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A Common Fixed Point of Integral Type Contraction in Generalized Metric Spacess

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Abstract
In this paper, we present a common fixed point theorem for two self-mappings satisfying a contractive condition of integral type in G- metric spaces. Our result generalizes some well-known results.

1. Introduction and Preliminaries

Mustafa and Sims [9] introduced the concept of G – metric spaces in the year 2004 as a generalization of the metric spaces. In this type of spaces a non- negative real number is assigned to every triplet of elements. In [11] Banach contraction mapping principle was established and a fixed point results have been proved. After that several fixed point results have been proved in these spaces. Some of these works may be noted in [2-4, 10-13] and [14]. Several other studies relevant to metric spaces are being extended to G- metric spaces. For instances we may note that a best approximation result in these type of spaces established by Nezhad and Mazaheri in [15] .the concept of w- distance, which is relevant to minimization problem in metric spaces [8], has been extended to G-metric spaces by Saadati et al .[23]. Also one can note that the fixed point results in G- metric spaces have been applied to proving the existence of solutions for a class of integral equations [26].

Definition 1.1. G-metric Space

Let X be a nonempty set and let G :X $\times$X$\times$ X→ R^+ be a function satisfying the following :

(1) G(x, y, z) = 0 if x=y=z.
(2) G(x, x, y) > 0 ; for all x,y,z $\in X$, with x$\neq y$.
(3) G(x, x, y) $\leq$ G(x, y, z); for all x, y , $z \in X$, with z$\neq y$.
(4) G(x, y, z)= G(x, z, y)= G(y, z, x)= - - - - --
(5) G(x, y, z) $\leq$ G(x, a, a) + G(a, y, z) ; for all x ,y ,z, a $\in X$

Then the function is called a generalized metric, or a G- metric on X and the pair (X, G) is a G-metric space.

Definition 1.2 Let (X, G) be a G –metric space and $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ is G-convergent to x$\in X$ if
$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$.
That is for any $\epsilon > 0$, there exists N$\in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all n,m $\geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n,\to\infty} x_n$ = x.

Definition 1.3 Let (X, G) be a G –metric space. A sequence $\{x_n\}$ is called a G- Cauchy sequence if, for any $\epsilon > 0$, there exists N$\in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$, for all
l, n,m $\geq N$. That is $G(x_n,x_m,x_l) \to 0$ as n, m→ ∞.

Definition 1.4 A G-metric space (X,G) is called G –complete if every G- Cauchy sequence is G-convergent in (X,G) .

Every G-metric on X will define a metric d_G on X by

d_G(X, y) = G(x, y,y) + G(y,x,x) , for all x,y $\in X$

Proposition 1.1 Let (X, G) be a G –metric space. The following are equivalent:

(1) (x_n) is G-convergent to x ;
(2) G$(x_n,x_n,x) \to 0$ as n → ∞;
(3) G$(x_n,x,x) \to 0$ as n → ∞;
(4) G $(x_n,x_n,x) \to 0$ as n, m→ ∞.

Proposition 1.2 Let (X, G) be a G –metric space .Then, for any x, y, z, a $\in X$ it follows that

(1) If G(x, y, z) = 0 then x = y =z.
(2) G(x, y, z) $\leq$ G(x, x, y) + G(x, x, z)
(3) G(x, y, y) $\leq$ 2G(y, x, x) ,
(4) G(x, y, z) $\leq$ G(x, a, z) + G(a, y, z) ,

There has been a considerable interest to study common fixed point for a pair of mappings satisfying some contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors .It was the turning point in the "fixed point arena" when the notion of commutativity was introduced by G.jungck [5] to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors.In one direction Jungck [6] introduced the compatibility in 1986. It has also been noted that fixed point problems of non-compatible mappings are also important and have been considered in a number of works. A few may be noted in [7,18]. In another direction weaker version of commutativity has been considered in a large number of works. One such concept is R-weakly commutativity. This is an extension of weakly commuting mappings [16, 24].Some other references may be noted in [17-20] and [22].

