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Multiset Path Orderings and Their Application to Termination of Term Rewriting Systems

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Abstract

In this expository paper, a comprehensive study of multiset orderings, nested multiset orderings and multiset path orderings is presented. In particular, it is illustrated how multiset path orderings admit the use of relatively simple and intuitive termination functions that lead to termination of a class of term rewriting systems.

1. Introduction

Termination is one of the most important properties of *term rewriting systems* (trs_s, for short) which, in general, is known to be undecidable (Huet and Lankford, 1978; Dershowitz, 1987). In the recent years, some powerful methods have been developed to prove termination of a large class of trs_s. Broadly, these methods can be divided into *direct* and *transformational* methods (Zankl, 2006). The direct methods are further classified into *syntactical* and *semantical* fragments. The *recursive path orders* (rpo_s), such as *lexicographic path orders* (lpo_s) and *multiset path orders* (mpo_s), are purely syntactical; while *Polynomial orders* and *Knuth-Bendix order* (kbo) are semantical. In *transformational* method, termination proof of a given *trs* is accomplished by *reducing* it to an appropriate equivalent form for which proving termination is relatively easier. Some examples of this class are the *dependency pair method*, *semantic labeling* and *freezing* (Terese, 2003; Singh *et al*. 2012a).
More often than not, designing an appropriate *reduction order* and *termination function* for trs_s is found quite an involved problem. The main objective of this paper is to demonstrate how the application of mpo_s permits the use of relatively simple and intuitive termination functions that help achieving termination of a class of trs_s.

2. Preliminaries

In what follows, abstracting from various expositions on trs_s, specially from (Ohlebusch, 2002; Dershowitz, 1982), we describe some basic concepts in order to make the paper self-contained.

Let $\mathcal{F}$ or Σ denote a *signature* (a finite set of function symbols with natural numbers as their arities) and $\mathcal{V}$ a countably infinite set of variables with $\mathcal{F} \cap \mathcal{V} = \emptyset$. A *constant* is a function symbol having no arguments. A *term* is formed from function symbols, constants and variables. A term without variables is called a *ground* or *closed* term and, the set of ground terms is denoted by $\mathcal{T}(\mathcal{F})$. Also, let $\mathcal{T}(\mathcal{F},\mathcal{V})$ denote the set of all terms built over $\mathcal{F}$ and $\mathcal{V}$. It is assumed that $\mathcal{F}$ contains at least one constant. The set of variables occurring in a term t is denoted by $\mathcal{V}ar(t)$. Terms are usually denoted by $r, s, t, \ldots$, possibly extended by subscripts. Occasionally, we write t to denote a sequence of terms $t_1, t_2, \ldots, t_k$ and, g to denote a sequence of function symbols $g_1, g_2, \ldots, g_k$. The size of a term t, denoted by $|t|$, represents the number of symbols in t. A *trs* is called *length preserving* if $|\sigma(l)| = |\sigma(r)|$ for all rules $l \longrightarrow r$ and all ground substitutions σ. It is not difficult to prove that any length preserving trs is simply terminating.

A *rewrite rule* is an ordered pair (l, r) of terms l and r such that $l \notin \mathcal{V}$ and the variables which occur in the right-hand side r, also occur in the left-hand side l. A *rewrite rule* (l, r) is usually written as $l \longrightarrow r$. A term rewriting system is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. A *trs* is often presented as a set of rewrite rules, without explicitly mentioning its signature, assuming that the signature consists of the function symbols occurring in the rewrite rules. We assume that $\mathcal{F}$ is finite, unless stated otherwise. Also, $l \longrightarrow r \in \mathcal{R} \Longrightarrow l\sigma \longrightarrow r\sigma$, where σ is a *substitution*. A subterm is successively replaced by an equal term until no further rewriting is possible.
Essentially, the main objective of the method of term rewriting is to apply a set of rewrite rules to terms to reduce them to their simplest forms. Formally, a *trs* $\mathcal{R}$ is called *terminating* if there is no infinite rewriting

sequence $t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow . . . .$ In other words, a *trs* is *terminating* if and only if all terms t have only rewrite sequences of finite length. A rewrite relation that is also a partial order is called a *rewrite order*. A well-founded rewrite order is called a *reduction order*. More explicitly, a reduction order is an order which is *well founded*, *monotonic*, and *stable* (closed under contexts and substitution). A trs $(\mathcal{F}, \mathcal{R})$ and a partial order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are said to be *compatible* if $\mathcal{R}$ is contained in $\succ$; that is, $l \succ r$ for every rewrite rule $l \longrightarrow r$ of $\mathcal{R}$. The rewrite relation induced by a rewrite system $\mathcal{R}$ is denoted by $\longrightarrow_{\mathcal{R}}$ and, its transitive closure by $\longrightarrow_{\mathcal{R}}^*$. It is easy to see that a *trs* is terminating if and only if it is compatible with a reduction order. The classical approach to prove termination of a particular trs consists in constructing a reduction order $\succ$ such that $l \succ r$ for each rewrite step $l \longrightarrow r$.

Definition 1.

A partial ordering $\succ$ is a *simplification ordering* for a set of terms $\mathcal{T}$ if it possesses the following three properties:

(i)	$t \succ t'$ implies $f(. . . t . . .) \succ f(. . . t' . . .)$	. . . (*replacement*)
(ii)	$f(. . . t . . .) \succ t$	. . . (*subterm*)
(iii)	$f(. . . t . . .) \succ f(. . . \ \ . . .)$	. . . (*deletion*)

where $f(. . . t . . .), f(. . . t' . . .),\ . . .$ belong to $\mathcal{T}$.

