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A Class of Three Stage Implicit Rational Runge-Kutta Schemes for Approximation of Second Order Ordinary Differential Equations

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Abstract
In this paper, 3 – stage Implicit Rational Runge – Kutta methods are derived using Taylor and Binomial series expansion for the direct solution of general second order initial value problems of ordinary differential equations with constant step length. The basic properties of the developed method were investigated and found to be consistent and convergent. The efficiency of the method were tested on some numerical examples and found to give better approximations than the existing methods.
Keywords: Java Programming Language, Implicit Rational Runge Kutta scheme, Second order equations.

1. Introduction

Runge-Kutta schemes are important family of implicit and explicit iterative methods for approximation of solution of ordinary differential equations. Consider the numerical approximation of second order initial value problems of the form:

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0 \quad , \; a \le x \le b \tag{1}$$

The general $s-$stage Runge-Kutta scheme for general second order initial value problems of ordinary differential equations of the form (1) as defined by Jain (1984) is

$$y_{n+1} = y_n + hy'_n + \sum_{r=1}^{s} c_r k_r \tag{2}$$

and

$$y'_{n+1} = y'_n + \frac{1}{h}\sum_{r=1}^{s} c'_r k_r \tag{3}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij} k_j , y'_n + \frac{1}{h}\sum_{j=1}^{r} b_{ij} k_j \right), \quad i = 1(1)s \tag{4}$$

with

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij} \quad , \quad i(1)r \tag{5}$$

where $c_i, a_{ij}, b_{ij}, c_r, c'_r$ are constants to be determined. The derivative of suitable parameters requires extremely lengthy algebraic manipulations, except for small values of s (Sharp and Fine (1992) and Dormand *et al* (1987)). The fourth order Runge-Kutta method for the solution of (1) is given in Jain (1984).

It should be noted that the methods considered above were Runge-Kutta method of second order ordinary differential equations but Much attention have not been given to Runge – Kutta method for the solution of general second order ODEs of the form (1), Lambert (1973). Much work was done in Runge-Kutta-Nystrom method of Special Second Order ODE of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0 \tag{6}$$

with numerical solution

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i k_i \tag{7}$$

and

$$y'_{n+1} = y'_n + h\sum_{i=1}^{s} b'_i k_i \tag{8}$$

where

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{i} a_{ij} k_j\right), \quad i = 1,2,\ldots,s \tag{9}$$

Runge-Kutta-Nystrom methods are direct extension of Runge-Kutta method to second order differential equations in (6). Work such as Sharp and Fine (1992), Dormand *et al.* (1987), Fudziah (2003), Fudziah (2009), Senu *et al.* (2011), more recently Okunuga *et al.* (2012) discussed the general techniques for solving equation of the form (6) directly without first reducing it to systems of first order ODEs. The above authors observed that the direct solution of second order equations is of greater advantage over reduction to systems of first order equations to increase efficiency and reduced storage requirement.

Hong (1982) proposed the use of rational function of Runge – Kutta method and adapted by Okunbor (1987) which investigated the use of the rational function of the Runge-Kutta scheme of first order initial value problems

$$y_{n+1} = \frac{y_n + h\sum_{i=1}^{r} w_i K_i}{1 + h y_n \sum_{i=1}^{r} v_i H_i} \tag{10}$$

where

$$k_i = f\left(x_n + c_i h, y_n + h\textstyle\sum_{i=1}^{r} a_{i-1j}\, k_j\right),\ i = 1(1)r \tag{11}$$

and

$$H_i = g\left(x_n + d_i h, z_n + h\textstyle\sum_{i=1}^{r} b_{i-1j} H_j\right),\ i = 1(1)r \tag{12}$$

in which

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \qquad \text{and} \qquad z_n = \frac{1}{y_n}$$

where c_i, a_{ij}, b_{ij}, d_i are arbitrary constants to be determined. Since then many new rational Runge – Kutta schemes have been developed for the solution of first order initial value problems and found out to give better estimates. Among these authors are: Ademuluyi and Babatola (2000), Odekunle (2001), Odekunle *et al.* (2004), Bolarinwa (2005), Babatola *et al.* (2007), Bolarinwa *et al.* (2012) and Abhulimen and Uluko (2012) The schemes are absolutely stable, consistent and convergent and was used to approximate a variety of first order differential equations. However, the methods are presently receiving more attention as efficient schemes for the solutions of various types of first order initial value problems are considered.

