Fixed Point Result in Menger Space with EA Property

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ABSTRACT
This paper's main objective is to define Menger space (PQM) and the concept of weakly compatible by using the notion of property (EA) & JSR maps to define new property to prove a common fixed point theorem for 4 self maps in Menger space (PQM).
Key Words: Fixed Point, Probabilistic Metric Space, Menger space, JSR mappings, property EA
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1. INTRODUCTION

The notion of probabilistic metric space is introduced by Menger in 1942 [10] and the first result about the existence of a fixed point of a mapping which is defined on a Menger space is obtained by Sehgel and Barucha-Reid.

A number of fixed point theorems for single valued and multivalued mappings in menger probabilistic metric space have been considered by many authors [2],[3],[4],[5],[6],[7]. In 1998, Jungck [8] introduced the concept weakly compatible maps and proved many theorems in metric space. Hybrid fixed point theory for nonlinear single valued and multivalued maps is a new development in the domain of contraction type multivalued theory ([4], [7], [11], [12], [13], [14]).Jungck and Rhoades [8] introduced the weak compatibility to the setting of single valued and multivalued maps. Singh and Mishra introduced (IT)-commutativity for hybrid pair of single valued and multivalued maps which need not be weakly compatible. Recently, Aamri and El Moutawakil [1] defined a property (EA) for self maps which contained the class of noncompatible maps. More recently, Kamran [9] extended the property (EA) for a hybrid pair of single valued and multivalued maps and generalized the (IT) commutativity for such pair.

The aim of this paper is to define a new property which contains the property (EA) for hybrid pair of single valued and multivalued maps and give some common fixed point theorems under hybrid contractive conditions in probabilistic space.

2. PRELIMINARIES

Now we begin with some definition

Definition 2.1: Let R denote the set of reals and R^+ the non-negative reals. A mapping $F: R \to R^+$ is called a distribution function if it is non decreasing left continuous with $\inf_{t\in R} F(t) = 0 \quad and \quad \sup_{t\in R} F(t) = 1$

Definition 2.2: A probabilistic metric space is an ordered pair (X, F) where X is a nonempty set, L be set of all distribution function and $F: X \times X \to L$. We shall denote the distribution function by $F\,(p,q)$ or $F_{p,q}$; $p,q \in X$ and $F_{p,q}\,(x)$ will represents the value of $F\,(p,q)$ at $x \in R$. The function $F(p,q)$ is assumed to satisfy the following conditions:

1. $F_{p,q}(x) = 1 \; for\ all\ x > 0\ if\ and\ only\ if\ p = q$
2. $F_{p,q}(0) = 0 \; for\ every\ p,q \in X$
3. $F_{p,q} = F_{q,p} \; for\ every\ p,q \in X$
4. $F_{p,q}(x) = 1 \; and\ F_{q,r}(y) = 1 \; then\ F_{p,r}(x+y) = 1 \; for\ every\ p,q,r \in X.$

In metric space (X, d), the metric d induces a mapping $F: X \times X \to L$ such that $F_{p,q}(x) = F_{p,q} = H\,(x - d(p,q))$ for every p, q ∈ X and x ∈ R, where H is the distribution function defined as

$$H(x) = \begin{cases} 0, if\ x \le 0 \\ 1, if\ x > 0 \end{cases}$$

Definition 2.3: A mapping Δ: $[0,1] \times [0,1] \to [0,1]$ is called t-norm if

1. $\Delta\,(a,1) = a \;\forall\, a \in [0,1]$
2. $\Delta\,(0,0) = 0,$
3. $\Delta\,(a,b) = \Delta\,(b,a),$

4. $\Delta (c,d) \geq \Delta (a,b)$ for $c \geq a, d \geq b$, and
5. $\Delta (\Delta (a,b), c) = \Delta (a, \Delta(b,c))$

Example: (i) $\Delta (a,b) = ab$, (ii) $\Delta (a,b) = min(a,b)$
(iii) $\Delta (a,b) = max(a+b-1; 0)$

Definition 2.4: A Menger space is a triplet (X,F,Δ) where (X,F)a PM-space and Δ is is a t-norm with the following condition

$$F_{u,w}(x+y) \geq \Delta(F_{u,v}(x), F_{v,w}(y)$$

The above inequality is called Menger's triangle inequality.