Definition 2.

A partial ordering $\succ$ is called *monotonic* if it possesses the *subterm* property $f(. . . t . . .) \succ t$, and the *deletion* property $f(. . . t . . .) \succ f(. . . \ \ . . .)$, for all terms in $\mathcal{T}$.

A well-founded monotonic order satisfying the subterm property is called a *simplification ordering*. By iterating the subterm property, it is easy to see that every term is also greater than any (not necessarily immediate) of its subterms. The deletion condition implies that deleting subterms of an operator of variable arity reduces the size of the term in the ordering; if an operator f is of fixed arity, the deletion condition is superfluous. Further, simultaneously holding of these conditions imply that *syntactically simpler* terms are smaller in the ordering (Dershowitz, 1982; Singh *et al*. 2012b).
A *trs* over a finite signature is called *simply terminating* if is compatible with a simplification order.

Definition 3. Let $\mathcal{V}$ be a set of variables. *The homeomorphic embedding* $\trianglerighteq_{emb}$, a binary relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, is defined as follows:
$s \trianglerighteq_{emb} t$ if and only if one of the followingh conditions holds:

1. $s = x = t$ for a variable $x \in \mathcal{V}$.

2. $s = f(s_1, . . . , s_n)$ and $t = f(t_1, . . . , t_n)$ for a function symbol $f \in \mathcal{F}^{(n)}$, and $s_1 \trianglerighteq_{emb} t_1, . . . , s_n \trianglerighteq_{emb} t_n$.

3. $s = f(s_1, . . . , s_n)$ for a function symbol $f \in \mathcal{F}^{(n)}$, and $s_j \trianglerighteq_{emb} t$ for some $j, 1 \leq j \leq n$.

For example,

$$f(f(h(a), h(x)), f(h(x), a)) \trianglerighteq_{emb} f(f(a, x), x).$$

Definition 4.

An infinite sequence $t_1, t_2, t_3, . . .$ of terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is *self-embedding* if there exist $1 \leq i < j$ such that $t_i \trianglelefteq_{emb} t_j$. Homeomorphic embedding $\trianglerighteq_{emb}$ could also be defined as the reduction relation $\longrightarrow_{\mathcal{R}_{emb}}^*$ induced by the rewrite system

$$\mathcal{R}_{emb} := \{f(x_1, . . . , x_n) \longrightarrow x_i | n \geq 1, f \in \mathcal{F}^{(n)}, 1 \leq i \leq n\}.$$

Since $\mathcal{R}_{emb}$ is obviously terminating, this shows that $\longrightarrow_{\mathcal{R}_{emb}}^* = \trianglerighteq_{emb}$ is a well-founded partial order. In fact, in view of Kruskal's Tree Theorem (Kruskal, 1960), $\trianglerighteq_{emb}$ satisfies a stronger property called *well partial order*

(Baader and Nipkow, 1998) for finite $\mathcal{F}$ and $\mathcal{V}$. Simplification orderings cannot be used to prove termination of *self- embedding* systems (Dershowitz, 1987).

Lemma 1

Let $\succ$ be a simplification ordering on a set of terms $\mathcal{T}(\mathcal{F},\mathcal{V})$, and $s,t \in \mathcal{T}(\mathcal{F},\mathcal{V})$. Then $s \trianglerighteq_{emb} t$ implies $s \succcurlyeq t$.

Proof Assume that $s \trianglerighteq_{emb} t$. We consider the three cases in the definition of $\trianglerighteq_{emb}$, and prove $s \succcurlyeq t$ by induction on $|s|$.
(i) If $s = x = t$ then $s \succcurlyeq t$, because $\succcurlyeq$ is reflexive. (ii) Assume that $s = f(s_{1,}. . . , s_n)$ and $t = f(t_1, . . . , t_n)$ for a function symbol $f \in \mathcal{F}^{(n)}$ and $s_1 \trianglerighteq_{emb} t_1, . . . , s_n \trianglerighteq_{emb} t_n$. By induction, we obtain $s_1 \succcurlyeq t_1, . . . , s_n \succcurlyeq t_n$. Since $\succ$ is a rewrite order, we have $f(s_{1,}. . . , s_n) \succcurlyeq f(t_1, . . . , t_n)$. (iii) Assume that $s = f(s_{1,}. . . , s_n)$ for a function symbol $f \in \mathcal{F}^{(n)}$ and $s_j \trianglerighteq_{emb} t$ for some $j, 1 \le j \le n$. By induction, we obtain $s_j \succcurlyeq t$. In addition, the subterm property of $\succ$ yields $s \succ s_j$, and thus $s \succ t$.

Lemma 2

Let $\mathcal{R}$ be a trs over a finite signature $\mathcal{F}$. Then every simplification order is a reduction order.

Proof.

