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On The Performance of the Logistic-Growth Population Projection Models

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Abstract
In this paper, we proposed two mathematical Models for population projection, the exponential growth and the Logistic Model. Thereafter, both Models were transformed and since Logistic Model is not linear in its parameters hence a surgery was performed on the data values separated by a fixed time say τ. This enables the Logistic parameters to be sometimes estimated by Least-Squares. It was shown that the estimates $\hat{z}$, $\dot{z}_1$ and $\tilde{z}_1$ were quite close to the data set from the National Population Commission of Nigeria.

Keywords: Carrying capacity, Logistic Model, Exponential growth, Polyfit, Least- Squares, Population Projection.

1.1 Introduction

One important area where data collection for human population is so vital is basically for national planning or budgeting. This is while government of nations placed emphasis on Census to determine the population count. Though, this exercise is quite expensive, time consuming and yet is still one of the major ground for population count. This explains the seriousness attached to Census by government for developmental purposes.

Population data gives an insight about the Social- economic conditions of a country; hence the need for projection is usually based on mathematical models which are significant. These models use vital assumptions about its parameters values that reflect the true nature of the population for which these assumptions were made.

Choi (2010) proposed an application of the component model to development of local population projections derived from the housing unit method. Haque et al. (2012) analyzed fourth order Rung-Kutta Scheme for the numerical of the non-autonomous and non-linear model to incorporate the growth rate as a function of time. Wali et al. (2012) focuses on the application of logistic equation to model the population growth of Uganda using data from 1980 to 2010. Earterling et al. (2000) introduced the integral projection model which eliminates the need for division into discrete classes, without requiring any additional biological assumptions. National Population Commission of Nigeria (2005), National Policy on Population for Sustainable Development. Neverova and Frisman (2012) investigate a two-component model of population dynamics with seasonal reproduction. Akamine and Suda (2011) analyzed the population projection matrix models in random walk models and the growth rate of the mean population size, which is equal to the maximum eigenvalue of the mean matrix, is better than the average of the intrinsic rates of natural increase calculated by computer simulation. Simpson reviews the mechanisms which can incorporate into a cohort component population projection incomplete and estimated data since the base year, as well as targets for population, housing and employment.

2.1 The Growth Model

Consider the differential equation,

$$\frac{dy}{dt} = ry \qquad (1)$$

$$\frac{dy}{y} = rdt \qquad (2)$$

$$y(t) = e^{rt+c}, where\ e^c = A \qquad (3)$$

$$At\ t = 0, y(0) = Ae^{r(0)} \qquad (4)$$

$A = y(0) = y_0$ is the initial population

$$\therefore\ y(t) = y_0 e^{rt} \qquad (5)$$

From equation (5)

$\log y = rt + \log A$ …………………………………………………..(5a)

$=> \log y = rt + b, where \log A = b$, $\log y = \frac{1}{z}$

$\log y = mt + b$ ………………………………………………...........(5b)

Equation (5) gives the solution to the differential equation (1) and equation (5b) is Linear in its parameters. Equation (5) gives the population size accurately after some time t, thereafter the population goes unbounded resulting to explosion in the population. Hence, we now consider the Logistic Model.

2.2 Logistic Model

Consider the model by Verhulst in 1845, Yeargers et al. (1996),

$\frac{dy}{dt} = ry(1 - \frac{y}{k})$ ………………………………………………………..(6)

Where, y = population size

$\frac{dy}{dt} = per\ capita\ growth\ rate$

K = demographic indicator (Carrying Capacity)

r = growth Rate.

It is easy to see that $\frac{dy}{dt} = 0$, when y = 0 or y = k. These are the stationary points of the equation (6). The stationary point y = k, at which the per capita growth rate becomes zero is called the carrying capacity of the environment.