EXAMPLE: Let $X = R, \Delta (a,b) = min(a,b) \forall a,b \in (0,1)$ and

$$F_{u,v}(x) = \begin{cases} H(x) & for\ u \neq v \\ 1 & for\ u = v \end{cases}$$

where $H(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$

Then (X,F,Δ) is a Menger space.

Definition 2.5: Let (X,F,Δ) be a Menger space. If $u \in X, \varepsilon > 0, \lambda \in (0,1)$, then an (ε,λ) neighbourhood of u, denoted by $U_u(\varepsilon,\lambda)$ is defined as

$$U_u(\varepsilon,\lambda) = \{v \in X; F_{u,v}(\varepsilon) > 1-\lambda\}.$$

If (X,F,Δ) be a Menger space with the continuous t-norm t, then the family$U_u(\varepsilon,\lambda); u \in X; \varepsilon > 0, \lambda \in (0,1)$ of neighbourhood induces a hausdorff topology on X and if $sup_{a<1} \Delta(a,a) = 1$, it is metrizable.

Definition 2.6: A sequence $\{p_n\}$ in (X,F,Δ) is said to be convergent to a point $p \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $\boldsymbol{N} = \boldsymbol{N}(\varepsilon,\lambda)$ such that $p_n \in U_p(\varepsilon,\lambda)$ for all $n \geq \boldsymbol{N}$ or equivalently $F_{x_n,x}(\epsilon) > 1-\lambda$ for all $n \geq \boldsymbol{N}$.

Definition 2.7: A sequence $\{p_n\}$ in (X,F,Δ) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $\boldsymbol{N} = \boldsymbol{N}(\varepsilon,\lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1-\lambda$ for all $n,m \geq \boldsymbol{N}$.

Definition 2.8: A Menger space (X,F,Δ) with the continuous t-norm Δ is said to be complete if every Cauchy sequence in X converges to a point in X.

Lemma 2.9 [14]: Let $\{p_n\}$ be a sequence in Menger space (X,F,Δ) where Δ is continuous and $\Delta(x,x) \geq x$ for all $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that $x > 0$ and $n \in \boldsymbol{N}$ $F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x)$, then $\{p_n\}$ is a Cauchy sequence.

Definition 2.10: Let $s: X \to X$ and $T: X \to CB(X)$ be mappings in Menger space (X, F, Δ) then,

(1) s is said to be T weakly commuting at $x \in X$ if $ssx \in Tsx$.

(2) s and T are weakly compatible if they commute at their coincidence points,

i.e. if $sTx = Tsx$ whenever $sx \in Tx$.

(3) s and T are (IT) commuting at $x \in X$ if $sTx \subset Tsx$ whenever $sx \in Tx$.

Definition 2.11: Let (X, F, Δ) be a Menger space. Maps $f, g: X \to X$ are said to satisfy the property (EA) if there exists a sequence $\{x_n\}$ in x such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X.$$

Definition 2.12: -Maps $f: X \to X$ and $T: X \to CB(X)$ are said to satisfy the property (EA) if there exists a sequence $\{x_n\}$ in X, some z in X and A in CB(X) such that

$$\lim_{n\to\infty} fx_n = z \in A = \lim_{n\to\infty} Tx_n.$$

Definition 2.13: Let $f, g, S, G: X \to X$ be mappings in Menger space. The pair (f, S) and (g, G) are said to satisfy the common property (EA) if there exist two sequences $\{x_n\}, \{y_n\}$ in X and some z in X such that

$$\lim_{n\to\infty} Gy_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z.$$

Definition 2.14: Let $f, g: X \to X$ and $S, G: X \to CB(X)$ be mappings on Menger space. The maps pair (f, S) and (g, G) are said to satisfy the common property (EA) if there exist two sequences $\{x_n\}, \{y_n\}$ in X and some z in X, and A, B in CB(X) such that

$$\lim_{n\to\infty} Sx_n = A \text{ and } \lim_{n\to\infty} Gy_n = B, \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in A \cap B.$$

