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Convergence of Ishikawa Iteration Process for General Class of Function

KOTI N.V.V.VARA PRASAD[1], G.V.V.Jagannadha Rao[2] and K.V.Narasimham[3]
1. Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur,India
2. Department of Mathematics, The ICFAI University, Raipur, India.
3. Department of Mathematics, Lendi Institute of Technology, Visakhapatnam, India.

Abstract
In this paper, we introduce a general class of function and prove the convergence result of Ishikawa iteration considered in Banach spaces.

1 Introduction
The last four decades many papers have been published on the iterative approximation of fixed points for certain classes of operators using the Picard and Krasnoselskij iteration methods. Those papers were motivated by the fact that under weaker contractive conditions the Picard iteration need not converge to the fixed point of the operators.
Let E be a normed linear space, T: E→E a given operators. Let $x_0 \epsilon E$ be arbitrary.

(i). For any $\lambda \in (0,1)$, the sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$x_{n+1} = T_\lambda x_n = (1-\lambda)x_n + \lambda_n T x_{n,} \quad n = 0,1,2,\ldots, \quad (1)$$

is called Krasnoselskij iteration.

(ii). The sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$x_{n+1} = (1-\alpha)x_n + \alpha_n T x_{n,} \quad n = 0,1,2,\ldots, \quad (2)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence satisfying $0 \le \alpha_n < 1$, n=0,1,2,…, is called Mann iteration.

(iii). The sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$x_{n+1} = (1-\alpha)x_n + \alpha_n T y_{n,} \quad n = 0,1,2,\ldots,$$
$$y_n = (1-\beta)x_n + \beta_n T x_{n,} \quad (3) \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences of reals satisfying $0 \le \alpha_n, \beta_n < 1$, n=0,1,2,…, is called the Ishikawa iteration.

Remark:
1. If β_n = 0, then Ishikawa iteration reduces to Mann iteration.
2. If $\alpha_n = \lambda$, then Mann iteration reduces to Krasnoselskij iteration.
3. If $\alpha_n = 1$, then Mann iteration reduces to Picard iteration.

In 1972, Zamfirescu [14] obtained the following theorem.

Theorem 1. [14] Let (X,d) be a complete metric space and T: X → X a mapping for which there exists real numbers a,b,c satisfying $a \,\epsilon\, (0,1) and\ b, c \,\epsilon (0, \frac{1}{2})$ such that for pair $x, y \,\epsilon\, X$ at least one of the following conditions hold:

(z_1) $d(Tx, Ty) \le a\, d(x, y)$
(z_2) $d(Tx, Ty) \le b\, [d(x, Tx) + d(y, Ty)]$
(z_3) $d(Tx, Ty) \le c\, [d(x, Ty) + d(y, Tx)]$

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = T\, x_n \,, n = 0,1,2, \ldots \quad (4)$$

Converges to p, for any $x_0 \in X$.
An operator T which satisfies the contractive condition (z_1)– (z_2) of Theorem 1 will be called Zamfirescu operators.
Berinde [3] introduced a new class of operators on an arbitrary Banach spaces X satisfying

$$d(Tx, Ty) \le 2\delta\, [d(x, Tx) + \delta d(x, y)] \quad (5)$$

For all x, y $\in X$ and $\delta \in [0,1)$.
He proved that this class is wider than the class of Zamfirescu operators and used the Ishikawa iteration process to approximate fixed points of this class of operators in an arbitrary Banach spaces.The condition (z_1) of

Zamfirescu operator is the well known contraction condition introduced by Banach [1], the condition (z_2) of zamfirescu operator is called a Kannan mapping, while the mapping satisfying the condition (z_3) is called Chatterjea operator.

Several authors including Rhoades [12,13] employed the Zamfirescu condition to establish several interesting convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space. Berinde [3] extends the results of Rhoades [12,13] to an arbitrary Banach spaces for the same fixed point iteration processes.

In 1995, Osilike [10] considered the following contractive condition: there is $L \geq 0, a \in [0,1)$ such that for each x, y $\in E$

$\|Tx - Ty\| \leq L\|x - Tx\| + a\|x - Tx\|$ (6)

and established T – stability for such maps with respect to Picard, Mann and Ishikawa iterations.

In 2003, Imoru and Olatinwo[5] extended the results of Osilike [10] and proved some stability results for Picard and Mann iteration process using the following contractive condition: there exists $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that for each x, y $\in E$,

$\|Tx - Ty\| \leq \varphi\|x - Tx\| + a\|x - y\|$ (7)

A lot of “generalizations” and contractive conditions similar to (2) and (3) were also employed by several authors Olatinwo [19 from111], for more details see [2, 3-14].

Our aim in this paper is to be introduced the following general class of function considered in Banach spaces. We shall employ the following contractive condition. Let $(E, \|.\|)$ be Banach space, $T: E \to E$ a self map of E, with a fixed point p such that for each y $\in E$ and M>0, $0 \leq a < 1$, and $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that for each x, y $\in E$,

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}\left[\frac{\varphi\|x - Tx\| + a\|x - y\|}{1 + M\|x - Tx\|}\right] \tag{8}$$

This contractive condition is called “general class of function”.

