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Block by Block and Finite Difference Hybrid Method to Solve Linear and Nonlinear Volterra Integro-Partial Differential Equations

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Abstract

In this paper a hybrid consistency of Block by Block method and finite difference method was proposed to solve two-dimensional linear and nonlinear Volterra integro-partial differential equations. This hybridization is based on using both finite difference forward formal method for derivatives required for the formulation and then applying Block by Block method. The numerical solution of examples and the comparison with the corresponding exact solution are performed and the results revealed that the method gives good accuracy and an efficiency of this hybridization.

1. Introduction

The purpose of this paper is to employ the two-dimensional nonlinear Volterra integro-partial differential equations, which are often encountered in many branches of physics, chemistry and engineering. Different methods are used to integral equations which are investigated from many physical applications such as the mixed problems in the theory of elasticity. There are many methods to solve this type of problems such as Moss (1982) in [38] used numerical solution of one-dimensional integral equations with convolution kernel. Ladopoulos (1988) in [31] studies numerical solution of the multi-dimensional singular integrals and integral equations. Atkinson (1992) in [8] gave a survey of numerical methods for solving nonlinear one-dimensional integral equations. Su and Sakar (1999) in [47] used the moment method for solving the two-dimensional Fredholm integral equations of the first kind. Brunner et al (1999) in [1] applied splitting methods for partial Volterra integro-differential equations. Yalçinbas and Sezer (2000) in [2] studies the approximate solution of high-order linear Volterra-Fredholm integro-differential equations under the mixed conditions in terms of Taylor polynomials about any point. El-Borai (2006) in [3] gave some stochastic Fractional integro-differential equations mild solutions of the nonlocal Cauch problem for the considered classes are studied. (2007) in [4] applied the stability of some stochastic integro-partial differential equations consider the Black-Scholes market consists of a bank account or a bond and a stock. Babolian et al (2008) in [5] gave new direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equations using operational matrix with Block-puise. Mohoseni Moghadam and Saeedi (2010) in [6] using application of differential transforms for solving two-dimensional Volterra integro-partial differential equations.

2. The Proposed Problem Method

In this section, we present a numerical method to solve

$$\alpha\frac{\partial^2\phi(x,t)}{\partial x^2}+\beta\frac{\partial^2\phi(x,t)}{\partial x\,\partial t}+\gamma\frac{\partial^2\phi(x,t)}{\partial t^2}+\delta\frac{\partial\phi(x,t)}{\partial x}+\varepsilon\frac{\partial\phi(x,t)}{\partial t}+\mu\phi(x,t)$$
$$=f(x,t)+\lambda\int_0^t\int_a^b F(t,\tau)k(x,y)G(\tau,y,\phi(y,\tau))dyd\tau,\quad 0\le t\le T<1 \qquad \ldots\ldots(2.1)$$

This method consists of three phases.

In phase one, we rewrite this equation as a system of linear Volterra integro-partial differential equations. To do so, we choose a sufficiently small step size h_x, and we assume $a=x_0<x_1<\ldots<x_{N_x}=b$ is a partition of the interval $[a,b]$ with $h_x=(b-a)/N_x$.

Then, at each point x_n, (1.1) becomes

$$\alpha\frac{\partial^2\phi_n(t)}{\partial x^2}+\beta\frac{\partial^2\phi_n(t)}{\partial x\,\partial t}+\gamma\frac{\partial^2\phi_n(t)}{\partial t^2}+\delta\frac{\partial\phi_n(t)}{\partial x}+\varepsilon\frac{\partial\phi_n(t)}{\partial t}+\mu\phi_n(t)$$
$$=f_n(t)+\lambda\int_0^t\int_a^b F(t,\tau)k(x_n,y)G(\tau,y,\phi(y,\tau))dyd\tau,\quad 0\le t\le T \qquad \ldots\ldots (2.2)$$

Where

$$\phi_n(t)=\phi(x_n,t),\quad f_n(t)=f(x_n,t),\quad n=0,1,\ldots,N_x \qquad \ldots\ldots \quad (2.3)$$