2.3 Solution to the Model

$\frac{dy}{dt} = ry(1 - \frac{y}{k})$ ……………………………………………………….(6)

$\frac{dy}{y(1-\frac{y}{k})} = rdt$……………………………………………………………(7)

Decomposing the L.H.S of equation (7) by partial fraction

$\left(\frac{1}{y} + \frac{\frac{1}{k}}{\left(1-\frac{y}{k}\right)}\right) dy = rdt$………………………………………….......(8)

$\frac{y}{1-\frac{y}{k}} = Ae^{rt}$…………………………………………………….....……(9)

Reciprocating both sides,

$\frac{1}{y} - \frac{1}{k} = \frac{1}{Ae^{rt}}$………………………………………………………….(10)

Isolating y,

$y = \frac{Ae^{rt}}{1+\frac{Ae^{rt}}{k}}$ ………………………………………………………….....(11)

Equation (12) is the solution to the differential equation (6).But, equation (6) with initial population $y_0 = y(0)$ can be obtained.

$y\ (t) = \frac{k}{1+\frac{e^{-rt}(k-y_0)}{y_0}}$ …………………………………………………….(12)

$\therefore\ y(t) = \frac{ke^{rt}y_0}{k+y_0\,(e^{rt}-1)}$ ………………………………………………….(13)

Logistic parameters can sometimes be estimated by Least Squares. Equation (12) is not Linear in its parameters A, r and K. If the data values are separated by fixed time τ, then Least Squares can be applicable.

Suppose the data points are $(t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)\ with\ t_i = t_{i-1} + \tau, i = 2, \ldots, n.\ Then\ t_i = t_1 + (i-1)\tau$ and the predicted value of $\frac{1}{y_i}$ from equation (10)

$\frac{1}{y_i} = \frac{1}{Ae^{rt_1}e^{(i-1)rt}} + \frac{1}{k}$ ………………………………………………...(14)

$= \frac{1}{e^{rt}}\left(\frac{1}{Ae^{rt_1}e^{(i-2)rt}} + \frac{e^{r\tau}}{k}\right)$ ………………………………………………(15)

$but, \frac{e^{rt}}{k} = \frac{e^{rt}-1}{k} + \frac{1}{k}$ ………………………………………..………(16)

$where, \frac{1}{y_{i-1}} = \frac{1}{Ae^{rt_1(i-2)r\tau}} + \frac{1}{k} => \frac{1}{Ae^{rt_1(i-2)r\tau}} = \frac{1}{y_{i-1}} - \frac{1}{k}$ ……..……...(17)

Substitute equation (17) into equation (15)

$\frac{1}{y_i} = \frac{1}{e^{rt}}\left(\frac{1}{Ae^{rt_1}e^{(i-2)rt}} + \frac{e^{r\tau}}{k}\right)$

$\frac{1}{y_i} = \frac{1}{e^{rt}}\left(\frac{1}{y_{i-1}} - \frac{1}{k} + \frac{e^{r\tau}}{k}\right)$ ……………………………………………..(18)

$\frac{1}{y_i} = \frac{1}{e^{rt}}\left(\frac{1}{y_{i-1}} + \frac{e^{r\tau}-1}{k}\right)$ ……………………………………………........(19)

$\frac{1}{y_i} = \frac{1}{e^{rt}} . \frac{1}{y_{i-1}} + \frac{1-e^{-r\tau}}{k}$ ……………………………………...……….(20)

Set $z = \frac{1}{y}$ in equation (20)

$z_i = \frac{1}{e^{rt}} . z_{i-1} + \frac{1-e^{-r\tau}}{k}$ ……………………………………..………..(21)

$\therefore\ z_i = e^{-r\tau} . z_{i-1} + \frac{1-e^{-r\tau}}{k}$ ………………………………..….........(22)

Therefore, a Least-Square is performed on the points, $(z_1, z_2), (z_2, z_3), \ldots, (z_{i-1}, z_i)$ to determine r and k. With r and k known, A can be determined from equation (10) to obtain an estimate of z.

From equation (22), set $\rho = e^{-r\tau}$ and $\beta = \frac{1-e^{-r\tau}}{k}$,

$=> z_i = \rho z_{i-1} + \beta$……………………………………………………(23)

We shall perform Least-Squares on equation (23) with the set of data $(z_0, z_1), (z_1, z_2), \ldots, (z_{i-1}, z_i)$ using MATLAB inbuilt function.

3.1 Analysis of Model for Data I

CASE 1: Consider the solution from equation (5b)

$\log y = mt + b$ and applying MATLAB inbuilt function on the set of data for the population projection system of Male in KANO STATE, NIGERIA.