By definition of simplification orders, it remains to be shown that every simplification order is well-founded. Assume that $\succ$ is a simplification order on $\mathcal{T}(\mathcal{F},\mathcal{V})$, and $t_1 \succ t_2 \succ t_3 \succ . . .$ is an infinite chain in $\mathcal{T}(\mathcal{F},\mathcal{V})$.
First, we show by contradiction that $\mathcal{V}ar(s_1) \supseteq \mathcal{V}ar(s_2) \supseteq \mathcal{V}ar(s_3) . . .$ holds. Assume that there exists a variable $x \in \mathcal{V}ar(t_{i+1}) - \mathcal{V}ar(t_i)$. Define a substitution $\sigma = \{x \longmapsto t_i\}$ such that on one hand, $t_i = \sigma(t_i)$ (since x does not occur in t_i) and $\sigma(t_i) \succ \sigma(t_{i+1})$ (since $\succ$ is a rewrite order). On the other hand, since t_i is a subterm of $\sigma(t_{i+1})$, it follows from the subterm property that $\sigma(t_{i+1}) \succcurlyeq t_i$. If we combine the two inequalities, we obtain $t_i \succ t_i$, which is a contradiction. The first part of the proof shows that, for the finite set $X = \mathcal{V}ar(t_i)$, all terms in the sequence $t_1, t_2, t_3, . . .$ belong to $\mathcal{T}(\mathcal{F},\mathcal{V})$. Since $\mathcal{F}$ and $\mathcal{V}$ are finite, Kruskal's Theorem implies that this sequence is good. i.e., there exist $i < j$ such that $t_i \trianglelefteq_{emb} t_j$. Now, Lemma 1 yields $t_i \preccurlyeq t_j$, which is a contradiction since we know that $t_i \succ t_{i+1} \succ . . . \succ t_j$.

A direct consequence of Kruskal's theorem (Kruskal, 1960) is that any simplification order over a finite signature is well-founded as shown above.

Theorem 1.

Simplification orders are well-founded on terms over finite signature $\mathcal{F}$.

Proof follows by Kruskal's Tree Theorem and Lemma 2 above.

3. The Recursive Path Ordering

An important syntactical technique to prove termination of term rewriting is by using the recursive path ordering (rpo) defined by Dershowitz (Dershowitz, 1987). For defining rpo, there underlies a well-founded ordering on the set of function symbols. At the first step, two terms are compared by comparing their root symbols, and then recursively, the collections of their immediate subterms are compared. These collections can be seen as *unordered multisets* (giving rise to the notion of *multiset path order*), introduced by Dershowitz (1982), or as *ordered tuples* (giving rise to the notion of *lexicographic path order*), introduced by Kamin and Levy (1980), or one can employ a combination of the two (giving rise to a *recursive path order with status*). In this work, we confine ourselves to the case where the *arguments are compared as multisets*.

In order to determine if a term s is greater than a term t using *rpo*, the outmost operators of the two terms are compared first. If the outermost operators happen to be equal, then those (immediate) subterms of t that are not also subterms of s must each be smaller(recursively in the term ordering) than some subterm of s. If the outmost operator of s is greater than that of t, then s must be greater than each subterm of t; while if the outmost operator of s is neither equal to nor greater than that of t,then some subterm of s must be greater than or equal to t.

Formally, *rpo* can be defined as follows:

Definition 5.

Let $\mathcal{F}$ be a signature and $\succ$ be a strict ordering (precedence) on $\mathcal{F}$. Then the rpo, $\succ_{rpo}$ on the set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of terms over $\mathcal{F}$ is defined recursively as follows:

$$s = f(s_1, . . . , s_m) \succ_{rpo} g(t_1, . . . , t_n) = t,$$

if and only if

(i) $f = g$ and $\{s_1, . . . , s_m\} \succ_{rpo} \{t_1, . . . , t_n\}$, or

(ii) $f \succ g$ and $\{s\} \succ_{rpo} \{t_1, . . . , t_n\}$, or

(iii) $f \not\succeq g$ and $\{s_1, . . . , s_m\} \succeq_{rpo} \{t\}$.

In other words, $s \succ_{rpo} t$ if either,

(i) f equals g but the tuple $(s_1, ..., s_n)$ accompanying f in s is bigger than the one accompanying g in t (i.e., $t_1, ..., t_n$)), or

(ii) s is bigger than all the *immediate subterms* t_i of t when $f \succ g$, or

(iii) when the outmost operator of s is neither equal to nor greater than that of t, then some subterm s_i of s must be bigger than or equal to t.

Example .

For the set of operators $\mathcal{F} = \{ack, s, 0\}$ with $ar(ack) = 2, ar(s) = 1, ar(0) = 0$ and the precedence $\succ$ given by $ack \succ s$, we have:

(i) $ack(0, M) \succ_{rpo} s(M)$ because $ack(0, M) \succ_{rpo} M$ (this is because M is a subterm of $ack(0, M)$), and $ack \succ s$.

(ii) $ack(s(M), 0) \succ_{rpo} ack(M, s(0))$.

On one hand, $ack(s(M), 0) \succ_{rpo} M$, because M is a subterm of $ack(s(M), 0)$ and $ack(s(M), 0) \succ_{rpo} s(0)$ (because $ack \succ s$ and $ack(s(M), 0) \succ_{rpo} 0$).

On the other hand, we have, $(s(M), 0) \succ_{rpo}^{lex} (M, s(0))$.

The *mpo* method for proving termination is based on the following:

3.1 Multiset Orderings

Multiset ordering was invented in the '70s to prove termination of programs (Dershowitz and Manna, 1979). It has been used in devising termination techniques like mpo, rpo and recently, in combination with the size-change principle of Lee *et al.* (2001), in the form of SCNP (SCT in NP) reduction pairs.