2. Derivation of the Scheme

The rational form of (2) and (3) can be defined as

$$y_{n+1} = \frac{y_n + h y'_n + \sum_{r=1}^{s} w_r K_r}{1 + y'_n \sum_{r=1}^{s} v_r H_r} \tag{2.1}$$

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \tag{2.2}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + h c_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j\right),\ i = 1(1)s \tag{2.3}$$

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + h d_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j\right),\ i = 1(1)s \tag{2.4}$$

with constraints

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j}^{i} b_{ij},\quad i = 1(1)r$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j}^{i} \beta_{ij}, \quad i = 1(1)r$$

in which

$$g(x_n, z_n, z'_n) = -z_n^2 f(x_n, y_n, y'_n) \quad and\ z_n = \frac{1}{y_n}$$

The constraint equations are to ensure consistency of the method, h is the step size and the parameters $a_{ij}, b_{ij}, c_i, d_{i,}\alpha_{ij}, \beta_{ij}$ are constants called the parameters of the method.

Following Abhulimen and Uloku (2012) and Bolarinwa *etal* (2012), the following procedures are adapted.

i. Obtain the Taylor series expansion of K_r and H_r about the point (x_n, y_n, y'_n) and binomial series expansion of right side of (2.1) and (2.2).
ii. Insert the Taylor series expansion into (2.1) and (2.2) respectively.
iii. Compare the final expansion of K_r and H_r about the point (x_n, y_n, y'_n) to the Taylor series expansion of y_{n+1} and y'_{n+1} about (x_n, y_n, y'_n) in the powers of h.

Normally the numbers of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of the following conditions are satisfied.

i. Minimum bound of local truncation error exists.
ii. The method has maximized interval of absolute stability.
iii. Minimized computer storage facilities are utilized.

In this paper, we shall consider the implicit scheme where all the a_{ij} and $b_{ij} \neq 0$ for at least one $j > i$. In equations (2.1), (2.2), (2.3) and (2.4) setting $s = 3$ we have

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{r=1}^{3} w_r K_r}{1 + y'_n \sum_{r=1}^{3} v_r H_r} \tag{2.5}$$

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{3} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{3} v'_r H_r} \tag{2.6}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{3} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{3} b_{ij} K_j \right), \ i = 1(1)3 \tag{2.7}$$

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{3} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{3} \beta_{ij} H_j \right), \ i = 1(1)3 \tag{2.8}$$

with constraints

$$c_1 = a_{11} + a_{12} + a_{13} = \frac{1}{2}(b_{11} + b_{12} + b_{13})$$
$$c_2 = a_{21} + a_{22} + a_{23} = \frac{1}{2}(b_{21} + b_{22} + b_{23})$$
$$c_3 = a_{31} + a_{32} + a_{33} = \frac{1}{2}(b_{31} + b_{32} + b_{33})$$
$$d_1 = \alpha_{11} + \alpha_{12} + \alpha_{13} = \frac{1}{2}(\beta_{11} + \beta_{12} + \beta_{13})$$
$$d_2 = \alpha_{21} + \alpha_{22} + \alpha_{23} = \frac{1}{2}(\beta_{21} + \beta_{22} + \beta_{23})$$
$$d_3 = \alpha_{31} + \alpha_{32} + \alpha_{33} = \frac{1}{2}(\beta_{31} + \beta_{32} + \beta_{33})$$

Now by adopting a binomial expansion on equations (2.5) gives

$$y_{n+1} = y_n + hy'_n + w_1K_1 + w_2K_2 + w_3K_3 - y_n^2(v_1H_1 + v_2H_2 + v_3H_3) - hy_ny'_n(v_1H_1 + v_2H_2 + v_3H_3) - y_n[(w_1K_1 + w_2K_2 + w_3K_3)(v_1H_1 + v_2H_2 + v_3H_3)] \tag{2.9}$$