Applying MATLAB inbuilt function to estimate equation (5b)

```
>> t=[0 1 2 3 4 5 6 7 8 9 10 11];
>> z=[4947952 5113958 5285534 5462867 5646149 5835580 6031367 6233723        6442867 6659029
6882443 7113353];
>> logy=(z);
>> p=polyfit(t,logy,1)
p =
  1.0e+006 *
    0.1965    4.8903
```

$where\ m = 196500\ and\ b = 4890300$

$=> \hat{z} = mt + b$

$\therefore\ \hat{z} = 196500t + 4890300$ ……………………………………………...(24)

CASE 2: Consider the solution from equation (23)

$z_i = \rho z_{i-1} + \beta$ ………………………………………………………..…...(23)

applying Least-Squares on $(z_0, z_1), (z_1, z_2), \ldots, (z_{i-1}, z_i)$ and using MATLAB inbuilt function on the set of data for the population projection system of Male in KANO STATE, NIGERIA.As in Table 1.

$set\ z_{i-1} = x\ and\ z_i = y\ (dummy\ variables)$

```
>> x=[4947952 5113958 5285534 5462867 5646149 5835580 6031367 6233723 6442867 6659029 6882443];
>> y=[5113958 5285534 5462867 5646149 5835580 6031367 6233723 6442867 6659029 6882443 7113353];
>> p=polyfit(x,y,1)
p =
   1.0336   -0.8600
>> xp=4947952:10000:6882443;
>> zp=polyval(p,xp)
>> plot(x,z,'o',xp,zp)
```

From equation (23)

$\dot{z}_i = 1.0336\dot{z}_{i-1} + (-0.8600)$ ……………………………………….(25)

Recall, $\rho = e^{-r\tau}$ and $\beta = \frac{1-e^{-r\tau}}{k}$

$r = -0.0330, k = 0.0390$ with r and k known A can be determined from equation (10).

CASE 3: Consider equation (10)

$at\ t\ \frac{1}{y} = \frac{1}{Ae^{rt}} + \frac{1}{k} => z_i = \frac{e^{-rt}}{A} + k^{-1}$ ……………………………….(10)

$= 0, z_0 = 4947952, k^{-1} = 25.63297102\ and\ r = -0.0330$

$\therefore\ \frac{1}{A} = 4947926.367$

From equation (10), we have

$\tilde{z}_i = 4947926.367e^{-(-0.0330)t} + 25.63297102$ ……………….…….(26)

From equation (24), (25) and (26) we have,

$\hat{z} = 196500t + 4890300$ ………………………………………….......(24)

$\dot{z}_i = 1.0336\dot{z}_{i-1} + (-0.8600)$ ……………………………………..(25)

$\tilde{z}_i = 4947926.367e^{-(-0.0330)t} + 25.63297102$ ………………….......(26)

A projection can be obtained from the three equations above to estimate the given data z.

3.2 Analysis of Model for Data II

CASE 1: Consider the solution from equation (5b)

$\log y = mt + b$ and applying MATLAB inbuilt function on the set of data for the population projection system for Male and Female in KANO STATE, NIGERIA.

Applying MATLAB inbuilt function to estimate equation (5b)

```
>> t=[0 1 2 3 4 5 6 7 8 9 10 11];
>> z=[9401288 9716706 10042707 10379645 10727887 11087814 11459816
      11844299 12241682 12652397 13076891 13515628];
>> logy=(z);
>> p=polyfit(t,logy,1)
p =
  1.0e+006 *
   0.3734    9.2917
```

$where\ m = 373400\ and\ b = 9291700$

$=> \hat{z} = mt + b$

$\therefore\ \hat{z} = 373400t + 9291700$ ……………………………….…….(27)

CASE 2: Consider the solution from equation (23)

$z_i = \rho z_{i-1} + \beta$ …………………………………………………........(23)

applying Least-Squares on $(z_0, z_1), (z_1, z_2), \ldots, (z_{i-1}, z_i)$ and using MATLAB inbuilt function on the set of data for the population projection system for both Male and Female in KANO STATE, NIGERIA.As in Table 3.