A multiset (mset, for short) M over a set S can be defined as a function from S into $\mathbb{N}$, the set of natural numbers including zero. Let $M(x)$ denote the number of occurrences of an object x of S in M called the multiplicity of x in M. If $M(x) > 0$ for finitely many x in S, then M is called *a finite mset*.
Let $\mathcal{M}(S)$ denote the set of all finite msets built on S. The *additive union* (or *sum*) denoted $\uplus$ of two msets M_1 and M_2 is defined as follows:

$$(M_1 \uplus M_2)(a) = M_1(a) + M_2(a), \text{ for all } a \in S.$$

The *difference* of M_1 and M_2 is defined as follows:

$$(M_1 - M_2)(a) = \max\{M_1(a) - M_2(a), 0\}, \text{ for all } a \in S.$$

Following (Dershowitz and Manna, 1979), a partial ordering $\succ$ on a set S may be extended to a partial ordering $\succ_{mul}$ on finite multisets of elements of S. In this extended ordering, $M_1 \succ_{mul} M_2$, for two finite multisets M_1 and M_2 over S, if M_2 can be obtained from M_1 by replacing one or more elements in M_1 by any (finite) number of elements taken from S, each of which is smaller than one of the replaced elements. They also proved that every well-founded relation on a set S induces a well-founded relation on $\mathcal{M}(S)$. Note that well-foundedness is an essential property of orderings for proving termination of TRSs. Formally, given a partially ordered set $(A, \succ)$, the multiset ordering $\succ_{mul}$ is defined on $\mathcal{M}(A)$ as follows:

$M_1 \succ_{mul} M_2$,

if $\exists\ X, Y \in \mathcal{M}(A)$, where $\emptyset \neq X \subseteq M_1$, $M_2 = (M_1 - X) \uplus Y$

and for all $y \in Y$, $\exists\ x \in X$ such that $x \succ y$.

Thus, if $\succ$ is an ordering, we have $M_1 \succ_{mul} M_2$ if we can obtain M_1 from M_2 by either (i) removing some (possibly duplicated) elements $x \in X$, or (ii) replacing some elements $x \in X$ by new but smaller elements $y \in Y$.

For example,

$\{3, 5\} \succ_{mul} \{3, 4, 4, 2\}$, $\{3, 3, 4, 0\} \succ_{mul} \{3, 4\}$,$\{3, 3, 4, 0\} \succ_{mul} \{3, 2, 2, 1, 1, 1, 4, 0\}$,

$\{3, 3, 4, 0\} \succ_{mul} \{3, 3, 3, 3, 2, 2\}$, $\{3, 3, 4, 0\} \succ_{mul} \{\ \}$, etc.

Theorem 2.

If $\succ$ is irreflexive and transitive, then $\succ_{mul}$ is also irreflexive and transitive.

Proof.

To show irreflexivity, we must show that there can be no multiset M such that $M \succ_{mul} M$.

Suppose that $M \succ_{mul} M$, then there would be some nonempty finite multiset X, such that $M = X \cup Z$ and $(\forall y \in X)(\exists x \in X)\ x \succ y$. In other words, for every element of X there would be a distinct elements of X greater than it, which is impossible for a finite X.

To show transitivity of $\succ_{mul}$, consider the following irreflexive relation $\succ_{mul}{}'$ on multisets in $\mathcal{M}(S)$: $Z \cup \{x\} \succ_{mul}{}' Z \cup Y$ if $(\forall y \in Y)\ x \succ y$. In other words, a finite multiset is reduced in the relation $\succ_{mul}{}'$ by replacing a single element with zero or more smaller elements. Note that the multiset ordering $\succ_{mul}$ is the transitive closure of the relation $\succ_{mul}{}'$ i.e., $M \succ_{mul} M'$ if and only if M' can be obtained from M by replacing elements in M one by one. It follows that $\succ_{mul}$ is transitive.

The following theorem implies that multiset orderings yield much simpler proof of termination of *trs*$_s$.

Theorem 3.

The multiset ordering $(\mathcal{M}(S), \succ_{mul})$ over (S , $\succ$) is well-founded if and only if $(S, \succ)$ is well-founded.

Proof.

(a) “only if” part. Suppose $(S, \succ)$ is not well-founded, then there exists an infinite decreasing sequence $s_1 \succ_{mul} s_2 \succ_{mul} s_3 \succ_{mul} \ldots$ of elements in S. The corresponding sequence of singletons $\{s_1\} \succ_{mul} \{s_2\} \succ_{mul} \{s_3\} \succ_{mul} \ldots$ forms an infinite decreasing sequence of elements in $\mathcal{M}(S)$, and thus $(\mathcal{M}(S), \succ_{mul})$ is not well-founded.

(b) "if" part. Assume that $(S, \succ)$ is not well-founded. We first extend S by adding to it an element $\perp$, and extend the ordering $\succ$ on S to make $\perp$ the least element i.e., for every element $s \neq \perp$ in S, $s \succ \perp$. Thus, clearly S remains well-founded.

Now, suppose that $(\mathcal{M}(S), \succ_{mul})$ is not well-founded. Thus, there exists an infinite decreasing sequence $M_1 \succ_{mul} M_2 \succ_{mul} M_3 \succ_{mul} \ldots$ of multisets from $\mathcal{M}(S)$. We derive a contradiction by constructing the following tree. Each node in the tree is labelled with some element of S and at each stage of the construction; the set of all terminal nodes in the tree forms a multiset in $\mathcal{M}(S)$.

