Mathematical Formulation of Inverse Scattering and Korteweg-De Vries Equation

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Abstract

Inverse scattering refers to the determination of the solutions of a set of differential equations based on known asymptotic solutions, that is, the solution of Marchenko equation. Marchenko equation was derived using integral equation. The potential function derived from eigenvalues and scattering data seems to be the inverse method of scattering problem. The reflection coefficient with one pole and zero reflection coefficients has been chosen to solve inverse scattering problem. Again this paper deals with the connection between inverse scattering and the Korteweg-de Vries equation and describes variety of examples with Korteweg-de Vries equation: the single-soliton solution, the two-soliton solution and finally the N-soliton solution. Throughout the work, the primary objective is to study some mathematical techniques applied in analyzing the behavior of soliton in the KdV equations.

Keywords: Marchenko equation, KdV equation, Solitons, Scattering, Inverse Scattering, Canal.

1. Introduction

In the area of scattering theory in physics, the inverse scattering problem determines the characteristics of an object (its shape, internal constitution, etc.) from measurement data of radiation or particles scattered from the object. In physical terms the problem is essentially one of finding the shape (or perhaps mass distribution) of an object which is mechanically vibrated, from a knowledge of all the sounds that makes, i.e. from the energy or amplitude at each frequency. But in mathematics, inverse scattering refers to the determination of the solutions of a set of differential equations based on known asymptotic solutions, that is, the solution of Marchenko equation [1]. Examples of equations that have been solved by inverse scattering are the reflection coefficient with pole and zero reflection coefficient. It is the inverse problem to the direct scattering problem, which is determining the distribution of scattered radiation on the characteristics of the scatterer. Since its early statement for radio location, the problem has found vast number of applications, such as echolocation, geophysical survey, nondestructive testing, medical imaging, quantum field theory, to name just a few [12].

The Korteweg-de Vries equation [3] describes the theory of water waves in shallow Channels, such as canal. It is a non-linear equation which exhibits special solutions, known as solitons, which are stable and do not disperse with time. Furthermore there as solutions with more than one soliton which can move towards each other, interact and then emerge at the same speed with no change in shape (but with a time “leg” or “speed up”). The soliton phenomenon was first described by John Scott Russell (1808-1882) who observed a solitary wave in the Union Canal, reproduced the phenomenon in a wave tank, and named it the "Wave of Translation"[4]. Owing to its nonlinearity, the KdV equation resisted analysis for many years, and it did not come under series scrutiny until 1965, when Zabusky [5] and Kruskal (see also [5]) obtained numerical solutions, while investigating the Fermi-Pasta-Ulam problem [6] of masses coupled by weakly nonlinear springs. From a general initial condition, a solution to KdV develops into a series of solitary pulses of the varying amplitudes, which pass through one another without modification of shape or speed. The only lingering trace of the strong nonlinear interaction between these so-called solitons (like electrons, protons, barions and other particles ending with ‘ons’) [10] is a slight forward phase shift for the larger, faster wave and slight backward shift for the smaller, shower one. Soliton solutions have subsequently been developed for a large number of nonlinear equations, including equations of particle physics, and magneto hydrodynamics. Having yielding a starling new type of wave behaviour, the KdV equation soon stimulated a further mathematical development, when in equation soon stimulated a further mathematical development, when in 1967, Garden[2], Green, Kruskal and Miura(see also [2]) introduced the inverse scattering transform method for determining the solitons that arise from arbitrary initial conditions. This technique represented a major advance in the mathematical theory of PDFs. As it made it possible obtain closed-form solutions to nonlinear evolution equations that were previously beyond the reach of analysis. This breakthrough initiated a period of rapid developments both in describing the properties of KdV solitons and in generalizing the approach to other nonlinear equations including the Sine-Gordon, non linear Schrödinger and Boussinesq equation.