$set\ z_{i-1} = x\ and\ z_i = y\ (dummy\ variables)$

```
>> x=[9401288 9716706 10042707 10379645 10727887 11087814 11459816    11844299 12241682 12652397
13076891];
>> y=[9716706 10042707 10379645 10727887 11087814 11459816 11844299   12241682 12652397 13076891
13515628];
>> p=polyfit(x,y,1)
p =
   1.0336   -0.2401
>> xp=9401288:10000:13076891;
>> yp=polyval(p,xp)
>> plot(x,y,'o',xp,yp)
```

From equation (25),

$\dot{z}_i = \rho z_{i-1} + \beta$

$\dot{z}_i = 1.0336 z_{i-1} - 0.2401$……………………………………………(28)

Recall, $\rho = e^{-r\tau}\ and\ \beta = \frac{1-e^{-r\tau}}{k}$

$r = -0.0330,\ k = 0.13973569$ with r and k known A can be determined from equation (10).

CASE 3: Consider equation (10)

$\frac{1}{y} = \frac{1}{Ae^{rt}} + \frac{1}{k} \Rightarrow z_i = \frac{e^{-rt}}{A} + k^{-1}$ ………………………………………(10)

$at\ t = 0, z_0 = 9401288, k^{-1} = 7.156367839\ and\ r = -0.0330$

$\therefore\ \frac{1}{A} = 9401295.156$

From equation (10), we have

$\tilde{z}_i = 940295.156 e^{-(-0.0330)t} + 7.156367839$ …………………………..(29)

Therefore,

$\hat{z} = 373400t + 9291700$ …………………………………………………(27)

$\dot{z}_i = 1.0336 z_{i-1} - 0.2401$…………………………………………………(28)

$\tilde{z}_i = 940295.156 e^{-(-0.0330)t} + 7.156367839$ …………………………...(29)

3.3 Analysis of Model for Data III

CASE 1: Consider the solution from equation (5b)

$\log y = mt + b$ and applying MATLAB inbuilt function on the set of data for the population projection system for both Male and Female in AKWA IBOM STATE, NIGERIA.

Applying MATLAB inbuilt function to estimate equation (5b)

```
>> t=[0 1 2 3 4 5 6 7 8 9 10 11 12];
>> z=[3902051 4037001 4176620 4321066 4470509 4625119 4785077 4950567
      5121781 5298916 5482177 5671776 5867932];
>> logy=(z);
>> p=polyfit(t,logy,1)
p =
  1.0e+006 *
   0.1635    3.8430
```

$where\ m = 163500\ and\ b = 3843000$

$\Rightarrow\ \hat{z} = mt + b$

$\therefore\ \hat{z} = 163500t + 3843000$ …………………………………………..(30)

CASE 2: Consider the solution from equation (23)

$z_i = \rho z_{i-1} + \beta$ ……………………………………………………….(23) applying Least-Squares on $(z_0, z_1), (z_1, z_2), \dots, (z_{i-1}, z_i)$ and using MATLAB inbuilt function on the set of data for the population projection system for both Male and Female in AKWA-IBOM STATE, NIGERIA.As in Table 5.

$set\ z_{i-1} = x\ and\ z_i = y\ (dummy\ variables)$

```
>> x=[3902051 4037001 4176620 4321066 4470509 4625119 4785077 4950567
5121781 5298916 5482177 5671776];
>> y=[4037001 4176620 4321066 4470509 4625119 4785077 4950567 5121781
5298916 5482177 5671776 5867932];
>> p=polyfit(x,y,1)
p =
   1.0346   -0.8325
>> xp=3902051:10000:5671776;
>> yp=polyval(p,xp)
>> plot(x,y,'o',xp,yp)
```

$\dot{z}_i = 1.0346\dot{z}_{i-1} - 0.8325$ …………………………………………….(31)

Recall, $\rho = e^{-r\tau}\ and\ \beta = \frac{1-e^{-r\tau}}{k}$

$r = -0.0340, k = 0.041543071$ with r and k known A can be determined from equation (10).