Proposition 1.3 Let f and g be weakly compatible self-mappings on a set X. If f and g have unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.

Definition 1.5 Let f and g be two self-mappings on a metric space (X,d).The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fg\,x_n, gfx_n) = 0$. Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

In particular, now we look in the context of common fixed point theorem in G-metric spaces. Start with the following contraction conditions:

Definition 1.6 Let (X, G) be a **G**-metric space and T: $X \to X$ be a self-mapping on (X, G). Now T is said to be a contraction if

$$G(Tx,Ty,Tz) \leq \alpha G(x,y,z) \text{ for all } x,y,z \in X \text{ where } 0 \leq \alpha < 1. \quad (1.1)$$

It is clear that every self-mapping T: $X \to X$ satisfying condition (1.1) is continuous. Now we focus to generalize the condition (1.1) for a pair of self-mappings S and T on X in the following way :

$$G(Sx,Sy,Sz) \leq \alpha\, G(Tx,Ty,Tz) \text{ for all } x,y,z \in X \text{ where } 0 \leq \alpha < 1. \quad (1.2)$$

Definition 1.7 Let f and g be two self-mappings on a G-metric space (X,G). The mappings f and g are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$. Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Theorem 1.1 Let (X,G) be a complete G-metric space and f ,g be two self-mappings on (X,G) satisfies the following conditions :

1. F(X) $\subseteq$g(X) ,
2. F or g is continuous,
3. $G(fx, fy, fz) \leq \alpha\, G(fx, gy, gz) + \beta\, G(gx, fy, gz) + \gamma\, G(gx,gy,fz)$

   For every $x, y,z \in X$ and $\alpha, \beta, \gamma \geq 0$ with $0 \leq \alpha+3\beta+3\gamma < 1$.

   Then f and g have a unique common fixed point in X provided f and g are compatible maps.

   In 2002, Branciari [21] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality. After the paper of

Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mapping satisfying various known properties. The aim of this paper is to extend and modified above theorem in integral type mapping.

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Theorem 1.1 Let (X,G) be a complete G-metric space and f ,g be two self-mappings on (X,G) satisfies the following conditions :

(1) F(X) $\subseteq$g(X) , (2.1)

(2) F or g is continuous, (2.2)

(3) $$\int_0^{G(fx,fy,fz)} \varphi(t)dt \leq \alpha \int_0^{G(fx,gy,gz)} \varphi(t)dt + \beta \int_0^{G(gx,fy,gz)} \varphi(t)dt + \gamma \int_0^{G(gx,gy,fz)} \varphi(t)dt + \eta \int_0^{G(gx,gy,gz)} \varphi(t)dt \quad (2.3)$$

For every $x, y, z \in X$ and$\alpha, \beta, \gamma \geq 0$ with $0 \leq 3\alpha+3\beta+\eta < 1$.

And $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be arbitrary in X. Since F(X) $\subseteq$g(X), choose $x_1 \in X$ such that $gx_1 = fx_0$.

Continuing this process, we choose x_{n+1} such that

$y_n = gx_{n+1} = fx_n, \forall n \in \mathbb{N}$.

By Inequality (2.3), we have

$$\int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt = \int_0^{G(fx_{n+1},fx_{n+1},fx_n)} \varphi(t)dt$$
$$\leq \alpha \int_0^{G(fx_{n+1},gx_{n+1},gx_n)} \varphi(t)dt + \beta \int_0^{G(gx_{n+1},fx_{n+1},gx_n)} \varphi(t)dt$$
$$+\gamma \int_0^{G(gx_{n+1},gx_{n+1},fx_n)} \varphi(t)dt + \eta \int_0^{G(gx_{n+1},gx_{n+1},gx_n)} \varphi(t)dt$$
$$= \alpha \int_0^{G(y_{n+1},y_n,y_{n-1})} \varphi(t)dt + \beta \int_0^{G(y_n,y_{n+1},y_{n-1})} \varphi(t)dt$$
$$+\gamma \int_0^{G(y_n,y_n,y_n)} \varphi(t)dt + \eta \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$$

