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Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space

A.S.Saluja [1], Alkesh Kumar Dhakde [2], Devkrishna Magarde [3]
1.J.H.Govt.P.G.College Betul (M.P.) India-460001
2.IES College of Technology Bhopal (M.P.) India-462044
3.Patel College of Sci. & Technology Bhopal (M.P.) India-462044

Abstract
The purpose of this paper is to present some fixed point theorem in dislocated quasi metric space for expansive type mappings.
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Introduction and Preliminaries:
It is well known that Banach Contraction mappings principle is one of the pivotal results of analysis. Generalizations of this principle have been obtained in several directions .Dass and Gupta [1] generalized Banach's Contraction principle in metric space. Also Rhoades [2] established a partial ordering for various definitions of contractive mappings. In 2005, Zeyada Salunke [4] proved some results on fixed point in dislocated quasimetric spaces. In 2005, Zeyada et al.[3] established a fixed point theorem in dislocated quasimetric spaces. In 2008, Aage and Salunke [4] proved some results on fixed point in dislocated quasimetric spaces. Recently, Isufati [5], proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric spaces. The following definitions will be needed in the sequel.

Definition 1.1(See [3]). Let X be a nonempty set, and let $d : X \times X \to [0,\infty)$ be a function, called a distance function. One needs the following conditions:

(M1) $d(x,x)=0$,

(M2) $d(x,y)=d(y,x)=0$, then $x=y$

(M3) $d(x,y)=d(y,x)$,

(M4) $d(x,y)\le d(x,z)+d(z,y)$,

(M4)$'$ $d(x,y)\le \max\{d(x,z),d(z,y)\}$, for all $x,y,z\in X$.

If d satisfies conditions (M1)-(M4), then it is called a metric on X. If d satisfies conditions (M1), (M2), and (M4), it is called a quasimetric on X. If it satisfies conditions (M2)-(M4) ((M2) and (M4)), it is called a dislocated metric (or simply d-metric) (a dislocated quasimetric (or simply dq-metric)) on X, respectively. If a metric d satisfies the strong triangle inequality (M)$'$, then it is called an ultrametric.

Definition 1.2 (See [3]). A sequence $\{x_n\}_{n\in N}$ in dq-metric space (dislocated quasimetric space) (X,d) is called a Cauchy sequence if , for given $\varepsilon>0$, there exists $n_0\in N$ such that $d(x_m,x_n)<\varepsilon$ or $d(x_n,x_m)<\varepsilon$, that is , $\min\{d(x_m,x_n),d(x_n,x_m)\}<\varepsilon$ for all $m,n\ge n_0$.

Definition 1.3 (See [3]). A sequence $\{x_n\}_{n\in N}$ in dq-metric space [d-metric space] is said to be d-converge to $x\in X$ provided that

$$\lim_{n\to\infty} d(x_n,x)=\lim_{n\to\infty} d(x,x_n)=0 \qquad (1.1)$$

In this case, x is called a dq-limit [d-limit] of $\{x_n\}$ and we write $x_n \to x$.

Definition 1.4 (See [3]). A dq-metric space (X,d) is called complete if every Cauchy sequence in it is a dq-convergent.

Main Results
In this paper, we prove some fixed point theorem for continuous mapping satisfying expansion condition in complete dq-metric space.

Theorem 2.1: Let (X,d) be a complete dislocated metric space and T a continuous mappings satisfying the following condition:

$$d(Tx,Ty)+\alpha\left[\frac{d(x,Ty)+d(y,Tx)}{1+d(x,Ty).d(y,Tx)}\right]\geq\beta\frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}+\gamma d(x,y) \tag{2.1}$$

For all $x,y\in X$, $x\neq y$,where $\alpha,\beta,\gamma\geq 0$ are real constants and $\beta+\gamma>1+2\alpha$, $\gamma>1+\alpha$ .Then T has a fixed point in X .