Definition 2.15:- Let (X, F, Δ) be a Menger space. Let f and g be two self maps of a Menger space. The pair {f, g} is said to be f-JSR mappings iff

$$\mu F(fgx_n, gx_n; p) \geq \mu F(ffx_n, fx_n; p)$$

where μ = lim Sup or lim inf and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some } z \in X \text{ and for all } \Delta(p,p) > p.$$

Example Let X = [0, 1] with $d(x,y) = |x-y|$ and f, g are two self mapping on X defined by $fx = \frac{2}{x+2}$, $gx = \frac{1}{x+1}$ for $x \in X$. Now the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}, n \in N$ then we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1$

$$|fgx_n - gx_n| \to \frac{1}{3} \text{ and } |ffx_n - fx_n| \to \frac{2}{3} \text{ as } n \to \infty.$$

Clearly we have $|fgx_n - gx_n| < |ffx_n - fx_n|$.

Thus pair {f, g} is f-JSR mapping. But this pair is neither compatible nor weakly compatible or other non commuting mapping S. Hence pair of JSR mapping is more general then others.

Let $f: X \to X$ self map of a Menger space (X, F, Δ) and $S: X \to CB(X)$ be multivalued map. The pair {f, S} is said to be hybrid S-JSR mappings for all Δ (p, p) > p if and only if

$$\mu F(Sfx_n, fx_n; p) \geq \mu F(SSx_n, Sx_n; p)$$

where μ = lim Sup or lim inf and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = z \in A = \lim_{n\to\infty} Sx_n.$$

Let $\phi: R \to R^+$ be continuous and satisfying the conditions

(i) ϕ is nonincreasing on R,

(ii) $\phi(t) > t$, for each $t \in (0, \infty)$.

3. MAIN RESULTS

Theorem 3.1: Let (X, F, Δ) be a Menger space. Let $f, g: X \to X$ and $S, G : X \to CB(X)$ such that

(3.1.1) (f, S) and (g, G) satisfy the common property (EA),

(3.1.2) $f(X)$ and $g(X)$ are closed,

(3.1.3) Pair (f, S) is $S-JSR$ maps and pair (g, G) is $G-JSR$ maps,

(3.1.4) $F_{Sx,Gy}(kp) \geq \phi[\min\{F_{fx,gy}(p), F_{fx,Sx}(p), F_{gy,Gy}(p), F_{fx,Gy}(p), F_{Sx,gy}(p)\}]$

Then *f, g, S* and *G* have a common fixed point in *X*.

Proof: By (3.1.1) there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $u \in X$, A, B in CB(X) such that

$$\lim_{n\to\infty} Sx_n = A \text{ and } \lim_{n\to\infty} Gy_n = B,$$

$$\text{and } \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

Since $f(X)$ and $g(X)$ are closed, we have $u = fv$ and $u = gr$ for some $v, r \in X$.

Now by (3.1.4) we get

$$F_{Sx_n,Gr}(kp) \geq \phi\left[\min\left\{\begin{matrix} F_{fx_n,gr}(p), F_{fx_n,Sx_n}(p), \\ F_{gr,Gr}(p), F_{fx_n,Gr}(p), F_{Sx_n,gr}(p)\end{matrix}\right\}\right]$$

On taking limit $n \to \infty$, we obtain

$$\begin{aligned} F_{A,Gr}(kp) &\geq \phi[\min\{F_{fv,gr}(p), F_{fv,A}(p), F_{gr,Gr}(p), F_{fv,Gr}(p), F_{A,gr}(p)\}] \\ &\geq \phi\, F_{gr,Gr}(p) \\ &> F_{gr,Gr}(p) \end{aligned}$$

Since $gr = fv \in A$ and $F_{gr,Gr}(p) \geq F_{A,Gr}(p) > F_{gr,Gr}(p)$.