Similarly the binomial expansion of (2.6) gives

$$y'_{n+1} = y'_n - \frac{1}{h} y'^2_n (v'_1 H_1 + v'_2 H_2 + v'_3 K_3) + \frac{1}{h}(w'_1 K_1 + w'_2 K_2 + w'_3 K_3) - \frac{1}{h^2} y'_n (w'_1 K_1 + w'_2 K_2 + w'_3 K_3)(v'_1 H_1 + v'_2 H_2 + v'_3 K_3) \qquad (2.10)$$

Now, expanding (2.7) using Taylor series of function of three variables gives

$$K_i = \frac{h^2}{2}\left[f_n + (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)f_y + 2c_i(b_{i1}K_1 + b_{i2}K_2 + b_{i3}K_3)f_{xy'} + (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^2 f_{yy} + 2c_i y'_n (b_{i1}K_1 + b_{i2}K_2 + b_{i3}K_3) f_{yy'}\right] + \frac{h^3}{2}\left[c_i f_x + c_i y'_n f_y + c_i(a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)f_{xy} + c_i y'_n\, (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3) f_{yy}\right] + \frac{h^4}{4}\left[{c_i}^2 f_{xx} + 2c_i y'_n f_{xy} + c_i^2 y'^2_n f_{yy}\right] + h(a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)(b_{i1}K_1 + b_{i2}K_2 + b_{i3}K_3) f_{yy'} + \frac{h}{2}(b_{i1}K_1 + b_{i2}K_2 + b_{i3}K_3) f_{y'} + \frac{1}{2}(b_{i1}K_1 + b_{i2}K_2 + b_{i3}K_3)^2 f_{y'y'} + 0(h^5) \qquad (2.11)$$

Equation (2.11) is implicit, which cannot be proceed by successive substitutions. We assume a solution for K_i which may be expressed as

$$K_i = h^2 B_i + h^3 C_i + h^4 D_i + 0(h^5) \qquad (2.12)$$

Substituting the values of K_i of (2.12) into equation (2.11) expand and re – arranging in powers of h gives

$$K_i = \frac{h^2}{2}\, f_n + \frac{h^3}{2}\left[c_i f_x + c_i y'_n f_y + \frac{1}{2}(b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3) f_{y'}\right] + \frac{h^4}{4}\left[c_i^2 f_{xx} + 2c_i y'_n f_{xy'} + c_i^2 y'^2_n f_{yy} + 2(b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3)^2 f_{yy'} + 2c_i(b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3) f_{xy'} + c_i y'_n (b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3) f_{yy'} + 2(b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3) f_y + 2(b_{i1}C_1 + b_{i2}C_2 + b_{i3}C_3) f_{y'}\right] + 0(h^5) \qquad (2.13)$$

On equating powers of h from equation (2.12) and (2.13), gives

$$\left.\begin{aligned} B_i &= \tfrac{1}{2} f_n \\ C_i &= \tfrac{1}{2}\left(c_i f_x + c_i y'_n f_y + (b_{i1}B_1 + b_{i2}B_2 + b_{i3}B_3) f_{y'}\right) = \tfrac{1}{2} c_i \Delta f_n \\ D_i &= \tfrac{1}{4}\left(c_i^2 \Delta^2 f_n + (b_{i1}C_1 + b_{i2}C_2 + b_{i3}C_3)\Delta f_n f_{y'} + c_i f_n f_y\right) \end{aligned}\right\} \qquad (2.14)$$

then

$$K_i = \frac{h^2}{2} f_n + \frac{h^3}{2} c_i \Delta f_n + \frac{h^4}{4}\left(c_i^2 \Delta^2 f_n + (b_{i1}C_1 + b_{i2}C_2 + b_{i3}C_3)\Delta f_n f_{y'} + c_i f_n f_y\right) + 0(h^5) \qquad (2.15)$$