2. Formulation of Inverse Scattering Problem

The inverse scattering technique is applicable to the KdV, Non linear Schrödinger and Sine-Gordon equations .We now describe how the inverse scattering transform can be used to construct the solution to the initial-value problem for KdV equation. In this section we shall summarize the results, and we discuss some specific examples.We wish to solve the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \qquad t > 0, \quad -\infty < x < \infty. \tag{1}$$

with $u(x,0) = f(x)$. It is assumed that f is a sufficiently well-behaved function in order to ensure the existence of a solution of the KdV equation and also of the Sturn-Liouville equation

$$\Psi_{xx} + (\lambda - u)\Psi = 0, \qquad -\infty < x < \infty. \tag{2}$$

The first stage to set $u(x,0) = f(x)$ and solve equation (2), at least to the extent of determining the discrete spectrum, $-k_n^2$, the normalization constants, $c_n(0)$, and the reflection coefficients, $b(k;0)$. The time evolution of these scattering data is then given by equations.

$$k_n = cons\tan t; \;\; c_n(t) = c_n(0)\exp(4k_n^3 t) \tag{3}$$

$$\text{and } b(k;t) = b(k;0)\exp(8ik^3 t). \tag{4}$$

The function F, defined by the equation

$$F(X) = \sum_{n=1}^{N} c_n^2 \exp(-k_n X) + \frac{1}{2\pi}\int_{-x}^{x} b(k)e^{ikX}dk.$$

Now the Marchenko equation for the above equation is

$$F(X;t) = \sum_{n=1}^{N} c_n^2(0)\exp(8k_n^3 t - k_n X) + \frac{1}{2\pi}\int_{-x}^{x} b(k;0)\exp(8ik^3 t + ikX)dk. \tag{5}$$

Note that F also depends upon the parameter t. The Marchenko equation for K(x,z;t) is therefore $K(x,z;t) + F(x+z;t) + \int K(x,y;t)F(y+z;t)dy = 0$. Finally the solution of the KdV equation can be expressed as $u(x,t) = -2\frac{\partial}{\partial x}K(x,t)$ and $K(x,t) = K(x,x;t)$.

3. Procedure

We first write the KdV equation in the convenient form

$$u_t - 6uu_x + u_{xxx} = 0. \tag{6}$$

Then one simple way of showing a connection with the Sturm-Liouville problem is to define a function u(x,t) such that

$$u = v^2 + v_x \tag{7}$$

Equation (7) is called the Miura Trasformation. Direct substitution of (7) into equation (6) then yields

$$2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0$$

which can be rearranged to give

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{xxx}) = 0 \tag{8}$$

Thus if v is a solution of $v_t - 6v^2 v_x + v_{xxx} = 0$ (9)

The above equation is called the Modified KdV equation (mKdV) .

Then equation (7) defines a solution of the KdV equation (6). Now we recognize equation (7) as a Riccati equation for v which therefore may be linearised by the substitution

$$v = \psi_x \psi \tag{10}$$

for some differentiable function $\psi(x;t) \neq 0.$ The fact that time (t) occurs only parametrically in equation (7) is accommodated in our notation for ψ by the use of the semicolon. Equation (7), upon the introduction of (10) becomes $\psi_{xx} - u\psi = 0$ (11)

which is almost the (time-dependent) Sturm-Liouville equation for ψ . The connection is completed when we observe that the KdV equation is Galilean invariant, that is

$$u(x,t) \to \lambda + u(x + 6\lambda t, t), \quad -\infty < \lambda < \infty.$$

leaves equation (6) unchanged for arbitrary (real) λ . Since the x-dependence is unaltered under this transformation (t plays the role of a parameter) we may equally replace u by $u - \lambda$. The equation of ψ now becomes $\psi_{xx} + (\lambda - u)\psi = 0$ (12)

which is the Sturm-Liouville equation with potential u and eigenvalue λ .