CASE 3: Consider equation (10)

$\frac{1}{y} = \frac{1}{Ae^{rt}} + \frac{1}{k} => z_i = \frac{e^{-rt}}{A} + k^{-1}$ ……………………………………..(10)

$at\ t = 0, z_0 = 3902051, k^{-1} = \frac{1}{k} = 24.7140282$

$\therefore\ \frac{1}{A} = 3902026.929$ and from equation (10) we have,

$\tilde{z}_i = 3902026.929e^{-(-0.0340)t} + 24.07140282$ ……………………….(32)

Therefore,

$\hat{z} = 163500t + 3843000$ ………………………………………………...(30)

$\dot{z}_i = 1.0346\dot{z}_{i-1} - 0.8325$ ……………………………………………....(31)

$\tilde{z}_i = 3902026.929e^{-(-0.0340)t} + 24.07140282$ ………………………...(32)

4. Conclusion

In this paper a mathematical estimation for the population projection of three states in Nigeria was analyzed based on two ordinary differential equations model which are the growth model and the logistic model. Firstly, the growth model was transformed to linear model whose parameters were estimated. Secondly, it was established that the solution to the logistic model is not linear in its parameters A, r and k. Therefore a surgery was performed on the data values which are separated by fixed time τ, and then Least Squares can be applicable. Thereafter, three equations were obtained to estimate the data from the National Population commission of Nigeria. The data values from the National population Commission of Nigeria compares well with the estimated population model $\hat{z}$, $\dot{z}_i$ and $\tilde{z}_i$ in each of the states considered. We also consider the MATLB inbuilt function to polyfit the data set, and obtained a linear plot for the data set. It was discovered that $\tilde{z}_i$ has the best approximation to the population data.

Appendix

Table 1: Population Data

YEAR (t)	MALE IN KANO STATE(DATA I)	MALE & FEMALE IN KANO STATE(DATA II)	MALE & FEMALE IN AKWA-IBOM STATE (DATA III)
2006	4947952	9401288	3902051
2007	5113958	9716706	4037001
2008	5285534	10042707	4176620
2009	5462867	10379645	4321066
2010	5646149	10727887	4470509
2011	5835580	11087814	4625119
2012	6031367	11459816	4785077
2013	6233723	11844299	4950567
2014	6442867	12241682	5121781
2015	6659029	12652397	5298916
2016	6882443	13076891	5482177
2017	7113353	13515628	5671776
2018	---------	-----------	5867932

Source: National Population Commission of Nigeria, **2005**.

Table 2: Population of Male in KANO STATE, NIGERIA.

YEAR (t)	POPULATION (z)
2006	4947952
2007	5113958
2008	5285534
2009	5462867
2010	5646149
2011	5835580
2012	6031367
2013	6233723
2014	6442867
2015	6659029
2016	6882443
2017	7113353

Table 3 : Population projection of Male in KANO STATE, NIGERIA.

YEAR (t)	POPULATION (z)	$\hat{z}$	$\dot{z}_t$	$\tilde{z}_t$
2006	4947952	4890300	4947952	4947952
2007	5113958	5086800	5114203	5113958
2008	5285534	5283300	5286039	5285533
2009	5462867	5479800	5463649	5462864
2010	5646149	5676300	5647227	5646146
2011	5835580	5872800	5836973	5835580
2012	6031367	6069300	6033094	6031362
2013	6233723	6265800	6235805	6233716
2014	6442867	6462300	6445328	6442860
2015	6659029	6658800	6661890	6659021
2016	6882443	6855300	6885728	6882434
2017	7113353	7051800	7117088	7113342

Table 4: Population for both Male and Female in KANO STATE, NIGERIA.

YEAR (t)	POPULATION (z)
2006	9401288
2007	9716706
2008	10042707
2009	10379645
2010	10727887
2011	11087814
2012	11459816
2013	11844299
2014	12241682
2015	12652397
2016	13076891
2017	13515628

Table 5: Population projection for both Male and Female in KANO STATE, NIGERIA.

YEAR (t)	POPULATION (z)	$\hat{z}$	$\dot{z}_t$	$\tilde{z}_t$
2006	9401288	9291700	9401288	9401302
2007	9716706	9665100	9603900	9716720
2008	10042707	10038500	9926591	10042721
2009	10379645	10411900	10260124	10379660
2010	10727887	10785300	10604864	10727903
2011	11087814	11158700	10961188	11087829
2012	11459816	11532100	11329483	11459832
2013	11844299	11905500	11710154	11844315
2014	12241682	12278900	12103615	12241698
2015	12652397	12652300	12510296	12652414
2016	13076891	13025700	12930642	13076909
2017	13515628	13399100	13365111	13515646

Table 6: Population for both Male and Female in AKWA-IBOM STATE, NIGERIA.