2. Main Result

Theorem 2.1. Let E be a arbitrary Banach spaces, K be an arbitrary closed convex subset of E and $T: K \to K$ a self map of E with a fixed point p, satisfying the condition (8). For $x_0 \in E$, let x_n be the Ishikawa iteration defined by (3) for $y_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in [0,1] with $\sum \alpha_n = \infty$.
then $\{x_n\}$ converges strongly to the fixed point of T.

Proof: Using the Ishikawa iteration (3), the condition (8) and triangle inequality, we get

$$\|x_{n+1} - p\| = \|(1 - \alpha_n)x_n + \alpha_n Ty_n - p\|$$
$$= \|(1 - \alpha_n)x_n + \alpha_n Ty_n - \alpha_n p + \alpha_n p - p\|$$
$$= \|(1 - \alpha_n)(x_n - p) + \alpha_n(Ty_n - p)\|$$

$\leq (1 - \alpha_n)\|(x_n - p)\| + \alpha_n\|(Ty_n - p)\|$ (9)

On taking x=p and y=y_n in (8), we have

$$\|Tp - Ty_n\| \leq e^{L\|p-Tp\|}\left[\frac{\varphi\|p - Tp\| + a\|p - y_n\|}{1 + M\|p - Tp\|}\right]$$

Since Tp=p, then we have

$\|p - Ty_n\| \leq a\|p - y_n\|$ (10)

Substitute (10) in (9), we get

$\|x_{n+1} - p\|$
$\leq (1 - \alpha_n)\|(x_n - p)\| + \alpha_n a\|y_n - p\|$ (11)

Since

$\|y_n - p\| = \|(1 - \beta_n)x_n + \beta_n Tx_n - p\|$
$= \|(1 - \beta_n)x_n + \beta_n Tx_n - \beta_n p + \beta_n p - p\|$

$$= \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\|$$

$\leq (1 - \beta_n)\|(x_n - p)\| + \beta_n\|(Tx_n - p)\|$ (12)

On taking x=p and y=x_n in (8), we have

$\|Tp - Tx_n\| \leq e^{L\|p-Tp\|}\left[\frac{\varphi\|p-Tp\|+a\|p-x_n\|}{1+M\|p-Tp\|}\right]$

Since Tp=p, then we have

$\|p - Tx_n\| \leq a\|p - x_n\|$ (13)

Substitute (13) in (9), we have

$\|y_n - p\|$
$\leq (1 - \beta_n)\|(x_n - p)\| + \beta_n a\|(x_n - p)\|$ (14)

Now from (11) and (14), we get

$$\|x_{n+1} - p\|$$
$$\leq \left[(1-\alpha_n) + \alpha_n a[1-\beta_n(1-a)]\right]\|(x_n - p)\|$$
$$= \left[1-\alpha_n\left(1-a\left(1-\beta_n(1-a)\right)\right)\right]\|(x_n - p)\|$$
$$= [1-(1-a)\alpha_n - \alpha_n\beta_n a(1-a)]\|(x_n - p)\|$$
$$\leq [1-(1-a)\alpha_n]\|(x_n - p)\|$$
$$\leq \prod_{k=0}^{n}[1-(1-a)\alpha_n]\|(x_n - p)\|$$
$$\leq \prod_{k=0}^{n} e^{-(1-a)\alpha_k}[1-(1-a)\alpha_n]\|(x_n - p)\| \tag{15}$$

as $n \to \infty$. Since $\sum_{k=0}^{n}\alpha_k = \infty$, $a \in]$ and from (12), we have
$\|x_n - p\| \to 0$ as $n \to \infty$,
which implies that Ishikawa iteration process converges to p.

Uniqueness: We take $\mathrm{p}_1, \mathrm{p}_2 \in \mathrm{F}_\mathrm{T}$, where F_T is the set of fixed points of T in E such that $\mathrm{p}_1 = \mathrm{Tp}_1$ and $\mathrm{p}_2 = \mathrm{Tp}_2$. Suppose on the contrary that $\mathrm{p}_1 \neq \mathrm{p}_2$. Then, by choosing $x = \mathrm{p}_1$ and $\mathrm{y} = \mathrm{p}_2$
In (8), we get easily uniqueness part.
This completes the proof.
Consequently, we have the following corollaries:

Corollary 2.2. Let E be a arbitrary Banach spaces, K be an arbitrary closed convex subset of E and T: K→K a self map of E with a fixed point p, satisfying the condition (8). For $x_0 \in E$, let x_n be the Mann iteration defined by (2) for $y_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in [0,1] with $\sum\{\alpha_n\} = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.

Corollary 2.3. Let E be a arbitrary Banach spaces, K be an arbitrary closed convex subset of E and T: K→K a self map of E with a fixed point p, satisfying the condition (8). For $x_0 \in E$, let x_n be the Krasnoselskij iteration defined by (1) for $y_0 \in K$, where $\lambda \in [0,1]$. Then, the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Corollary 2.4. Let E be a arbitrary Banach spaces, K be an arbitrary closed convex subset of E and T: K→K a self map of E with a fixed point p, satisfying the condition (8). For $x_0 \in E$, let x_n be the Picard iteration defined by (4), then, the sequence $\{x_n\}$ converges strongly to the fixed point of T.

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