Thus, if we able to solve for ψ , we can then recover u from equations (10) and (7). However, the procedure is far from straightforward since equation (12) already involves the function u which we wish to determine. The way to avoid this dilemma is to interpret the problem in terms of scattering data by the potential $u.$ These data are described by the behaviors of the eigenfunction, ψ , in the form

$$\psi(x;k) \approx \begin{cases} e^{-ikx} + b(k)e^{ikx} & as \quad x \to +\infty \\ a(k)e^{-ikx} & as \quad x \to -\infty \end{cases}$$

for $\lambda > 0,$ with $k = \lambda^{1/2}$ for the continuous spectrum,

and $\psi_n(x) \approx c_n \exp(k_n x)$ as $x \to +\infty$

for $\lambda < 0,$ with $k_n = (-\lambda)^{1/2}$ for each discrete eigenvalue $(n = 1,2,..........,N)$

We then show $u(x) = -2\dfrac{d}{dx}K(x,z)$ where $K(x,z)$ is the solution of the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,t)F(y+z)dy = 0 \tag{13}$$

and F is defined by $$F(X) = \sum_{n=1}^{N} c_n^2 \exp(-k_n X) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx}dk \tag{14}$$

Let $u(x,t)$ be the solution of $u_t - 6uu_x + u_{xxx} = 0$ with $u(x,0) = f(x)$ given: this defines the initial-value problem for the KdV equation. Further, let us introduce the function ψ which satisfies the equation $\psi_{xx} + (\lambda - u)\psi = 0$ for some λ , and by virtue of the parametric dependence on t we must allow $\lambda = \lambda(t).$

The solution of the KdV equation can therefore be described as follows. At $t = 0$ we are given $u(x,0) = f(x)$ and so (provided ψ exists) we may solve the scattering problem for this potential, yielding expressions for $b(k),$ k_n and c_n $(n = 1..........,N).$ If the time evolution of these scattering data can be determined then we shall know the scattering data at any later time. This information therefore allows us to solve the inverse scattering problem and so reconstruct $u(x,t)$ for $t > 0.$ The procedure is represented schematically in the following figure, where $S(t)$ denotes the scattering data, i.e. $b(k,t),$ $k_n(t)$ and $c_n(t)$ $(n = 1..........,N).$

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{Scattering}"] \arrow[d, dashed, "\text{KdV}"'] & S(0) \arrow[d, "\text{Time Evaluation}"'] \\
u(x,t) & S(t) \arrow[l, "\text{Inverse Scattering}"']
\end{tikzcd}$$

Representation of the inverse scattering transform for the KdV equation

It is clear that the success or failure of this approach now rests on whether, or not, the time evolution of S can be determined. Furthermore, it is to be hoped that the evolution is fairly straightforward so that application of this technique does not prove too difficult. We shall demonstrate in the next section how $S(t)$ can be found and else show that it takes a surprisingly simple form. However, before we start this, we note the parallel between the scheme represented in the above figure and the use of the Fourier transform for the solution of linear partial differential equations.
Consider the equation

$$u_t + u_x + u_{xxx} = 0$$

which is one linearization of the KdV equation. If $u(x,0) = f(x)$ then we can write

$$f(x) = \int_{-x}^{x} A(k)e^{ikx}dk \quad \text{or} \quad A(k) = \frac{1}{2\pi}\int_{-x}^{x} f(x)e^{-ikx}dx$$

and $A(k)$ is then analogous to the scattering data S(0). Further, if $u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-wt)}dk$ where $w = w(k),$ then $w(k) = k - k^3$ and the term in w expresses the time evolution of the 'scattering data'.

4. Reflectionless potentials

The inverse scattering transform method is best exemplified by choosing the initial profile, $u(x,0)$ to be a $\sec h^2$ function and in particular one of those which corresponds to a reflection less potential (i.e. $b(k) = 0$ for all k).Although the solitary wave is already known to be an exact solution of the KdV equation, it is possible to obtain this solution by passing a suitable initial value problem without taking the assumption that the solution takes the form of a steady progressing wave. This example then affords a simple introduction to the application of the technique.