Let us begin with a root node with children corresponding to each element of M.

Then, since $M_1 \succ_{mul} M_2$, there must exist multisets X, Y, Z, such that $M_1 = X \cup Z, M_2 = Y \cup Z$ are not empty, and $(\forall y \in Y)(\exists x \in X)\, x \succ y$. Then for each $y \in Y$, add a "son" labelled y to the corresponding node. In addition, grow a child $\perp$ from each of the elements of X. Since X is nonempty, growing $\perp$ ensures that even if Y is empty, at least one node is added to the tree. Since Y is finite, the nodes corresponding to X each have a finite number of sons. Repeat the process for $M_2 \succ_{mul} M_3$, $M_3 \succ_{mul} M_4$, and so on.

Since at least one node is added to the tree for each multiset M, in the sequence, were the sequence infinite, the tree corresponding to the sequence would also be infinite. But, by Konig's Infinity Lemma, an infinite tree with a finite number of children for each node must have an infinite path. On the other hand, by our construction, all paths in the tree are descending in the well-founded ordering $\succ$ on S, and must be finite. Thus, we have derived a contradiction, implying there cannot be an infinite sequence of multisets $M_1, M_2, M_3, \ldots$ satisfying $M_1 \succ_{mul} M_2 \succ_{mul} M_3 \ldots$

Definition 6.

Given a quasi-ordered set$(S, \succsim)$, the multiset quasi-ordering $\succsim_{mul}$ on $\mathcal{M}(S)$ is defined as follows: $M_1 \succsim_{mul} M_2$ if and only if, for some multisets M_1 and $M_2, Y \in \mathcal{M}(S)$, $M_2 \approx (M_1 - X) \cup Y$, and for all $y \in Y$, there is an $x \in X$ such that $x \succ y$, where the two multisets are considered equivalent if the equivalence classes of their elements (under $\approx$) are the same.

Remark 1.

An ordering $\succ$ over a set A can be extended to an ordering $\succ_{mul}$ on tuples in A^n (for some $n > 0$) as follows:

$(x_1, \ldots, x_n) \succ_{mul} (y_1, \ldots, y_n)$ if $\{x_1, \ldots, x_n\} \succ_{mul} \{y_1, \ldots, y_n\}$.

If $(S, \succ)$ is totally ordered, then for any two multisets $M_1, M_2 \in \mathcal{M}(S)$, one may decide whether $M_1 \succ_{mul} M_2$ by first sorting the elements of both M_1 and M_2 in descending order (with respect to the relation $\succ$) and then comparing the two sorted sequences lexicographically.

For example, in order to compare the multisets $\{3, 3, 4, 0\}$ and $\{3, 2, 1, 2, 0, 4\}$, one may compare the sorted sequences $(4, 3, 3, 0)$ and $(4, 3, 2, 2, 1, 0)$. Since $(4, 3, 3, 0)$ is lexicographically greater than $(4, 3, 2, 2, 1, 0)$, it follows that $\{3, 3, 4, 0\} \succ_{mul} \{3, 2, 1, 2, 0, 4\}$.

Multisets ordering enjoys the following *minimality property*:

Theorem 4 (Lescanne and Jouannaud, 1982):

For a given partial ordering $\succ$ on a set S, any partial ordering $\succ_{mul}'$ on $\mathcal{M}(S)$ that satisfies the property

$$s \succ s' \text{ implies } \{\ldots s \ldots\} \succ_{mul}' \{\ldots s' \ldots\}$$

is contained in the multiset ordering $\succ_{mul}$.

Remark 2.

If $(S, \succ)$ is of order type α, then the multiset ordering $(\mathcal{M}(S), \succ_{mul})$ over $(S, \succ)$ is of order type ω^{α}.This follows from the fact that there exists a mapping ψ from $\mathcal{M}(S)$ onto ω^{α} that is one-to-one and order-preserving, i.e., if $M \succ_{mul} M'$ for $M, M' \in \mathcal{M}(S)$,then the ordinal $\psi(M)$ is greater than $\psi(M')$. Such a mapping ψ can be defined $\psi(M) = \sum_{m\in M} \omega^{\varphi(m)}$ where Σ denotes the natural (commutative) sum of ordinals, and φ is the one-to-one order-preserving map from S onto α.

Remark 3.

Let us consider the special case where there is a bound k on the number of *replacement elements* i.e., $|Y| < k$. Any termination proof using this *bounded* multiset ordering over ℕ may be translated into a proof using $(\mathbb{N}, >)$. This may be done using the order-preserving function

$$\psi(M) = \sum_{n\in M} k^n$$

which maps multisets over the natural numbers into the natural numbers by summing the number k^n for every natural number n in a multiset M. Two special cases of interest are the following:

(i) If $|Y| \le |X|$ i.e., the size of the multiset is not increased, then the simpler function

$$\psi(M) = \sum_{n\in M} n + |M|$$

is order-preserving.

(ii) If $|Y| = |X|$ i.e., the size of the multiset is constant, then

$$\psi(M) = \sum_{n\in M} n$$

is order-preserving.

3.2 Nested Multiset Ordering

By a nested multiset, we mean that the elements of the multiset may be elements of the ground set S, or may be multisets of elements of S, or may be multisets containing both elements of S and multisets of elements of S, and so on.

For example,

$\{\{1,1\},\{\{0\},1,2\},0\}$ is a nested multiset.