In a similar manner

$$H_i = h^2 M_i + h^3 N_i + h^4 R_i + 0(h^5) \qquad (2.16)$$

Where

$$\left.\begin{aligned} M_i &= \tfrac{1}{2} g_n \\ N_i &= \tfrac{1}{2} d_i \Delta g f_n \\ R_i &= \tfrac{1}{4}(d_i^2 \Delta^2 g_n + (\beta_{i1}d_1 + \beta_{i2}d_2 + \beta_{i3}d_3)\Delta g_n g_{z'} + d_i g_n g_z) \end{aligned}\right\} \qquad (2.17)$$

And also,

$$H_i = \frac{h^2}{2} g_n + \frac{h^3}{2} d_i \Delta g f_n + \frac{h^4}{4}(d_i^2 \Delta^2 g_n + (\beta_{i1}d_1 + \beta_{i2}d_2 + \beta_{i3}d_3)\Delta g_n g_{z'} + d_i g_n g_z) + 0(h^5) \qquad (2.18)$$

Substituting equations (2.12) and (2.16) into equations (2.9) and (2.10) re – arranging and compare the resulting equation with the Taylor's series expansion of y_{n+1} about x_n

$$y_{n+1} = y_n + hy'_n + \frac{h^2 y_n{}''}{2!} + \frac{h^3 y_n{}'''}{3!} + \frac{h^4 y_n^{iv}}{4!} + \ \ldots \qquad (2.19)$$

and

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2 y_n'''}{2!} + \frac{h^3 y_n^{iv}}{3!} + \ldots \qquad (2.20)$$

where

$$\left.\begin{aligned} y_n'' &= f(x_n, y_n, y_n') = f_n \\ y_n''' &= f_x + y' f_y + f_n f_{y'} = \Delta f_n \\ y_n'^{v} &= f_{xx} + y'^2_n f_{yy} + f^2 f_{y'y'} + 2y' f_n f_{yy'} + 2 f_n f_{xy'} + f_{y'} \Delta f_n \\ y_n'^{v} &= \Delta^2 f_n + f_{y'} \Delta f_n + f_n f_y \\ \text{Since}\quad \Delta &= \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y'} \end{aligned}\right\} \qquad (2.21)$$

gives the following

$$w_1 B_1 + w_2 B_2 + w_3 B_3 - y_n^2 (v_1 M_1 + v_2 M_2 + v_3 M_3) = \tfrac{1}{2} f_n$$
$$w_1 C_1 + w_2 C_2 + w_3 B_3 - y_n^2 (v_1 N_1 + v_2 N_2 + v_3 N_3) = \tfrac{1}{6} \Delta f_n$$
$$w_1 D_1 + w_2 D_2 + w_3 D_3 - y_n^2 (v_1 R_1 + v_2 R_2 + v_3 R_3) = \tfrac{1}{24} (\Delta^2 f_n + \Delta f_n f_{y'} + f_n f_y)$$
$$w_1' B_1 + w_2' B_2 + w_3' B_3 - y_n^2 (v_1' M_1 + v_2' M_2 + v_3' M_3) = f_n$$
$$w_1' C_1 + w_2' C_2 + w_3' C_3 - y_n'^2 (v_1' N_1 + v_2' N_2 + v_3' N_3) = \tfrac{1}{2} \Delta f_n$$
$$w_1' D_1 + w_2' D_2 + w_3' D_3 - y_n'^2 (v_1' R_1 + v_2' R_2 + v_3' R_3) = \tfrac{1}{6} (\Delta^2 f_n + \Delta f_n f_{y'} + f_n f_y)$$