4 (a): Single-soliton solution of KdV equation

The initial profile is taken to be $u(x,0) = -2\sec h^2 x$ (15)

and so the Sturm-Liouville equation, at $t = 0$ becomes

$$\psi_{xx} + (\lambda + 2\sec h^2 x)\psi = 0 \tag{16}$$

which is conveniently transformed by the substitution $T = \tanh x$ (so that $-1 < T < 1$ for $-\infty < x < \infty$).

Thus $$\frac{d}{dx} \equiv \sec h^2 x \frac{d}{dT} = (1-T^2)\frac{d}{dT}$$

and so $$(1-T^2)\frac{d}{dT}\left\{(1-T^2)\frac{d\psi}{dT}\right\} + \left\{\lambda + 2(1-T^2)\right\}\psi = 0$$

or $$\frac{d}{dT}\left\{(1-T^2)\frac{d\psi}{dT}\right\} + \left(2 + \frac{\lambda}{1-T^2}\right)\psi = 0$$

which is the associated Legendre equation. The only bounded solution for $\lambda = -k^2 (< 0)$ occurs if $k = k_1 = 1$ and the solution is proportion to the associated Legendre function $P_1^1(\tanh x)$ i.e. the corresponding eigenfunction is $\psi_1(x) \propto P_1^1(\tanh x) = -\sec hx.$

and since $\int_{-\infty}^{\infty} \sec h^2 x dx = 2$ the normalized eigenfunction becomes $\psi_1(x) = 2^{-1/2} \sec hx$.

(The sign of ψ_1 is irrelevant.) Then the asymptotic behavior of this solution yields

$\psi_1(x) \approx 2^{1/2} e^{-x}$ as $x \to +\infty$ so that $c_1(0) = 2^{1/2}$, and then equation (1.3) gives $c_1(t) = 2^{1/2} e^{4t}$. This transformation is sufficient to the reconstruction of $u(x,t)$ since we have chosen an initial profile for which $b(k) = 0$ for all k. Now from equation (1.5) we obtain $F(X;t) = 2e^{8t-x}$ which incorporates only one term form the summation, the contribution form the integral being zero. The Marchenko equation is therefore

$$K(x,z;t) + 2e^{8t-(x+z)} + 2\int_x^{\infty} K(x,y;t) e^{8t-(x+z)} dy = 0$$

which implies $K(x,z;t) = L(x,t) e^{-z}$

form some function L such that $L + 2e^{8t-x} + 2Le^{8t} \int_x^{\infty} e^{-2x} dy = 0$

This can be solved directly to yield $L(x,t) = \dfrac{-2e^{8t-x}}{1+e^{8t-2x}}$

and then $u(x,t) = 2\dfrac{\partial}{\partial x}\left(\dfrac{2e^{8t-2x}}{1+e^{8t-2x}}\right) = \dfrac{8e^{2x-8t}}{\left(1+e^{2x-8t}\right)^2} = -2\sec h^2(x-4t).$

which is the solitary wave of amplitude -2 and speed of propagation 4.

Coding and Output: By using programming language MATHEMATICA.

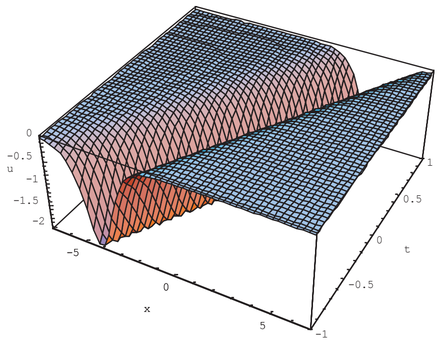

Fig.1

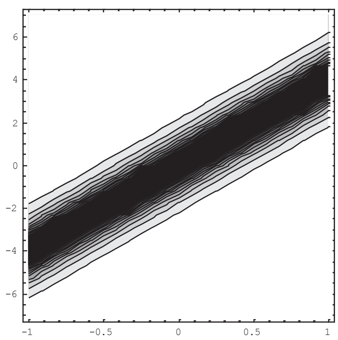

Fig.2

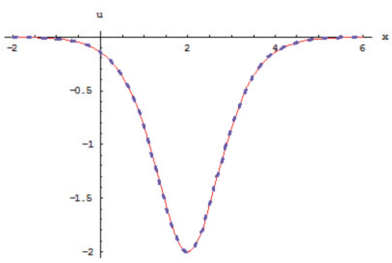

Fig.3

4 (b): Two-soliton solution of KdV equation

We consider the problem for which the initial profile is $u(x,0) = -6\sec h^2 x$ (17)