Further, given a partially ordered set $(S, \succ)$, a nested multiset over S is either an element of S or else it is a finite multiset of nested multisets over S. Let $\mathcal{M}^*(S)$ denote the set of nested multisets over S.

Nested multiset ordering $\succ_{mul}{}^*$ on $\mathcal{M}^*(S)$ which is a recursive version of the standard multiset ordering, is defined as follows:

Definition 7 (Dershowitz and Manna, 1979):

For any two elements $M, M' \in \mathcal{M}^*(S)$, $M \succ_{mul}{}^* M'$ if

(i) $M, M' \in S$ and $M \succ M'$ i.e., two elements of the base set are compared using $\succ$, or

(ii) $M \notin S$ and $M' \in S$ i.e., a multiset is greater than an element of the base set, or

(iii) $M, M' \notin S$, and for some $X, Y, Z \in \mathcal{M}^*(S)$ where $X \neq \emptyset$, $M = X \cup Z$ and $M' = Y \cup Z$

and $(\forall y \in Y)(\exists x \in X)$ $x \succ_{mul}{}^* y$.

For example,

$\{\{1,1\},\{\{0\},1,2\},0\}$ is greater than $\{\{1,0,0\},5,\{\{0\},1,2\},0\}$, since $\{1,1\}$ is greater than both $\{1,0,0\}$ and 5, and also $\{\{1,1\},\{\{0\},1,2\},0\}$ is greater than $\{\{\{\},1,2\},\{5,5,2\},5\}$, since $\{\{0\},1,2\}$ is greater than each of the three elements $\{\{\},1,2\}$, $\{5,5,2\}$ and 5 .

Remark 4.

Let $\mathcal{M}^{i}(S)$ denote the set of all nested multisets of depth i. In other words,

$\mathcal{M}^{0}(S)=S$ and $\mathcal{M}^{i+1}(S)$ contains the multisets whose elements are taken from $\mathcal{M}^{0}(S),\mathcal{M}^{1}(S),\ldots,\mathcal{M}^{i}(S)$, with at least one element taken from $\mathcal{M}^{i}(S)$. Thus, the set $W^{*}(S)$ is the infinite union of the disjoint sets $\mathcal{M}^{0}(S),\mathcal{M}^{1}(S),\mathcal{M}^{2}(S),\ldots$. The following property holds:

Theorem 5

For nested multisets M and M′ , if the depth of M is greater than the depth of M′ , then $M \succ_{mul}{}^{*} M'$. That is, the elements of $\mathcal{M}^{i}(S)$ are all greater than the elements of $\mathcal{M}^{j}(S)$, for any $j<i$.

Proof.

The proof follows by induction on depth. It trivially holds for M of depth 0 . For the inductive step, let us assume that the nested multisets of depth i are greater than the nested multisets of depth less than i. That is, we need to show that a nested multiset M of depth i is greater than any nested multiset M′ of lesser depth. If the depth of M′ is 0, then $M'\in S$ while $M\notin S$, and therefore $M \succ_{mul}{}^{*} M'$, as desired. If the depth of M′ is less than i but greater than 0 , then each of the elements in M′ is of depth less than $i-l$. The nested multiset M , on the other hand, is of depth $i+l$ and must therefore contain some element of depth i, which by the inductive hypothesis, must be greater than each of the elements in M′. It follows that $M \succ_{mul}{}^{*} M'$.

It is easy to see that the partial ordering $\succ_{mul}{}^{*}$ is irreflexive and transitive. The proof that it is well-founded is the following theorem:

Theorem 6

The nested multiset ordering $(\mathcal{M}^{*}(S),\succ_{mul}{}^{*})$ over $(S,\succ)$ is well-founded if and only if $(S,\succ)$ is well-founded.

Proof.

(a) "only if" part. If $(S,\succ)$ is not well-founded, then there exists an infinite decreasing sequence $s_1 \succ s_2 \succ s_3 \succ \ldots$ of elements in S . This sequence is also an infinite decreasing sequence of elements in $\mathcal{M}^{*}(S)$ under $\succ_{mul}{}^{*}$, and $(\mathcal{M}^{*}(S),\succ_{mul}{}^{*})$ is therefore not well-founded.

(b) "if" part. In order to show that $(\mathcal{M}^{*}(S),\succ_{mul}{}^{*})$ is well-founded, it suffices to show that each $\mathcal{M}^{i}(S)$ is itself well-founded under $\succ_{mul}{}^{*}$. If $\mathcal{M}^{*}(S)$ were not well-founded, then there would exist an infinite decreasing sequence of nested multisets $M_1 \succ_{mul}{}^{*} M_2 \succ_{mul}{}^{*} \ldots$. By theorem 5 above, it follows that the depth of any nested multiset M_{i+1} in the sequence cannot be greater than the depth of its predecessor M_i . Since the sequence is infinite, it must have an infinite subsequence of nested multisets all of the same depth i , which contradicts the well-foundedness of $\mathcal{M}^{i}(S)$.