YEAR (t)	POPULATION (z)
2006	3902051
2007	4037001
2008	4176620
2009	4321066
2010	4470509
2011	4625119
2012	4785077
2013	4950567
2014	5121781
2015	5298916
2016	5482177
2017	5671776
2018	5867932

Table 7: Population projection for both Male and Female in AKWA-IBOM STATE, NIGERIA.

YEAR (t)	POPULATION (z)	$\hat{z}$	$\dot{z}_\iota$	$\tilde{z}_\iota$
2006	3902051	3840300	4037061	3902051
2007	4037001	4006500	4176742	4037001
2008	4176620	4170000	4321257	4176618
2009	4321066	4333500	4470771	4321065
2010	4470509	4497000	4625459	4470506
2011	4625119	4660500	4758499	4625116
2012	4785077	4824000	4951077	4785073
2013	4950567	4987500	5122383	4950562
2014	5121781	5151000	5299617	5121774
2015	5298916	5314500	5482983	5298908
2016	5482177	5478000	5672693	5482167
2017	5671776	5641500	5868967	5671765
2018	5867932	5805000	6072033	5867920

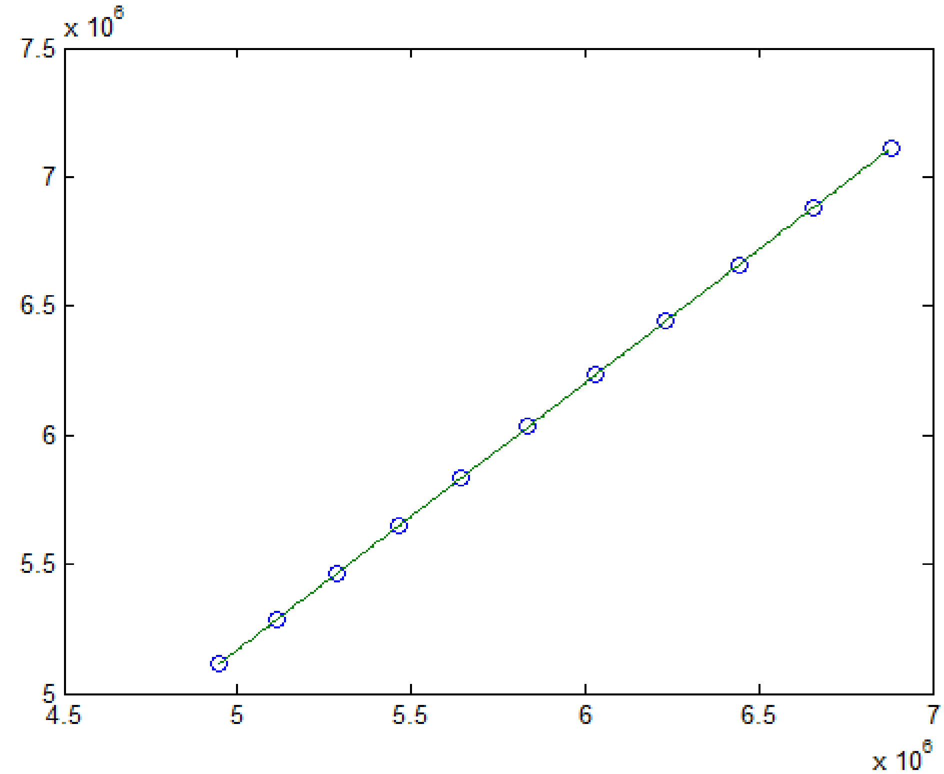

Figure 1: Linear plot for $\dot{z}_t$ in data I.