Hence $gr \in Gr$

Similarly

$$F_{Sv,Gy_n}(kp) \geq \phi[\min\{F_{fv,gy_n}(p), F_{fv,Sv}(p), F_{gy_n,Gy_n}(p), F_{fv,Gy_n}(p), F_{Sv,gy_n}(p)\}]$$

$$F_{Sv,B}(kp) \geq \phi[\min\{F_{fv,gr}(p), F_{fv,Sv}(p), F_{gr,B}(p), F_{fv,B}(p), F_{Sv,gr}(p)\}]$$

$$\geq \phi F_{fv,Sv}(p)$$

$$> F_{fv,Sv}(p)$$

Since $fv = gr \in B$ and $F_{fv,Sv}(p) \geq F_{B,Sv}(p) > F_{fv,Sv}(p)$,

We get $fv \in Sv$.

Now as pair (f, S) is an S-JSR map therefore $fp \in Sp$

and similarly as pair (g, G) is G-JSR maps therefore $gu \in Gr$

$$F_{fx_n,gu}(p) \geq F_{Sx_n,Gu}(kp)$$

$$\geq \phi[\min\{F_{fx_n,gu}(p), F_{fx_n,Sx_n}(p), F_{gu,Gu}(p), F_{fx_n,Gu}(p), F_{Sx_n,gu}(p)\}]$$

On taking limit $n \to \infty$,we obtain

$$F_{u,gu}(p) \geq \phi[\min\{F_{u,gu}(p), F_{u,A}(p), F_{gu,Gu}(p), F_{u,Gu}(p), F_{A,gu}(p)\}]$$

$$\geq \phi\left[\min\left\{\begin{matrix} F_{u,gu}(p), F_{u,A}(p), F_{gu,Gu}(p), \\ F_{u,Gu}(p), F_{A,u}(p/2), F_{u,gu}(p/2)\end{matrix}\right\}\right]$$

By triangular inequality and as $u \in A \cap B$, we obtain

$$F_{u,gu}(p) \geq F_{u,gu}(p)$$

$$\Rightarrow gu = u.$$

Again

$$F_{fu,gx_n}(p) \geq F_{Su,Gx_n}(kp)$$

$$\geq \phi[\min\{F_{fu,gx_n}(p), F_{fu,Su}(p), F_{gx_n,Gx_n}(p), F_{fu,Gx_n}(p), F_{Su,gx_n}(p)\}]$$

On taking limit $n \to \infty$,we obtain

$$F_{fu,u}(p) \geq \phi[\min\{F_{fu,u}(p), F_{fu,Su}(p), F_{u,Gu}(p), F_{fu,B}(p), F_{Su,u}(p)\}]$$

$$\geq \phi\left[\min\left\{\begin{matrix} F_{fu,u}(p), F_{fu,Su}(p), F_{u,Gu}(p), \\ F_{fu,u}(p/2), F_{u,B}(p/2), F_{Su,u}(p)\end{matrix}\right\}\right]$$

By triangular inequality and as $u \in A \cap B$, we obtain

$$F_{fu,u}(p) \geq F_{fu,u}(p)$$

$$\Rightarrow fu = u.$$

Hence $u = fu \in Su$ and $u = gu \in Su$.

Example: Let X = [1,∞) with usual metric. Define $S: X \to X$ as $Sx = \frac{2+x}{3}$ and $T: CB(X) \to X$ as $Tx = [1, 2+x]$. Consider the sequence $\{x_n\} = \left\{3 + \frac{1}{n}\right\}$. Then all conditions are satisfies of the theorem and hence 3 is the common fixed point.

Theorem 3.2: Let (X, F, Δ) be a Menger space. Let $f, g: X \to X$ and $S_i, G_j : X \to CB(X)$ such that

(3.2.1) (f, S_i) and (g, G_j) satisfy the common property (EA),

(3.2.2) $f(X)$ and $g(X)$ are closed,

(3.2.3) Pair (f, S_i) is $S_i - JSR$ maps and pair (g, G_j) is $G_j - JSR$ maps,

(3.2.4) $F_{S_i x, G_j y}(kp) \geq \phi[\min\{F_{fx,gy}(p), F_{fx,S_i x}(p), F_{gy,G_j y}(p), F_{fx,G_j y}(p), F_{S_i x,gy}(p)\}]$

Then $f, g, S_i and\ G_j$ have a common fixed point in X.

Proof: Same as theorem *3.1* for each sequence $S_i and\ G_j$.

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