Now, $G(y_{n-1}, y_n, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_n, y_{n+1})$

$$\leq G(y_{n-1}, y_n, y_n) + 2\,G(y_n, y_{n+1}, y_{n+1})$$

(By using Proposition 1.2)

Then,

$$\int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt \leq (\alpha+\beta) \int_0^{G(y_{n-1},y_n,y_{n+1})} \varphi(t)dt$$
$$+ \eta \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$$
$$\leq (\alpha+\beta)\left[\int_0^{G(y_{n-1},y_n,y_n)} \varphi(t)dt + \int_0^{2G(y_n,y_{n+1},y_{n+1})} \varphi(t)dt \right]$$
$$+ \eta \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$$
$$= (\alpha+\beta+\eta) \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$$
$$+ (2\alpha+2\beta) \int_0^{G(y_n,y_{n+1},y_{n+1})} \varphi(t)dt$$

$(1\text{-}2\alpha\text{-}2\beta) \int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt \leq (\alpha+\beta+\eta) \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$ $\int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt \leq k \int_0^{G(y_n,y_n,y_{n-1})} \varphi(t)dt$ where $k = \frac{\alpha+\beta+\eta}{1-2\alpha-2\beta} \leq 1$.

By induction, one can find

$$\int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt \leq k^n \int_0^{G(y_1,y_1,y_0)} \varphi(t)dt$$

Since $k \in [0,1)$, so $\lim_{n\to\infty} \int_0^{G(y_{n+1}y_{n+1},y_n)} \varphi(t)dt = 0$. By a property of function φ, we obtain $\lim_{n\to\infty} G(y_{n+1}y_{n+1},y_n) = 0$. (2.4)

Now, we shall show that $\{y_n\}$ is a G- Cauchy sequence in g(X).

Suppose to the contrary. Then there exist $\epsilon > 0$ and sequences of natural numbers (m(k)) and (l(k)) such that for every natural number k, m(k) > l(k) $\geq$ k and

$$G(y_{m(k)}, y_{m(k)}, y_{l(k)}) \geq \epsilon. \quad (2.5)$$

Now corresponding to l(k) we choose m(k) to be the smallest for which (2.5) holds. So

$$G(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)}) < \epsilon.$$

Using (2.5) and the rectangle inequality, we have

$$\epsilon \leq G(y_{m(k)}, y_{m(k)}, y_{l(k)})$$
$$\leq G(y_{m(k)}, y_{m(k)}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)})$$
$$< \epsilon + G(y_{m(k)}, y_{m(k)}, y_{m(k)-1}) .$$

Letting k → ∞ in the above inequality and using (2.4), we get

$$\lim_{k\to\infty} G(y_{m(k)}, y_{m(k)}, y_{l(k)}) = \epsilon . \quad (2.6)$$

Again, a rectangle inequality gives us

$$G(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)-1}) \leq G(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)})$$
$$+ G(y_{l(k)}, y_{l(k)}, y_{l(k)-1})$$
$$< \epsilon + G(y_{l(k)}, y_{l(k)}, y_{l(k)-1})$$

By (2.4), letting k → ∞ , we obtain

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)-1}) \leq \epsilon. \quad (2.7)$$