Substituting the values of $B_1, B_2, B_3, C_1, C_2, C_3, M_1, M_2, M_3, N_1, N_2, N_3, R_1, R_2$, and R_3 in the above equation and re-arranging, we have the following sets of non – liner differential equations

$$w_1 + w_2 + w_3 + v_1 + v_2 + v_3 = 1$$
$$w_1 c_1 + w_2 c_2 + w_3 c_3 + v_1 d_1 + v_2 d_2 + v_3 d_3 = \tfrac{1}{3}$$
$$w_1 c_1^2 + w_2 c_2^2 + w_3 c_3^2 + v_1 d_1^2 + v_2 d_2^2 + v_3 d_3^2 = \tfrac{1}{6}$$
$$w_1(c_1 b_{11} + c_2 b_{12} + c_3 b_{13}) + w_2(c_1 b_{21} + c_2 b_{22} + c_3 b_{23}) + w_3(c_1 b_{31} + c_2 b_{32} + c_3 b_{33}) + v_1(\beta_{11} d_1 + \beta_{12} d_2 +$$
β13d3+v2β21d1+β22d2+β23d3+v3β31d1+β32d2+β33d3=16

$$w_1' + w_2' + w_3' + v_1' + v_2' + v_3' = 2$$
$$w_1' c_1 + w_2' c_2 + w_3' c_3 + v_1' d_1 + v_2' d_2 + v_3' d_3 = 1$$
$$w_1' c_1^2 + w_2' c_2^2 + w_3' c_3^2 + v_1' d_1^2 + v_2' d_2^2 + v_3' d_3^2 = \tfrac{2}{3}$$
$$w_1'(c_1 b_{11} + c_2 b_{12} + c_3 b_{13}) + w_2'(c_1 b_{21} + c_2 b_{22} + c_3 b_{23}) + w_3'(c_1 b_{31} + c_2 b_{32} + c_3 b_{33}) + v_1'(\beta_{11} d_1 + \beta_{12} d_2 +$$
β13d3+v2′β21d1+β22d2+β23d3+v3′β31d1+β32d2+β33d3=23

with constraints

$$c_1 = a_{11} + a_{12} + a_{13} = \tfrac{1}{2}(b_{11} + b_{12} + b_{13})$$
$$c_2 = a_{21} + a_{22} + a_{23} = \tfrac{1}{2}(b_{21} + b_{22} + b_{23})$$
$$c_3 = a_{31} + a_{32} + a_{33} = \tfrac{1}{2}(b_{31} + b_{32} + b_{33})$$
$$d_1 = \alpha_{11} + \alpha_{12} + \alpha_{13} = \tfrac{1}{2}(\beta_{11} + \beta_{12} + \beta_{13})$$
$$d_2 = \alpha_{21} + \alpha_{22} + \alpha_{23} = \tfrac{1}{2}(\beta_{21} + \beta_{22} + \beta_{23})$$
$$d_3 = \alpha_{31} + \alpha_{32} + \alpha_{33} = \tfrac{1}{2}(\beta_{31} + \beta_{32} + \beta_{33})$$

This is fourteen (14) equations with fifty-four (54) unknowns. That means the expected scheme it not unique; we can have a family of 3 – stage schemes.

Choosing the Parameters

$$w_1 = w_2 = w_3 = 0, \quad v_1 = v_2 = v_3 = \tfrac{1}{3}, \quad c_1 = d_1 = a_{11} = b_{11} = \alpha_{11} = \beta_{11} = 0$$
$$w_1' = w_2' = \tfrac{1}{2}, \quad w_3' = 1, \quad v_1' = v_2' = v_3' = 0, \quad c_2 = c_3 = b_{21} = b_{31} = \tfrac{2}{3}$$
$$d_2 = d_3 = a_{12} = b_{12} = \beta_{12} = \beta_{21} = \beta_{31} = \tfrac{1}{2}, \quad a_{22} = a_{23} = a_{32} = a_{33} = \tfrac{1}{6},$$
$$a_{21} = a_{31} = b_{22} = b_{23} = b_{32} = b_{33} = \tfrac{1}{3}, \quad \alpha_{12} = \alpha_{21} = \alpha_{31} = \beta_{22} = \beta_{23} = \beta_{32} = \beta_{33} = \tfrac{1}{4}, \quad a_{13} =$$
$$b_{13} = \beta_{13} = -\tfrac{1}{2}, \quad \alpha_{13} = -\tfrac{1}{4}, \quad \alpha_{22} = \alpha_{23} = \alpha_{32} = \alpha_{33} = \tfrac{1}{8}$$