We prove that each $(\mathcal{M}^{i}(S),\succ_{mul}{}^{*})$ is well-founded by induction on i : The ordering $\succ_{mul}{}^{*}$ on $\mathcal{M}^{0}(S)=S$ is simply the ordering $\succ$ on S and hence it follows that $(\mathcal{M}^{0}(S),\succ_{mul}{}^{*})$ is well-founded. For the inductive step, assume that each $(\mathcal{M}^{j}(S),\succ_{mul}{}^{*}), j<i,$ is well-founded (note that each of the elements of $\mathcal{M}^{1}(S)$ is a member of the union of $\mathcal{M}^{0}(S),\mathcal{M}^{1}(S),\ldots,\mathcal{M}^{i-1}(S)$). By the induction hypothesis, each of these $\mathcal{M}^{j}(S)$ is well-founded under $\succ_{mul}{}^{*}$.Therefore their union under $\succ_{mul}{}^{*}$ also is well-founded. Furthermore, the ordering $\succ_{mul}{}^{*}$ on a pair of nested multisets from $\mathcal{M}^{i}(S)$ is exactly the standard multiset ordering over their union and since the union is well-founded, $\mathcal{M}^{i}(S)$ is well-founded.

Remark 5.

We have seen earlier that for $(\mathrm{S}, \succ)$ of order type α, the multiset ordering $(\mathcal{M}(\mathrm{S}), \succ_{mul})$ is of order type ω^{α}. In a similar manner, it can be shown that the order type of $(\mathcal{M}^{\mathrm{i}}(\mathrm{S}), \succ_{mul}{}^{*})$ is

$$\left.\begin{matrix}\omega^{\alpha}\\ \cdot\\ \cdot\\ \cdot\\ \omega^{\omega}\end{matrix}\right\} i\ times,$$

the limit of which is the ordinal ϵ_0, provided α is less than ϵ_0 . Consequently, if $(\mathrm{S}, \succ)$ is of order type less than ϵ_0 , then $(\mathcal{M}^{*}(\mathrm{S}), \succ_{mul}{}^{*})$ is of order type ϵ_0 (Dershowitz, 1982).

Definition 8.

Let $s = f(s_1, \ . \ . \ . \ . \ , s_n)$ and $t = f(t_{1, \ . \ . \ . \ . \ ,} \ t_n)$ be terms $\mathcal{T}$, then s and t are said to be equivalent *up to a permutation of arguments* if both s and t are variables, or if

(i) the top-level function symbols of s and t are identical, and

(ii) there is a one- to – one correspondence between the multisets $\{s_1, \ . \ . \ . \ . \ , s_n\}$ and $\{t_{1, \ . \ . \ . \ . \ ,} \ t_n\}$ such that if s_i and t_j correspond to each other, then s_i and t_j are equivalent up to a permutation of arguments.
We write $s \sim t$ if s and t are identical up to a permutation of arguments.

For example, $f(g(a, b), h(a, c))$ and $f(h(a, c), g(b, a))$ are equivalent up to a permutation of arguments.

Remark 6.

It is observed that the multiset ordering, nested multiset ordering (Dershowitz and Manna, 1979), and the simple path ordering (Plaisted ,1978a) can be considered as a special case of the recursive path ordering, in which the multiset constructor $\{. \ . \ .\}$ is greater than all other operators involved. The nested multiset ordering is a recursive path ordering on all terms constructed from one varyadic operator and with just that one operator of the order type ϵ_0. It was pointed out in (Dershowitz, 1987 and Paulson, 1984) that the nested multiset ordering has all the properties of simplification orderings.

3.3 Multiset Path Ordering

Plaisted (1978b) introduced a syntactic order on terms which consists in comparing terms by first comparing their root symbols according to a given *precedence*, and, in case of equality, recursively comparing the multisets of their immediate subterms. As these multisets ignore the positions of the subterms, this order is not sensitive to permutations of subterms. Multiset path ordering (*mpo*) is a binary relation on the first order terms. It was first introduced by Dershowitz (Dershowitz, 1982) for proving termination of rewriting systems. The *mpo* is a simplification ordering which is transitive and irreflexive, closed both under context application and substitution, and as well possesses the subterm property. Moreover, the subterm property contains homeomorphic embedding.

Definition 9.

If a term t is of the form $f(t_1, . \ . \ . \ , \ t_n)$ then f is called the root symbol of t. Also, $\{t_1, . \ . \ . \ , \ t_n\}$ are called the top-level subterms of t.

Definition 10.

A path is a sequence of operators, beginning with outmost one of the whole terms (say, the *root*, on viewing terms as trees) and taking subterms until a constant (*leaf*) is reached. In other words, if t is a ground term of the form $f(t_1, . \ . \ . \ , \ t_n)$, then a path in t is a sequence beginning with f and followed by a path from some top-level subterm of t. A *path of subterms* of t is the sequence consisting of t itself followed by a path of subterms for t_i for some i, $1 \leq i \leq n$. If t is a variable, then t itself is the only path of subterms for t.Thus, a path of subterms for $f(g(a, b), c)$ is the sequence $f(g(a, b), c)$, $g(a, b)$ and b. Also, a path of subterms of the term

$f(x, g(y, c))$ is the sequence $f(x, g(y, c))$, $g(y, c)$ and y. A *path order* is a well-founded relation on a set of terms; for example, *mpo* and *lpo*.

Definition 11.

A TRS $\mathcal{R} = (\mathcal{F}, R)$ is called *strict* mpo-*terminating* if and only if there exists a strict precedence $\succ$ such that $l \succ_{mpo} r$ for all $l \longrightarrow r \in R$.

Lemma 3.

Whenever $\succ$ is a strict precedence on a signature $\mathcal{F}$ then $\succ_{mpo}$ is a simplification order.

Definition 12.