Replacing the integral $\int_a^b k(x_n,y)\gamma(\tau,y,\phi(y,\tau))dy$ by a numerical integration rule of the form $\sum_{j=0}^{n} w_j k_{nj}\gamma_j(\tau,y_j,\phi_j(\tau))$, where , then an approximate to (2.1) can be found as a system of Volterra integro-partial differential equations:

$$\alpha\frac{\partial^2\phi_n(t)}{\partial x^2}+\beta\frac{\partial^2\phi_n(t)}{\partial x\,\partial t}+\gamma\frac{\partial^2\phi_n(t)}{\partial t^2}+\delta\frac{\partial\phi_n(t)}{\partial x}+\varepsilon\frac{\partial\phi_n(t)}{\partial t}+\mu\phi_n(t)$$
$$=f_n(t)+\lambda\sum_{j=0}^{n}w_j k_{nj}\int_0^t F(t,\tau)\gamma_j(\tau,y_j,\phi_j(\tau))d\tau,\quad n=0,1,\ldots,N_x \qquad \ldots\ldots (2.4)$$

In phase two, we use the block-by-block method to solve such system of integral equations of Volterra type. The following subsection explains how this method works.

***2.1** Block-by-Block Method*

Assume that we need to solve an equation of the form.

$$u(t)=\nu(t)+\int_0^t H(t,s,u(s))ds \qquad \ldots\ldots \quad (2.5)$$

The idea behind the block-by-block method is to divide the interval $[0,t]$ into a mesh $0=t_0<t_1<\ldots<t_n<\ldots<t_{N_t}=t<T$, and then we try to evaluate the value of the unknown function $u(t)$ at these points except at $t=0,$ where we have that $u=0=\nu(0).$

Using any known rule, say Simpson's rule, we have

$$u(t_2)=\nu(t_2)+\lambda\frac{h_t}{3}\{H(t_2,t_0,u(t_0))+4H(t_2,t_1,u(t_1))+H(t_2,t_2,u(t_2))\} \qquad \ldots\ldots (2.6)$$

To obtain a value for $u(t_1)$, we introduce the point $t_{1/2}=h_t/2,\quad h_t=t/N_t$, and then we use Simpson's rule rule again to obtain.

$$u(t_1)=\nu(t_1)+\lambda\frac{h_t}{3}\{H(t_1,t_0,u(t_0))+4H(t_1,t_{1/2},u(t_{1/2}))+H(t_1,t_1,u(t_1))\} \qquad \ldots\ldots (2.7)$$

Replacing the value $u(t_{1/2})$ by a quadratic interpolation using the values u_0,u_1 and u_2, we have

$$u(t_{1/2})=\frac{3}{8}u(t_0)+\frac{3}{4}u(t_1)-\frac{1}{8}u(t_2) \qquad \ldots\ldots \quad (2.8)$$

So that we can compute $u(t_1)$ by

$$u(t_1)=\nu(t_1)+\lambda\frac{h_t}{3}\left\{H(t_1,t_0,u(t_0))+4H\left(t_1,t_{1/2},\left[\frac{3}{8}u(t_0)+\frac{3}{4}u(t_1)-\frac{1}{8}u(t_2)\right]\right)+H(t_1,t_1,u(t_1))\right\} \qquad \ldots\ldots (2.9)$$

Equations (2.6) and (2.9) are a pair of simultaneous equations for $u(t_1)$ and $u(t_2)$. For sufficiently small h_t, $u(t_1)$ and $u(t_2)$ can be found uniquely using any procedure such as Newton's method.

In general, for $m=0,1,2,...,N_t-1$, the approximate solution of (2.5) is evaluated using the following two equations:

$$u(t_{2m+1})=v(t_{2m+1})+\lambda h_t\sum_{s=0}^{2m}k_sH(t_{2m+1},t_s,u(t_s))+\lambda\frac{h_t}{3}\{H(t_{2m+1},t_{2m},u(t_{2m}))+ 4H\left(t_{2m+1},t_{2m+1/2},\left[\frac{3}{8}u(t_{2m})+\frac{3}{4}u(t_{2m+1})-\frac{1}{8}u(t_{2m+2})\right]\right)+H(t_{2m+1},t_{2m+1},u(t_{2m+1}))\} \quad \text{...... (2.10)}$$