then equations (2.5) and (2.6) becomes

$$y_{n+1} = \frac{y_n + hy'_n}{1 + \frac{1}{3} y_n (H_1 + 2H_2)} \tag{2.22}$$

and

$$y'_{n+1} = y'_n + \frac{1}{2h}(K_1 + 3K_2) \tag{2.23}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y'_n) = \frac{h^2}{2} f_n$$
$$K_2 = \frac{h^2}{2} f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hy'_n + \frac{1}{3}(K_1 + K_2), y'_n + \frac{2}{3h}(K_1 + K_2)\right)$$

and

$$H_1 = \frac{h^2}{2} g(x_n,\ z_n, z'_n) = \frac{h^2}{2} g_n$$
$$H_2 = \frac{h^2}{2} g\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz'_n + \frac{1}{4}(H_1 + H_2), z'_n + \frac{1}{2h}(H_1 + H_2)\right)$$

3. CONVERGENCE

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size tends to zero.

Convergent $= \lim_{h\to 0} |y(x_{n+1}) - y_{n+1}|$

In other words, if the *discretiation* error at x_{n+1} tends to zero as $h \to \infty$, i.e if

$$e_{n+1} = |y(x_{n+1}) - y_{n+1}| \to 0 \qquad \text{as } n \to \infty \tag{3.1}$$

From equation (14),

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \tag{3.2}$$

while the exact solution $y'(x_{n+1})$ seems to satisfy the equation of the form

$$y'(x_{n+1}) = \frac{y(x_n) + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v'_r H_r} + T_{n+1} \tag{3.3}$$

Where T_{n+1} is a local truncation error.

Subtracting equation (3.3) from (3.2) gives

$$y'_{n+1} - y'(x_{n+1}) = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} - \frac{y(x_n) + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v'_r H_r} + T_{n+1} \tag{3.4}$$

Adopting equation (3.4) gives

$$e_{n+1} = \frac{\left(1 + \frac{1}{h} y'(x_n)\sum_{r=1}^{s} v'_r H_r\right)\left(y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r\right) - \left(1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r\right)\left(y(x_n) + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r\right)}{\left(1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r\right)\left(1 + \frac{1}{h} y'(x_n)\sum_{r=1}^{s} v'_r H_r\right)} + T_{n+1} \tag{3.5}$$

Expanding the brackets and re-arranging gives

$$e_{n+1} = \frac{e_n + \frac{1}{h^2}(y'_n - y'(x_n))\left[\left(\sum_{r=1}^{s} w'_r K_r \sum_{r=1}^{s} v'_r H_r\right)\right]}{\left(1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r\right)\left(1 + \frac{1}{h} y'(x_n)\sum_{r=1}^{s} v'_r H_r\right)} + T_{n+1}$$

This implies that

$$e_{n+1} = \frac{e_n + e_n \frac{1}{h^2}\left[\left(\sum_{r=1}^{s} w_r' K_r \sum_{r=1}^{s} v_r' H_r\right)\right]}{\left(1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r\right)\left(1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r' H_r\right)} + T_{n+1} \tag{3.6}$$

$$e_{n+1} = \frac{e_n\left[1 + \frac{1}{h^2}\left(\sum_{r=1}^{s} w_r' K_r \sum_{r=1}^{s} v_r' H_r\right)\right]}{\left(1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r\right)\left(1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r' H_r\right)} + T_{n+1} \tag{3.7}$$