so that the Sturm-Liouville equation, at $t = 0$ becomes $\psi_{xx} + (\lambda + 6\sec h^2 x)\psi = 0$ (18)

or
$$\frac{d}{dT}\left\{(1-T^2)\frac{d\psi}{dT}\right\} + \left(6 + \frac{\lambda}{1-T^2}\right)\psi = 0$$

which is the associated Legendre equation. Where $T = \tanh x$. This equation has bounded solutions, for $\lambda = -k^2 (< 0)$ occurs if $k_1 = 1$ or $k_2 = 0$ of the form

$$\psi_1(x) = \sqrt{\frac{3}{2}}\tanh x \sec hx; \quad \psi_2(x) = \frac{\sqrt{3}}{2}\sec h^2 x;$$

both of which have been made to satisfy the normalization condition. The asymptotic behaviors of these solutions are

$$\psi_1(x) \approx \sqrt{6e^{-x}}; \quad \psi_2(x) \approx 2\sqrt{3e^{-2}}; \qquad \text{as } x \to +\infty$$

so that $c_1(0) = \sqrt{6}$; $c_2(0) = 2\sqrt{3}$, and then equation (3) gives

$$c_1(t) = 2^{1/2} e^{4t}, \quad c_2(t) = 2\sqrt{3}e^{32t}.$$

As in the above example, the choice of initial profile ensures that $b(k) = 0$ for all k and so $b(k;t) = 0$ for all t. The function F then becomes

$$F(X;t) = 6e^{8t-x} + 12e^{64t-x}$$

(since there are two terms in the series),and the Marchenko equation is therefore

$$K(x,z;t) + 6e^{8t-(x+z)} + 12e^{64t-2(x+z)} + \int_x^\infty K(x,y;t)\{6e^{8t-(x+z)} + 12e^{64t-2(x+z)}\}dy = 0$$

It is clear that the solution for K must take the form

$$K(x,z;t) = L_1(x,t)e^{-z} + L_2(x,t)e^{-2z}$$

Since F is a separable function, collecting the coefficients of e^{-z} and e^{-2z} , we obtain the pair of equations

$$L_1 + 6e^{8t-x} + 6e^{8t}(L_1\int_x^\infty e^{-2x}dy + L_2\int_x^\infty e^{-3x}dy) = 0$$

$$L_2 + 12e^{64t-2x} + 12e^{64t}(L_1\int_x^\infty e^{-3x}dy + L_2\int_x^\infty e^{-4x}dy) = 0$$

for the functions L_1 and L_2. Upon the evaluation of the definite integrals these two equations becomes

$$L_1 + 6e^{8t-x} + 3L_1e^{8t-2x} + 2L_2e^{8t-3x} = 0$$

$$L_2 + 12e^{64t-2x} + 4L_1e^{64t-3x} + 3L_2e^{64t-4x} = 0$$

which are solved to yield

$$L_1(x,t) = 6(e^{72t-5x} - e^{8t-x})/D = 0$$

$$L_2(x,t) = -12(e^{64t-2x} + e^{72t-4x})/D = 0$$

where
$$D(x,t) = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}$$

The solution of the KdV equation can now be expressed as

$$u(x,t) = -2\frac{\partial}{\partial x}(L_1 e^{-x} + L_2 e^{-2x}) = 12\frac{\partial}{\partial x}\{(e^{8t-2x} + e^{72t-6x} + 2e^{64t-4x})/D\}$$

which can be simplified to give

$$u(x,t) = -12\frac{12 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\{3\cosh(x - 28t) + \cosh(3x - 36t)\}^2}. \tag{19}$$

which is the two soliton solution.