$$u(t_{2m+2})=v(t_{2m+2})+\lambda h_t\sum_{s=0}^{2m}k_sH(t_{2m+2},t_s,u(t_s))+\lambda\frac{h_t}{3}\{H(t_{2m+2},t_{2m},u(t_{2m}))+ 4H(t_{2m+2},t_{2m+1},u(t_{2m+1}))+H(t_{2m+2},t_{2m+2},u(t_{2m+2}))\} \quad \text{...... (2.11)}$$

where

$$k_s=\frac{1}{3}\{1,4,2,...,2,4,1\},\quad s=0,1,2,...,2m$$

$$t_{2m+1/2}=t_{2m}+\frac{h_t}{2} \quad \text{...... (2.12)}$$

Then the problem in VFIPDE as follows

$$\alpha\frac{\partial^2\phi_n(t)}{\partial x^2}+\beta\frac{\partial^2\phi_n(t)}{\partial x\,\partial t}+\gamma\frac{\partial^2\phi_n(t)}{\partial t^2}+\delta\frac{\partial\phi_n(t)}{\partial x}+\varepsilon\frac{\partial\phi_n(t)}{\partial t}+\mu\phi_n(t)=f_n(t_{2m+1})+ \lambda\sum_{j=1}^{n}w_jh_t\sum_{s=0}^{2m}k_sF(t_{2m+1},t_s)k(x_n,y_j)G_j(t_s,\phi_n(t_s))+\lambda\frac{h_t}{3}\{F(t_{2m+1},t_{2m})G_n(t_{2m},\phi_n(t_{2m}))+G_n(t_{2m},\phi_n(t_{2m})) 4F\left(t_{2m+1},t_{2m+1/2},\left[\frac{3}{8}+\frac{3}{4}G_n(t_{2m+1},\phi_n(t_{2m+1}))-\frac{1}{8}G_n(t_{2m+2},\phi_n(t_{2m+2}))\right]\right)+F(t_{2m+1},t_{2m+1})G_n(t_{2m+1},\phi_n(t_{2m+1}))\} \quad \text{...... (2.13)}$$

Now to find the left side of equation (2.13) assume that we need to solve an equation of the form

$$\left(\frac{\partial^2u}{\partial x^2}\right)_{n,m}+\left(\frac{\partial^2u}{\partial x\,\partial t}\right)_{n,m}+\left(\frac{\partial^2u}{\partial t^2}\right)_{n,m}+\left(\frac{\partial u}{\partial x}\right)_{n,m}+\left(\frac{\partial u}{\partial t}\right)_{n,m}+u_n(t_m)=\Psi_n(t_m) \quad \text{...... (2.14)}$$

where if m is odd number

$$\Psi_n(t_m)=f_n(t_{2m+1})+\lambda\sum_{j=1}^{n}w_jh_t\sum_{s=0}^{2m}k_sF(t_{2m+1},t_s)k(x_n,y_j)G_j(t_s,\phi_n(t_s))+\lambda\frac{h_t}{3}\{F(t_{2m+1},t_{2m})G_n(t_{2m},\phi_n(t_{2m}))+ 4F\left(t_{2m+1},t_{2m+1/2},\left[\frac{3}{8}+\frac{3}{4}G_n(t_{2m+1},\phi_n(t_{2m+1}))-\frac{1}{8}G_n(t_{2m+2},\phi_n(t_{2m+2}))\right]\right)+F(t_{2m+1},t_{2m+1})G_n(t_{2m+1},\phi_n(t_{2m+1}))\}$$

and if m is even number

$$\Psi_n(t_m)=f_n(t_{2m+2})+\lambda\sum_{j=1}^{n}w_jh_t\sum_{s=0}^{2m}k_sF(t_{2m+2},t_s)k(x_n,y_j)G_j(t_s,\phi_n(t_s))+\lambda\frac{h_t}{3}\{F(t_{2m+2},t_{2m})G_n(t_{2m},\phi_n(t_{2m})) +4F(t_{2m+2},t_{2m+1})G_n(t_{2m+2},\phi_n(t_{2m+1}))+F(t_{2m+2},t_{2m+2})G_n(t_{2m+2},\phi_n(t_{2m+2}))\}$$