From equations (3.7), setting

$$A_n = \left[1 + \frac{1}{h^2}\left(\sum_{r=1}^{s} w_r' K_r \sum_{r=1}^{s} v_r' H_r\right)\right], \quad B_n = \left[1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r\right], \quad C_n = \left[1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r' H_r\right]$$

and $T_{n+1} = T$

Then

$$e_{n+1} = \frac{A_n}{B_n C_n} e_n + T \tag{3.8}$$

Let $B = \max B_n > 0$, $C = \max C_n > 0$ and $A = \max A_n < 0$ then (3.8) becomes,

$$e_{n+1} \le \frac{A}{BC} e_n + T$$

Set $\frac{A}{BC} = K < 1$, then

$$e_{n+1} \le K e_n + T \tag{3.9}$$

If $n = 0$, then from (3.9),

$$e_1 = K e_0 + T$$

$$e_2 = K e_1 + T = K^2 e_0 + KT + T \text{ by substituting the value of } e_1$$

$$e_3 = K e_2 + T = K^3 e_0 + K^2 T + T$$

Continuing in this manner, we get the following

$$e_{n+1} = K^{n+1} e_0 + \sum_{t=0}^{n+1} K^t T \tag{3.10}$$

Since $\frac{A}{BC} = K < 1$, then one can see that as $n \to \infty$, $e_{n+1} \to 0$. This proves that the scheme converges.

4. CONSISTENCY

A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve as the step size ends to zero. To prove if equation (2.22) is consistent, subtract y_n from both side of (2.22), then

$$y_{n+1} - y_n = \frac{y_n + h y_n'}{1 + \frac{1}{3} y_n (H_1 + 2H_2)} - y_n \tag{4.1}$$

$$y_{n+1} - y_n = \frac{y_n + h y_n' - y_n\left(1 + \frac{1}{3} y_n (H_1 + 2H_2)\right)}{1 + \frac{1}{3} y_n (H_1 + 2H_2)}$$

Simplifying gives

$$y_{n+1} - y_n = \frac{h y_n' - \frac{1}{3} y_n^2 (H_1 + 2H_2)}{1 + \frac{1}{3} y_n (H_1 + 2H_2)} \tag{4.2}$$

But

$$H_1 = \frac{h^2}{2} g_n \text{, and}$$

$$H_2 = \frac{h^2}{2} f\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz_n' + \frac{1}{4}(H_1 + H_2), z_n' + \frac{1}{2h}(H_1 + H_2)\right)$$

Substituting H_1 and H_2 in equation (4.2), gives

$$y_{n+1}-y_n=\frac{hy_n'-\frac{1}{3}y_n^2\left(\frac{h^2}{2}g_n+2\left(\frac{h^2}{2}g\left(x_n+\frac{1}{2}h,z_n+\frac{1}{2}hz_n'+\frac{1}{4}(H_1+H_2),z_n'+\frac{1}{2h}(H_1+H_2)\right)\right)\right)}{1+\frac{1}{3}y_n\left(\frac{h^2}{2}g_n+2\left(\frac{h^2}{2}g\left(x_n+\frac{1}{2}h,z_n+\frac{1}{2}hz_n'+\frac{1}{4}(H_1+H_2),z_n'+\frac{1}{2h}(H_1+H_2)\right)\right)\right)}$$

Dividing all through by h and taking the limit as h tends to zero on both sides gives

$$\lim_{h\to 0}\frac{y_{n+1}-y_n}{h}=y_n'$$

Hence, the scheme in (2.22) is consistent.

Numerical Problems

Example 1.

Consider the equation $y''=(1+x^2)y,\quad y(0)=1,\ y'(0)=0,\ x\in[0,1]$

The exact solution is

$y(x)=e^{x^2/2}$

[Sources: Jain 1984]

Example 2.

Consider a non-linear ordinary differential equation

$$y''-x(y')^2=0,\quad y(0)=1,\ \ y'(0)=\frac{1}{2}$$

The exact solution is

$$y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

[Sources: Jacob (2010), Taiwo and Osilagun (2011)]

5. Discussions

Table 1 is the results obtained by applying the schemes to example 1, the variable coefficient differential equation. The result performed well and approximate the exact solution better as the step size goes to *h = 0.001*.

Example 2 is a non-linear ordinary differential equation which is also well approximated with results in table 2. The results show superiority over the results of Optimal Order Method of Jacob (2010) highlighted in table 3.