Coding and Output: By using programming language MATHEMATICA

(a) $t = -.5$

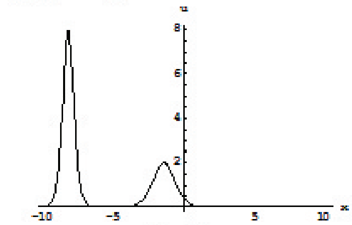

Fig.4

(b) $t = -.3$

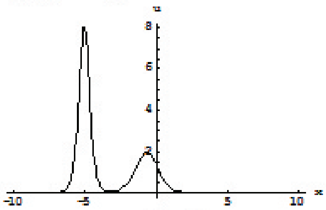

Fig.5

(c) $t = -.1$

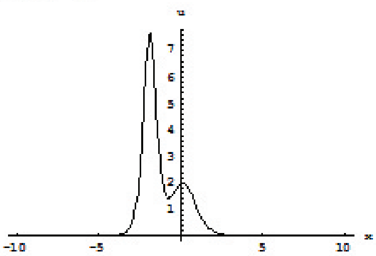

Fig.6

(d) $t = .1$

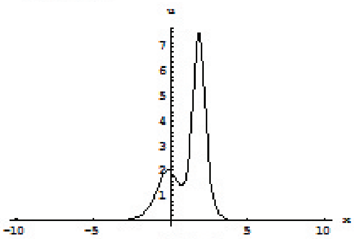

Fig.7

(e) $t = .3$

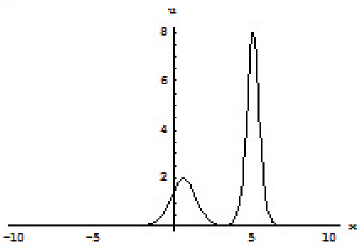

Fig.8

(f) $t = .5$

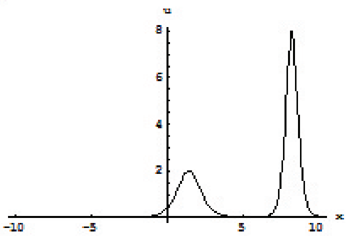

Fig.9

Next we plot the solution at time t=1

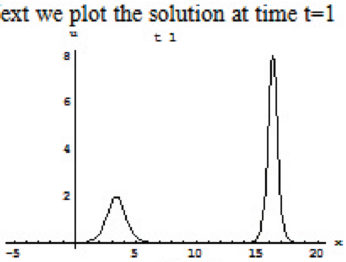

Fig.10

we see a canal of depth 8 and a canal of depth 2. To determine the speeds of these canal we locate the minima of the function at two different times, t=2 and t=3,

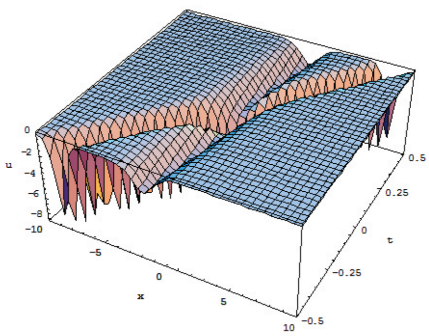

Fig. 11

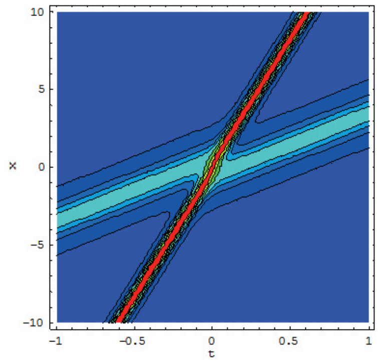

Fig.12

4 (c): N-soliton solution of KdV equation

We consider the problem for which the initial profile is

$$u(x,0) = -N(N+1)\sec h^2 x \tag{20}$$

then similarly the N -soliton solution is

$$u(x,t) \approx -2\sum_{n=1}^{N} n^2 \sec h^2 \{n(x-4n^2t) \pm x_n\} \quad \text{as } t \to \pm\infty$$

Coding and Output: By using programming language MATHEMATICA.