We use the forward finite difference formal to solve the equations (2.14)

$$\frac{1}{h_x^2}[u_{n+1}(t_m)-2u_n(t_m)+u_{n-1}(t_m)]+\frac{1}{h_xh_t}[u_{n+1}(t_{m+1})-u_{n+1}(t_m)-u_n(t_{m+1})+u_n(t_m)] +\frac{1}{h_t^2}[u_n(t_{m+1})-2u_n(t_m)+u_n(t_{m-1})]+\frac{1}{h_x}[u_{n+1}(t_m)-u_n(t_m)]+\frac{1}{h_t}[u_n(t_{m+1})-u_n(t_m)]+u_n(t_m)=\Psi_n(t_m) \quad \text{...... (2.15)}$$

***2.2** Hybriding of the Block-by-Block Method and Finite Difference Method on the NVFIPDE*

Combining the previous results, we obtain a new scheme defined by the following two equations:

$$\alpha\left\{\frac{1}{h_x^2}\left[\phi_{n+1}(t_{2m+1})-2\phi_n(t_{2m+1})+\phi_{n-1}(t_{2m+1})\right]\right\}+\beta\left\{\frac{1}{h_x h_t}\left[\phi_{n+1}(t_{2m+2})-\phi_{n+1}(t_{2m+1})-\phi_n(t_{2m+2})+\phi_n(t_{2m+1})\right]\right\}$$
$$+\gamma\left\{\frac{1}{h_t^2}\left[\phi_n(t_{2m+2})-2\phi_n(t_{2m+1})+\phi_n(t_{2m})\right]\right\}+\delta\left\{\frac{1}{h_x}\left[\phi_{n+1}(t_{2m+1})-\phi_n(t_{2m+1})\right]\right\}+\varepsilon\left\{\frac{1}{h_t}\left[\phi_n(t_{2m+2})-\phi_n(t_{2m+1})\right]\right\}$$
$$+\mu\phi_n(t_{2m+1})=f_n(t_{2m+1})+\lambda\sum_{j=1}^{n}w_j h_t\sum_{s=0}^{2m}k_s F(t_{2m+1},t_s)k(x_n,y_j)G_j(t_s,\phi_n(t_s))+\lambda\frac{h_t}{3}\{F(t_{2m+1},t_{2m})G_n(t_{2m},\phi_n(t_{2m}))+$$
$$4F\left(t_{2m+1},t_{2m+1/2},\left[\frac{3}{8}G_n(t_{2m},\phi_n(t_{2m}))+\frac{3}{4}G_n(t_{2m+1},\phi_n(t_{2m+1}))-\frac{1}{8}G_n(t_{2m+2},\phi_n(t_{2m+2}))\right]\right)+F(t_{2m+1},t_{2m+1})G_n(t_{2m+1},\phi_n(t_{2m+1}))\Big\}$$

...... (2.16)

$$\alpha\left\{\frac{1}{h_x^2}\left[\phi_{n+1}(t_{2m+2})-2\phi_n(t_{2m+2})+\phi_{n-1}(t_{2m+2})\right]\right\}+\beta\left\{\frac{1}{h_x h_t}\left[\phi_{n+1}(t_{2m+3})-\phi_{n+1}(t_{2m+2})-\phi_n(t_{2m+3})+\phi_n(t_{2m+2})\right]\right\}$$
$$+\gamma\left\{\frac{1}{h_t^2}\left[\phi_n(t_{2m+3})-2\phi_n(t_{2m+2})+\phi_n(t_{2m+1})\right]\right\}+\delta\left\{\frac{1}{h_x}\left[\phi_{n+1}(t_{2m+2})-\phi_n(t_{2m+2})\right]\right\}+\varepsilon\left\{\frac{1}{h_t}\left[\phi_n(t_{2m+3})-\phi_n(t_{2m+2})\right]\right\}$$
$$+\mu\phi_n(t_{2m+2})=f_n(t_{2m+2})+\lambda\sum_{j=1}^{n}w_j h_t\sum_{s=0}^{2m}k_s F(t_{2m+2},t_s)k(x_n,y_j)G_j(t_s,\phi_n(t_s))+\lambda\frac{h_t}{3}\{F(t_{2m+2},t_{2m})G_n(t_{2m},\phi_n(t_{2m}))+$$
$$4F(t_{2m+2},t_{2m+1})G_n(t_{2m+2},\phi_n(t_{2m+1}))+F(t_{2m+2},t_{2m+2})G_n(t_{2m+2},\phi_n(t_{2m+2}))\}$$

...... (2.17)

3. Examples

We use the block by block and finite differential method to solve numerically the following examples when

(i) Case 1:If the equations in linear form $\gamma(t,x,\phi(x,t))=\phi(x,t)$,

(ii) Case 2: If the equations in nonlinear form $\gamma(t,x,\phi(x,t))=\phi^2(x,t)$.