From (2.3), we have

$$\int_0^{G(y_{m(k)},y_{m(k)},y_{l(k)})}\varphi(t)dt = \int_0^{G(fx_{m(k)},fx_{m(k)},fx_{l(k)})}\varphi(t)dt$$
$$\leq \alpha \int_0^{G(fx_{m(k)},gx_{m(k)},gx_{l(k)})}\varphi(t)dt + \beta\int_0^{G(gx_{m(k)},fx_{m(k)},gx_{l(k)})}\varphi(t)dt + \gamma \int_0^{G(gx_{m(k)},gx_{m(k)},fx_{l(k)})}\varphi(t)dt + \eta\int_0^{G(gx_{m(k)},gx_{m(k)},gx_{l(k)})}\varphi(t)dt$$
$$= \alpha \int_0^{G(y_{m(k)},y_{m(k)-1},y_{l(k)-1})}\varphi(t)dt + \beta\int_0^{G(y_{m(k)-1},y_{m(k)},y_{l(k)-1})}\varphi(t)dt$$
$$+\gamma \int_0^{G(y_{m(k)-1},y_{m(k)-1},y_{l(k)})}\varphi(t)dt + \eta\int_0^{G(y_{m(k)-1},y_{m(k)-1},y_{l(k)-1})}\varphi(t)dt$$

Letting $k\to\infty$, we find using (2.6) and (2.7)

$$0 < \int_0^{\epsilon}\varphi(t)dt \leq (\alpha+\beta)\int_0^{\lim_{k\to\infty}G(y_{m(k)},y_{m(k)-1},y_{l(k)-1})}\varphi(t)dt$$
$$+\gamma \int_0^{\lim_{k\to\infty}G(y_{m(k)-1},y_{m(k)-1},y_{l(k)})}\varphi(t)dt$$
$$+\eta \int_0^{\lim_{k\to\infty}G(y_{m(k)-1},y_{m(k)-1},y_{l(k)-1})}\varphi(t)dt$$
$$\leq (\alpha+\beta+\gamma+\eta)\int_0^{\epsilon}\varphi(t)dt$$

Which is a contradiction, since $\alpha+\beta+\gamma+\eta \in [0,1)$. Thus, we proved that $\{ g x_n\}$ is a G- Cauchy sequence g(X) . Since g(X) is G- complete, we obtain that $\{ g x_n\}$ is G – convergent to some q $\in$ g(X). So there exists p $\in$ X such that gp = q. From Proposition 1.1, we have

$$\lim_{n\to\infty} G(g x_n, g x_n, gp) = \lim_{n\to\infty} G(g x_n, gp, gp) = 0 . \qquad (2.8)$$

We will show that gp = fp. Suppose that gp $\neq$ fp. By (2.3), we have

$$\int_0^{G(g x_n,fp,fp)}\varphi(t)dt = \int_0^{G(f x_{n-1},fp,fp)}\varphi(t)dt$$
$$\leq \alpha \int_0^{G(fx_{n-1},gp,gp)}\varphi(t)dt + \beta\int_0^{G(gx_{n-1},fp,gp)}\varphi(t)dt$$
$$+\gamma \int_0^{G(gx_{n-1},gp,fp)}\varphi(t)dt + \eta\int_0^{G(gx_{n-1},gp,gp)}\varphi(t)dt$$

Taking $k\to\infty$, we obtain

$$\int_0^{G(g p,fp,fp)}\varphi(t)dt \leq 0.$$

Which implies that $G(g p, fp, fp) = 0$, so gp = fp. We now show that f and g have a unique point of coincidence. Suppose that ft= gt for some t $\in$ X . By applying (2.3), it follows that

$$\int_0^{G(gt,gp,gp)}\varphi(t)dt = \int_0^{G(ft,fp,fp)}\varphi(t)dt$$
$$\leq \alpha \int_0^{G(ft,gp,gp)}\varphi(t)dt + \beta\int_0^{G(gt,fp,gp)}\varphi(t)dt$$
$$+\gamma \int_0^{G(gt,gp,fp)}\varphi(t)dt + \eta\int_0^{G(gt,gp,gp)}\varphi(t)dt$$

Which holds unless $G(gt, gp, gp) = 0$, so gt = gp , that is the uniqueness of coincidence point of f and g. From Proposition 1.3 , f and g have a unique common fixed point .

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