(a) $N = 3$

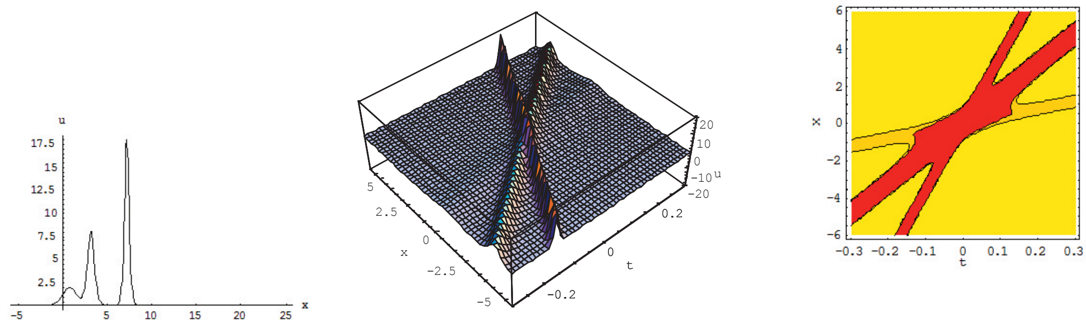

Fig.13

(b) $N = 4$

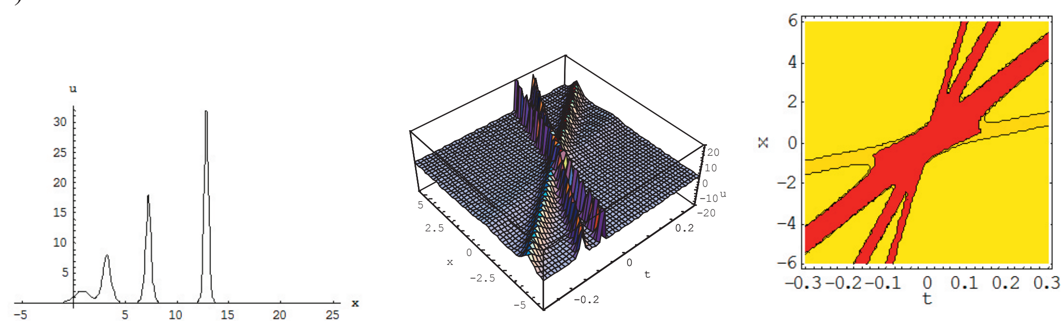

Fig.14

(c) $N = 5$

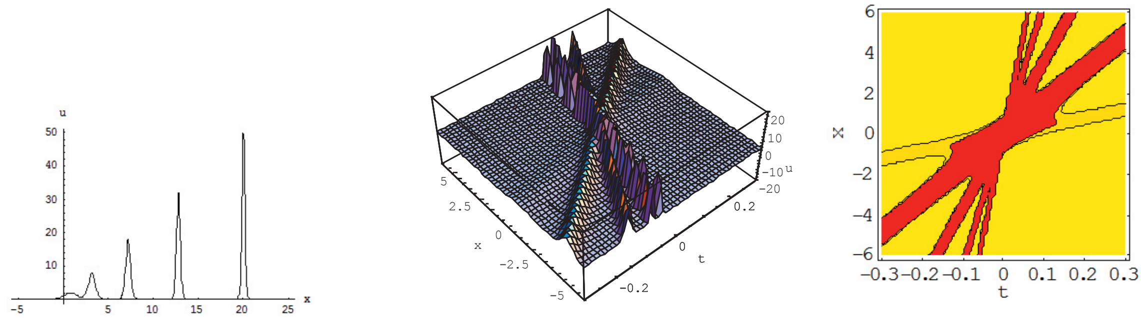

Fig.15

(d) $N = 6$

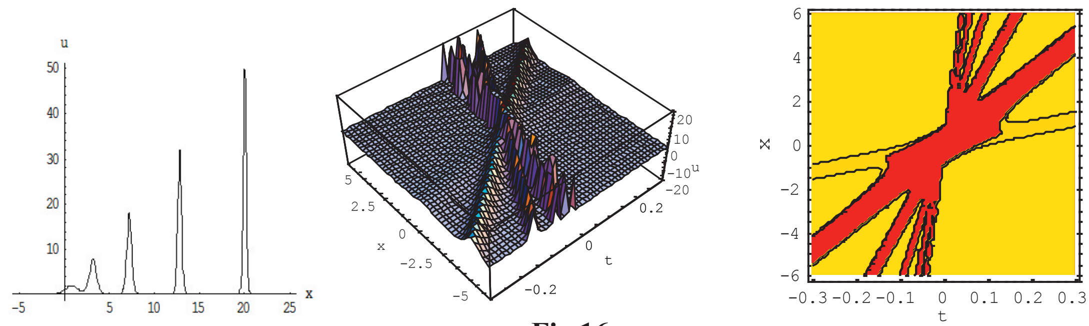

Fig.16

5. Result and discussion

It is clear from the **Fig.1** that the wave moves forward as t increases with depth 2 and speed 4 and not changing its shape. Plotting the solution shows the canal propagating to the right. A contour plot can also be useful in **Fig.2**. To verify that the numerical solution is the solution, we plot both for a particular value of t (t=0.5 here) which illustrated in **Fig.3**. We see that the two plots agree very well. In fact there is a whole family of single-soliton solutions parameterized by the depth of the channel. So the deeper the canal the faster the soliton moves and the narrower it is. We verified that this does satisfy the KdV equation. Since the solution is valid for positive and negative t, we may examine the development of the profile specified at $t = 0$. The wave profile, plotted as a function of x at six different times, is shown in **Fig.4-Fig.9**. Here we have chosen to plot $-u$ rather than u, this allows a direct comparison to be made with the application of the KdV equation to water waves. The solution shows two waves, which are almost solitary where the taller one catches the shorter, merges to form a single wave at $t = 0$ and then reappears to the right and moves away from the shorter one as t increases. Also we plotted the solution at time t=1 which showed in **Fig.10** for a canal of depth 8 and a canal of depth 2. Thus we have created two solitons of the type that we discussed in the previous section. However, there is no linear superposition, so the two-soliton solution is not the sum of the two individual solitons in the region where they overlap, as one