Example 3.1. Solve the VFIPDE

$$\frac{\partial^2\phi}{\partial t^2}+\frac{\partial\phi}{\partial x}=f(x,t)+\lambda\int_0^t\int_0^1\sin(x+y)\tau^2\gamma(t,x,\phi(x,t))dy\,d\tau;\ \text{where } \lambda=1,\ 0\le t\le T. \quad \dots (3.1)$$

Such that $\phi(x,0)=0,\quad \phi(0,t)=t^2$

The exact solution of this integral equation is $\phi(x,t)=t^2\sin(x)$. Tables 1 and Tables 2 list the error at different values of x and t for different value of N_x (number of partitions of the interval $[a,b]$ and $2N_t$) (number of partitions of the interval $[0,T]$; $T=0.5$).

Table 1 The error at different values of x and t for different value of N_x)

(x,y)	Appro. Case 1	error Case 1	Appro. Case 2	error Case 2
(0.5,0.1)	0.0073	0.0025	0.0082	0.0034
(0.5,0.2)	0.241	0.0049	0.0243	0.0051
(0.5,0.3)	0.0507	0.0076	0.0499	0.0068
(0.5,0.4)	0.0867	0.01	0.0853	0.0086
(0.5,0.5)	0.1321	0.0122	0.1291	0.0092
(0.5,0.6)	0.1869	0.0143	0.1825	0.0099
(0.5,0.7)	0.2507	0.0158	0.2462	0.0113
(0.5,0.8)	0.3233	0.0165	0.3192	0.0124
(0.5,0.9)	0.4053	0.0170	0.4017	0.0134
(0.5,1.0)	04972	0.0178	0.4949	0.0155

Example 3.2. Solve the VFIPDE

$$\frac{\partial^2\phi}{\partial t^2}+\frac{\partial\phi}{\partial x}=f\left(x,t\right)+\lambda\int_0^t\int_0^1 e^{x+y}\tau^2\gamma\left(t,x,\phi\left(x,t\right)\right)dyd\tau; \qquad \text{…… (3.2)}$$

where $\lambda=0.1,\ 0\le t\le T$.

Such that $\phi(x,0)=0,\quad \phi(0,t)=t^2$

The exact solution of this integral equation is $\phi(x,t)=t^2e^x$. Tables 1 and tables 2 list the error at different values of x and t for different value of N_x (number of partitions of the interval $[a,b]$ and $2N_t$) (number of partitions of the interval $[0,T]$; $T=0.5$).

Table 2 (The error at different values of x and t for different value of N_x)

(x,y)	Appro. Case 1	Error Case 1	Appro. Case 2	Error Case 2
(0.5,0.1)	0.0123	0.0012	0.0132	0.0021
(0.5,0.2)	0.0466	0.0024	0.0478	0.0036
(0.5,0.3)	0.1032	0.0037	0.1047	0.0052
(0.5,0.4)	0.1811	0.0043	0.1832	0.0064
(0.5,0.5)	02821	0.0058	0.2841	0.0078
(0.5,0.6)	0.4051	0.0072	0.4074	0.0095
(0.5,0.7)	0.5501	0.0086	0.5529	0.0114
(0.5,0.8)	0.7168	0.0095	0.7200	0.0127
(0.5,0.9)	0.9053	0.0101	0.9090	0.0138
(0.5,1.0)	1.1174	0.0122	1.1203	0.0151

4. Conclusion

Two rules of numerical integro-partial differential methods are used to solve (2.1). One of these rules is of order $O(h_x)$, the rectangle method, which is applied on the integral with respect to the variable x. The other method is of higher-order $O(h_t^4)$, the block-by-block method, and used finite difference formal to solve (2.1), and is applied on the integral with respect to the variable t where we used the fact that $\phi(x,0)=f(x,0)$. More information about the solution could allow us to use higher-order method with the block-by-block method.

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