6. Conclusion

The new numerical schemes derived follows the techniques of rational form of Runge – Kutta methods proposed by Hong (1982) which was adopted by Okunbor (1987) and Ademiluyi and Babatola (2000) by using Taylor and Binomial expansion in stages evaluation. The order condition obtained in this research is up to five (5) and the stage is up to three (3). This is an improvement on the work of earlier authors.

The new schemes are of high accuracy for direct numerical solution of general second order ordinary differential equations. The steps to the derivation of the new schemes are presented in the methodology while the analysis of the schemes proved to be consistent, convergent, the results proves to be good estimate of the exact equations. Thus, the scheme is effective and efficient, these suggest a wider application of the schemes for even more complicated physical problems; since the methods is used to solve equations of the form $y''=(x,y,y')$ and $y''=(x,y)$ favorably. Equations of variable coefficients is considered and the fact that is also used to solve non-linear problem.

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PRESENTATION OF THE RESULTS

The results below are the exact and the approximate solution to each problem.

(**Note that.** All results is in 10 decimal places)

Table 1: Results of example 1 at h = 0.001

	Exact Solutions		**Approximate Solutions**		**Errors**	
t	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y_n	y'_n
0.001	1.000000500	0.001000001	1.000000500	0.001000000	0.00E-00	0.00E-00
0.002	1.000002000	0.002000004	1.000002000	0.002000002	0.00E-00	2.28E-09
0.003	1.000004500	0.003000014	1.000004500	0.003000006	0.00E-00	7.14E-09
0.004	1.000008000	0.004000032	1.000008000	0.004000016	0.00E-00	1.58E-08
0.005	1.000012500	0.005000063	1.000012500	0.005000033	0.00E-00	2.94E-08
0.006	1.000018000	0.006000108	1.000018000	0.006000059	0.00E-00	4.87E-08
0.007	1.000024500	0.007000172	1.000024500	0.007000097	0.00E-00	7.49E-08
0.008	1.000032001	0.008000256	1.000032000	0.008000147	0.00E-00	1.09E-07
0.009	1.000040501	0.009000365	1.000040500	0.009000213	0.00E-00	1.52E-07
0.01	1.000050001	0.010000500	1.000050001	0.010000296	0.00E-00	2.05E-07

Table 2: Results of example 2 at $h = 0.001$

	Exact Solutions		Approximate Solutions		Errors	
t	$y(x_n)$	$y'(x_n)$	y_n	y'_n	y_n	y'_n
0.001	1.000500000	0.500000125	1.000500000	0.500000251	0.00E-00	-1.26E-07
0.002	1.001000000	0.500000500	1.001000000	0.500000752	0.00E-00	-2.52E-07
0.003	1.001500001	0.500001125	1.001500001	0.500001505	0.00E-00	-3.80E-07
0.004	1.002000003	0.500002000	1.002000003	0.500002509	0.00E-00	-5.09E-07
0.005	1.002500005	0.500003125	1.002500006	0.500003763	0.00E-00	-6.38E-07
0.006	1.003000009	0.500004500	1.003000010	0.500005269	-1.37E-09	-7.69E-07
0.007	1.003500014	0.500006125	1.003500016	0.500007026	-1.90E-09	-9.01E-07
0.008	1.004000021	0.500008000	1.004000024	0.500009033	-2.53E-09	-1.03E-06
0.009	1.004500030	0.500010125	1.004500034	0.500011292	-3.24E-09	-1.17E-06
0.01	1.005000042	0.500012500	1.005000046	0.500013802	-4.05E-09	-1.30E-06

Table 3: Comparing errors of example 4, Jacob (2010) and new scheme at $h = 3.125\times10^{-3}$

	Exact Solutions	Computed with New scheme	Absolute Errors in Jacob (2010)	Absolute error in new Scheme
t	$y(x_n)$	y_n	y_n	y_n
0.1	1.001562501	1.001562502	6.125E-08	4.41E-10
0.2	1.003125010	1.003125014	1.211E-07	3.51E-09
0.3	1.004687534	1.004687544	1.874E-07	9.31E-09
0.4	1.006250081	1.006250099	2.616E-07	1.79E-08
0.5	1.007812659	1.007812688	3.534E-07	2.95E-08