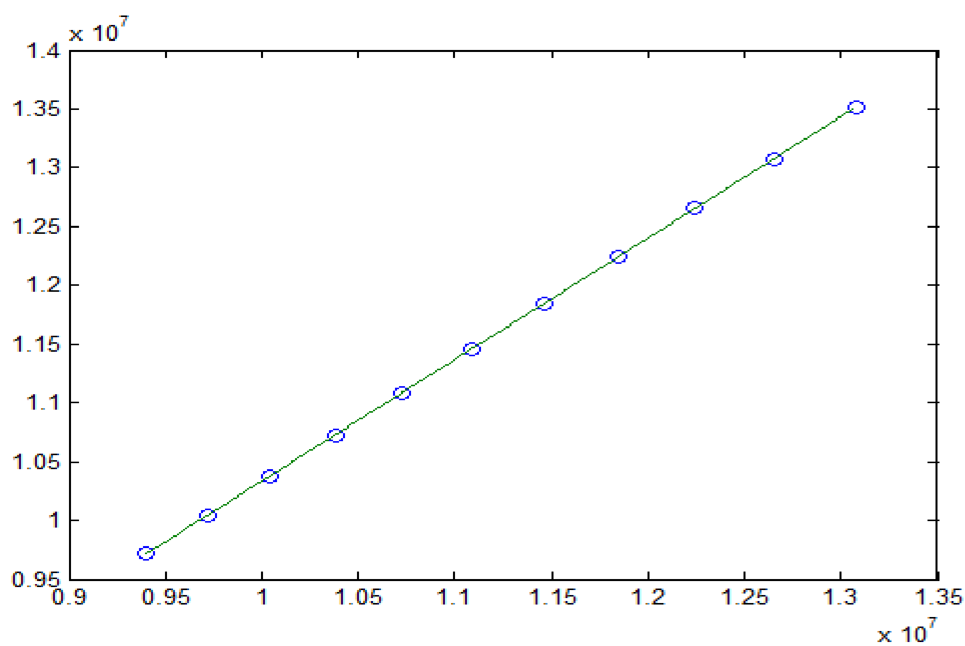

Figure 2: Linear plot for $\dot{z}_t$ in data II.

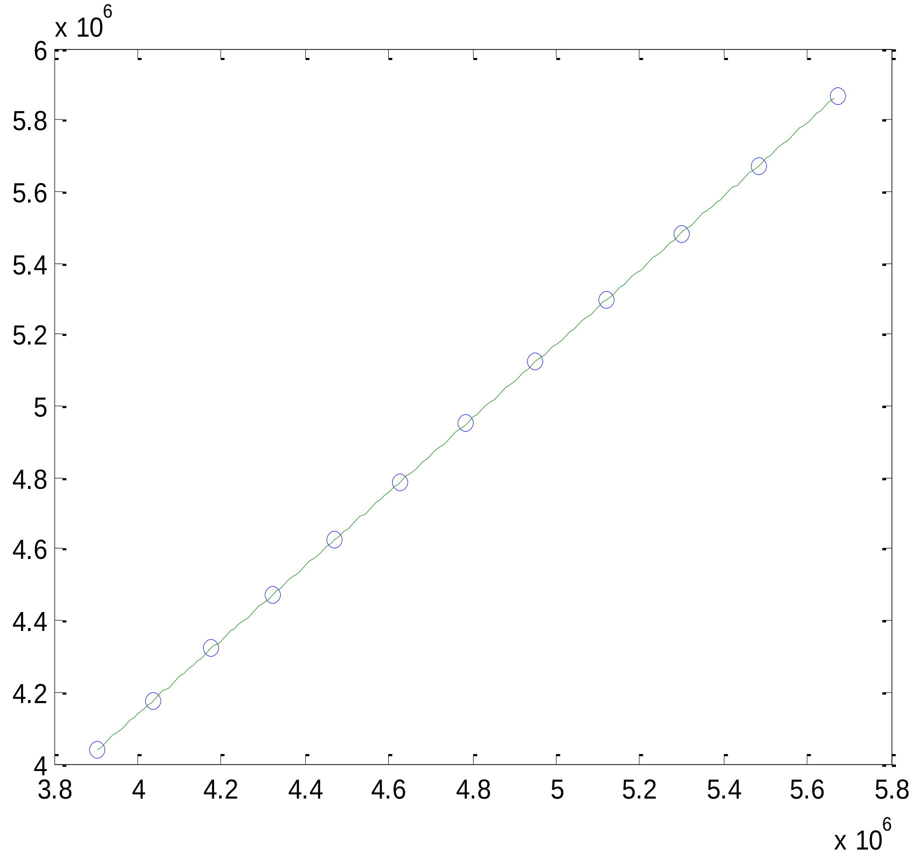

Figure 3: Linear plot for $\dot{z}_t$ in data III.

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