Proof: Choose $x_0\in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows and $Tx_n=x_{n-1}$ for $n=1,2,3......$ Then, using (2.1) we obtain

$$d(Tx_{n+1},Tx_{n+2})+\alpha\left[\frac{d(x_{n+1},Tx_{n+2})+d(x_{n+2},Tx_{n+1})}{1+d(x_{n+1},Tx_{n+2}).d(x_{n+2},Tx_{n+1})}\right]\geq\beta\frac{d(x_{n+1},Tx_{n+1})[1+d(x_{n+2},Tx_{n+2})]}{1+d(x_{n+1},x_{n+2})}$$
$$+\gamma d(x_{n+1},x_{n+2})$$
$$\Rightarrow d(x_n,x_{n+1})+\alpha\left[\frac{d(x_{n+1},x_{n+1})+d(x_{n+2},x_n)}{1+d(x_{n+1},x_{n+1}).d(x_{n+2},x_n)}\right]\geq\beta\frac{d(x_{n+1},x_n)[1+d(x_{n+2},x_{n+1})]}{1+d(x_{n+1},x_{n+2})}+\gamma d(x_{n+1},x_{n+2})$$
$$\Rightarrow d(x_n,x_{n+1})+\alpha d(x_{n+2},x_n)\geq\beta d(x_n,x_{n+1})+\gamma d(x_{n+1},x_{n+2})$$
$$\Rightarrow d(x_n,x_{n+1})+\alpha d(x_n,x_{n+1})+\alpha d(x_{n+1},x_{n+2})\geq\beta d(x_n,x_{n+1})+\gamma d(x_{n+1},x_{n+2})$$
$$\Rightarrow(1+\alpha-\beta)d(x_n,x_{n+1})\geq(\gamma-\alpha)d(x_{n+1},x_{n+2})$$

The last inequality gives

$$\begin{aligned}d(x_{n+1},x_{n+2})&\leq\left(\frac{1+\alpha-\beta}{\gamma-\alpha}\right)d(x_n,x_{n+1})\\&\leq kd(x_n,x_{n+1})\end{aligned} \tag{2.2}$$

Where $k=\dfrac{(1+\alpha-\beta)}{(\gamma-\alpha)}<1$. Hence by induction, we obtain

$$d(x_{n+1},x_{n+2})\leq k^{n+1}d(x_0,x_1)$$

Note that, for $m,n\in N$ such that $m>n$ we have

$$\begin{aligned}d(x_m,x_n)&\leq d(x_m,x_{m-1})+d(x_{m-1},x_{m-2})+...............+d(x_{n+1},x_n)\\&\leq\left[k^{m-1}+k^{m-2}+....................................+k^n\right]d(x_0,x_1)\\&\leq k^n\left(1+k+k^2+......................+k^{m-n-1}\right)d(x_0,x_1)\\&\leq k^n\sum_{r=0}^{\infty}k^r d(x_0,x_1)\\&=\frac{k^n}{1-k}d(x_0,x_1)\end{aligned} \tag{2.3}$$

Since $0\leq k<1$, then as $n\to\infty$, $k^n(1-k)^{-1}\to 0$. Hence, $d(x_m,x_n)\to 0$ as $m,n\to\infty$ .This forces that $\{x_n\}_{n\in N}$ is a Cauchy sequence in X . But X is a complete dislocated metric space; hence, $\{x_n\}_{n\in N}$ is d-converges. Call the d-limit $x^*\in X$ .Then, $x_n\to x^*$ as $n\to\infty$ .By continuity of T we have,

$$Tx^*=T\left(d-\lim_{n\to\infty}x_n\right)=d-\lim_{n\to\infty}Tx_n=d-\lim_{n\to\infty}x_{n-1}=x^* \tag{2.4}$$

That is, $Tx^*=x^*$; thus , T has a fixed point in X .

Uniqueness

Let y^* be another fixed point of T in X , then $Ty^*=y^*$ and $Tx^*=x^*$ .now ,

$$d(Tx^*,Ty^*)+\alpha\left[\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{1+d(x^*,Ty^*).d(y^*,Tx^*)}\right]\geq\beta\frac{d(x^*,Tx^*)[1+d(y^*,Ty^*)]}{1+d(x^*,y^*)}+\gamma d(x^*,y^*) \quad (2.5)$$

This implies that

$$d(x^*,y^*)+\alpha\left[\frac{d(x^*,y^*)+d(y^*,x^*)}{1+d(x^*,y^*).d(y^*,x^*)}\right]\geq\beta\frac{d(x^*,x^*)[1+d(y^*,y^*)]}{1+d(x^*,y^*)}+\gamma d(x^*,y^*)$$

$$\Rightarrow d(x^*,y^*)+\frac{2\alpha d(x^*,y^*)}{1+[d(x^*,y^*)]^2}\geq\gamma d(x^*,y^*)$$

$$\Rightarrow d(x^*,y^*)+[d(x^*,y^*)]^3+2\alpha d(x^*,y^*)\geq\gamma d(x^*,y^*)+\gamma[d(x^*,y^*)]^3$$

$$\Rightarrow(1+2\alpha-\gamma)d(x^*,y^*)\geq(\gamma-1)[d(x^*,y^*)]^3$$

$$d(x^*,y^*)\leq\left(\frac{1+2\alpha-\gamma}{\gamma-1}\right)^{\frac{1}{3}}d(x^*,y^*) \quad (2.6)$$

This is true only when $d(x^*,x^*)=0$.Similarly $d(y^*,x^*)=0$ .Hence $d(x^*,y^*)=d(y^*,x^*)=0$ and so $x^*=y^*$.Hence, T has a unique fixed point in X .

