_1, \dots, x_d)) - q_n((x_1, \dots, x_d))| \\
 &\leq |g((x_1, \dots, x_d))| + \|\tilde{g} - \hat{g}\| + \|\hat{g} - q_n\| \\
 &\leq C \psi_{r_0}^{-3} \omega_3^\varphi(f, \delta),
 \end{aligned}$$

and the easy inequality

$$\int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} \psi_{r_0}^v((u_1, \dots, u_d)) du_1 \dots du_d \leq C(h_{r_{01}} \dots h_{r_{0d}}), \quad v \geq 2, \tag{14}$$

Finally, in order to estimate E_4 , we need an estimate on $q_n'((\alpha_1, \dots, \alpha_d))$. To this end we observe that since q_n is convex in $[-1, y_{21}] \times \dots \times [-1, y_{2d}]$, then q_n' is monotone increasing there.

If $q_n'((\alpha_1, \dots, \alpha_d)) \geq 0$, then by (5), for some $\xi \in (\alpha_1, \alpha_1 + h_{r_{01}}) \times \dots \times (\alpha_d, \alpha_d + h_{r_{0d}})$,

$$\begin{aligned}
 0 &\leq q_n'((\alpha_1, \dots, \alpha_d)) \leq q_n'((\xi_1, \dots, \xi_d)) \\
 &= \frac{q_n((\alpha_1 + h_{r_{01}}, \dots, \alpha_d + h_{r_{0d}})) - q_n((\alpha_1, \dots, \alpha_d))}{(h_{r_{01}} \dots h_{r_{0d}})} \\
 &= (h_{r_{01}}^{-1} \dots h_{r_{0d}}^{-1}) q_n((\alpha_1 + h_{r_{01}}, \dots, \alpha_d + h_{r_{0d}})) \\
 &\leq C(h_{r_{01}}^{-1} \dots h_{r_{0d}}^{-1}) \omega_3^\varphi(f, \delta).
 \end{aligned}$$

And if $q_n'((\alpha_1, \dots, \alpha_d)) < 0$, then by (5), for some $\xi \in (\alpha_1 - h_{r_{01}}, \alpha_1) \times \dots \times (\alpha_d - h_{r_{0d}}, \alpha_d)$,

$$\begin{aligned}
 0 &\leq -q_n'((\alpha_1, \dots, \alpha_d)) \leq -q_n'((\xi_1, \dots, \xi_d)) \\
 &= \frac{q_n((\alpha_1 - h_{r_{01}}, \dots, \alpha_d - h_{r_{0d}})) - q_n((\alpha_1, \dots, \alpha_d))}{(h_{r_{01}} \dots h_{r_{0d}})} \\
 &= (h_{r_{01}}^{-1} \dots h_{r_{0d}}^{-1}) q_n((\alpha_1 - h_{r_{01}}, \dots, \alpha_d - h_{r_{0d}})) \\
 &\leq C(h_{r_{01}}^{-1} \dots h_{r_{0d}}^{-1}) \omega_3^\varphi(f, \delta).
 \end{aligned}$$

Hence by (9), (10) and (14)

$$\begin{aligned}
 E_4 &\leq |q'_n((\alpha_1, \dots, \alpha_d))| \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} |V_n((u_1, \dots, u_d)) - W_n((u_1, \dots, u_d))| du_1 \dots du_d \\
 &\leq C(h_{r_{01}} \dots h_{r_{0d}}) |q'_n((\alpha_1, \dots, \alpha_d))| \leq C \omega_3^\varphi(f, \delta),
 \end{aligned} \tag{15}$$

Combining (12), (13) and (15) we see that

$$\|f - P_n\| \leq C(s) \omega_3^\varphi(f, \delta).$$

References

- [1] K. A. Kopotun, "Coconvex polynomial approximation of twice differentiable functions", *J. Approx. Theory*, 83141-156, (1995).
- [2] M. A. Kareem, "On the multiapproximation in suitable spaces", Ph. M Dissertation, Babylon University, College of Education, (2011).
- [3] D. Leviatan, "Pointwise estimates for convex polynomial approximation", *Proc. Amer. Math. Soc.* 98 471-474, (1986).
- [4] D.C. Myers, "Comonotone and coconvex approximation", Ph. D. dissertation, Temple University :L. Raymon supervisor, 1975.
- [5] I. A. Shevchuk, "Approximation of monotone functions by monotone polynomials", *Russ. Akad. Nauk. Matem. Sbornik* 183 (1992), English transl. in *Russ. Acad. Sci. Sbornik Math.* 76, 51-64, (1993).
- [6] I. A. Shevchuk, "Approximation by polynomials and traces of functions continuous on an interval", (in Russian), *Naukova Dumka*, Kiev, Ukraine, 1992.