can see form the explicit solutions. It is also seen that these two solutions interact in the area of t=0. In **Fig.11** showed at negative times, the deeper soliton, which moves faster, approaches the shallower one. At t=0 they combine to give equation (17) (a single trough of depth 6) and, after the encounter, the deeper soliton has overtaken the lower one and both resume their original shape and speed. However, as result of the interaction, the lower soliton experiences a delay and the deeper soliton is speeded up. This is also easily seen in a **Fig.12** (contour plot). On the other hand, The asymptotic solution for N-soliton of KdV eqaution represents separate solitons, ordered according to their speeds; as $t \to +\infty$ the tallest (and therefore fastest) is at the front followed by progressively shorter solitons behind. All N solitons interact at $t = 0$ to form the single $\sec h^2$ which was specified as the initial profile at that instant. Finally some plots are illustrated as 2D plots, 3D plots & Density plots in **Fig.13-Fig.16** for different values of N $(i.e. N = 3,4,5,6)$ where interaction of *N*- solitons is easily seen.

6. Conclusions

In this paper our aim was to understand the mathematical formalism of the inverse scattering problems through non-linear differential equation. We have made all efforts to represent the mathematical concept along with examples as 3D -Plots, Density Plots, and 2D-Plots for discrete values of time of inverse scattering problems by using Computer programming package MATHMATICA [11]. Again we deal with the connection between inverse scattering and the Korteweg-de Vries equation. In this section we have described variety of examples with Korteweg-de Vries equation: the single-soliton solution, the two-soliton solution and finally the N-soliton solution.

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