Theorem 2.2: Let (X,d) be a complete dislocated metric space and T a sujective mapping satisfying the condition (2.1) for all $x,y\in X,x\neq y$, where $\alpha,\beta,\gamma\geq 0$ are real constants and $\beta+\gamma>1+2\alpha$, $\gamma>1+\alpha$. Then, T has a fixed point in X .

Proof : Choose $x_0\in X$ to be arbitrary , and define the iterative sequence $\{x_n\}_{n\in N}$ as follows : $Tx_n=x_{n-1}$ for $n=1,2,3,......$ Then , using (2.1) , we obtain , sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence in X .But X is a complete dislocated metric space ; hence $\{x_n\}_{n\in N}$ is a d-converges. Call the d-limit $x^*\in X$. Then, $x_n\to x^*$ as $n\to\infty$.

Existence of fixed point

Since T is a surjective map, so there exists a point y in X , such that $x=Ty$ .Consider

$$d(x_n,x)=d(Tx_{n+1},Ty)$$
$$\geq-\alpha\left[\frac{d(x_{n+1},Ty)+d(y,Tx_{n+1})}{1+d(x_{n+1},Ty).d(y,Tx_{n+1})}\right]+\beta\frac{d(x_{n+1},Tx_{n+1})[1+d(y,Ty)]}{1+d(x_{n+1},y)}+\gamma d(x_{n+1},y) \quad (2.7)$$

Taking $n\to\infty$, we get

$$d(x,x)\geq-\alpha\left[\frac{d(x,x)+d(y,x)}{1+d(x,x).d(y,x)}\right]+\beta\frac{d(x,x)[1+d(y,x)]}{1+d(x,y)}+\gamma d(x,y)$$

$$0\geq-\alpha d(x,y)+\gamma d(x,y) \quad (2.8)$$

$$\Rightarrow(\gamma-\alpha)d(x,y)\leq 0$$

$$\Rightarrow d(x,y)=0 \text{ as } \gamma>\alpha$$

Similarly, $d(y,x)=0$ .Hence $d(x,y)=d(y,x)=0$

This implies $x=y$ and so $Tx=x$, that is x is fixed point of T .

Uniqueness

Let y^* be another fixed point of T in X , then $Ty^*=y^*$ and $Tx^*=x^*$. Now,

$$d(Tx^*,Ty^*)+\alpha\left[\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{1+d(x^*,Ty^*).d(y^*,Tx^*)}\right]\geq\beta\frac{d(x^*,Tx^*)[1+d(y^*,Ty^*)]}{1+d(x^*,y^*)}+\gamma d(x^*,y^*)$$

This implies that

$$d(x^*,y^*)+\alpha\left[\frac{d(x^*,y^*)+d(y^*,x^*)}{1+d(x^*,y^*).d(y^*,x^*)}\right]\geq \beta\frac{d(x^*,x^*)[1+d(y^*,y^*)]}{1+d(x^*,y^*)}+\gamma d(x^*,y^*)$$

$$\Rightarrow d(x^*,y^*)+\frac{2\alpha d(x^*,y^*)}{1+[d(x^*,y^*)]^2}\geq \gamma d(x^*,y^*)$$

$$\Rightarrow d(x^*,y^*)+[d(x^*,y^*)]^3+2\alpha d(x^*,y^*)\geq \gamma d(x^*,y^*)+\gamma[d(x^*,y^*)]^2$$

$$\Rightarrow (1-2\alpha-\gamma)d(x^*,y^*)\geq(\gamma-1)[d(x^*,y^*)]^3$$

$$\Rightarrow d(x^*,y^*)\leq\left(\frac{1+2\alpha-\gamma}{\gamma-1}\right)^{\frac{1}{3}}d(x^*,y^*) \tag{2.9}$$

This is true only when $d(x^*,y^*)=0$. Similarly, $d(y^*,x^*)=0$. Hence $d(x^*,y^*)=d(y^*,x^*)=0$ and so $x^*=y^*$. Hence T has a unique fixed point in X.

The proof is completed.

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