Let $\succ$ be a strict precedence and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s, t \notin \mathcal{V}$ then $t = f(t_1, . . . , t_n)$ and $s = g(s_1, . . . , s_m)$.The *mpo*, $t \succ_{mpo} s$ is defined if one of the following holds:

(i) $f \prec g$ and $t_k \succcurlyeq_{mpo} s$, for a $k \in \{1, . . . , n\}$, or
(ii) $f \succ g$ and $t \succ_{mpo} s_k$, for all $k \in \{1, . . . , m\}$, or
(iii) $f = g$ and $\{t_1, . . . , t_n\} \succ_{mpo}^{mul} \{s_1, . . . , s_m\}$

where $\succ_{mpo}^{mul}$ stands for multiset extension of $\succ_{mpo}$.

3.4 Application of mpo in Proving Termination of TRSs

In what follows, a number of illustrations are provided to emphasize that *mpo* are amongst the simplest recursive path orderings developed so far for proving termination of trs_s.

Example 1. Consider the *trs*

$0 + y \longrightarrow y$ $\qquad$ $0 \times y \longrightarrow 0$

$s(x) + y \longrightarrow s(x + y)$ $\qquad$ $s(x) \times y \longrightarrow (y \times x) + y$.

It is immediate to see that *lpo* with precedence $\times > + > s$ cannot orient the last rule and *mpo* with precedence $\times > + > s$ orients all rules from left to right. We have

$$\{s(x), y\} \succ_{mpo}^{mul} \{y, x\}.$$

Example 2. The *trs* $\mathcal{R}$, consisting of the rule

$$f(x, g(y)) \longrightarrow f(y, x),$$

is mpo-terminating, but not lpo-terminating.

Since neither $x \gtrsim_{lpo} f(y, x)$ nor $g(y) \gtrsim_{lpo} f(y, x)$, both (i) and (ii) of definition 14 do not hold, we conclude that $f(x, g(y)) \succ_{lpo} f(y, x)$ is not lpo-terminating. Since the root symbols are equal case (iii) might apply and the arguments of the two terms could be compared lexicographically, but as x and y are incomparable, it does not lead to termination. For *mpo*, (iii) applies and $\{\{x, g(y)\}\} \succ_{mpo}^{mul} \{\{y, x\}\}$ is satisfied because $\forall s \in \{y\} \exists$ a $t \in \{g(y)\}$ such that $t \succ_{mpo} s$. That is, $g(y) \succ_{mpo}$ y, (rule (ii)).

Example 3. Consider the set T of arithmetic expressions constructed from some set of symbols and the single operator $+$. The *trs*

$$(\alpha + \beta) + \gamma \longrightarrow \alpha + (\beta + \gamma)$$

over T, contains just one rewrite rule which reparenthesizes a sum by associating to the right. For example, the expression $(p + q) + ((r + s) + t)$ becomes either $p + (q + ((r + s) + t))$ or $(p + q) + (r + (s + t))$, both of which become $p + (q + (r + (s + t)))$. Since the size of the expression remains constant when the rule is applied, some other measure is required to prove termination. This is as follows (Dershowitz and Mann, 1979):

Let us **c**onsider the multiset ordering over the natural numbers, $(\mathcal{M}(\mathbb{N}), \succ_{mul})$. Let the termination function $\mathcal{T}: T \longrightarrow \mathcal{M}(\mathbb{N})$ return the multiset of the sizes $|\alpha|$ of all the subexpressions of the form $\alpha + \beta$ in the expression (t). i.e.,

$$\mathcal{T}(t) = \{|\alpha|: \alpha + \beta \ in \ t\}.$$

For example,

$\mathcal{T}\left((p+q)+\left((r+s)+t\right)\right) = \{1,3,1,3\}$, since the left operands of the operator $+$ are $p, p+q,\ r$ and $r+s$.

Also, $\mathcal{T}(p+\left(q+\left(r+(s+t)\right)\right)) = \{1,1,1,3\}$.

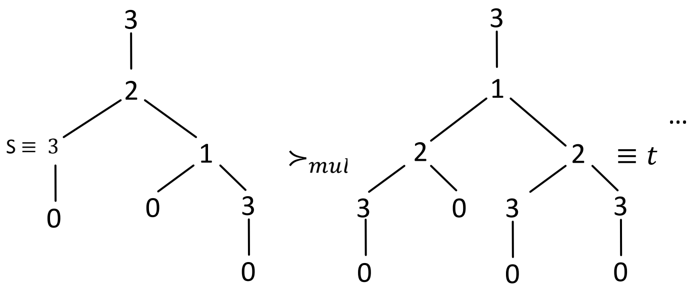
The value of the termination function τ decreases with each application of the rewrite rule,

i.e., $\mathcal{T}\left((\alpha+\beta)+\gamma\right) \succ_{mul} \mathcal{T}(\alpha+(\beta+\gamma))$.

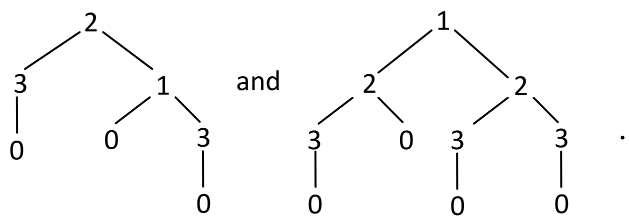
Thus, the system terminates.

Example 4. Determine if $s \succ_{mul} t$, where terms are denoted as trees (Dershowitz, 1982),

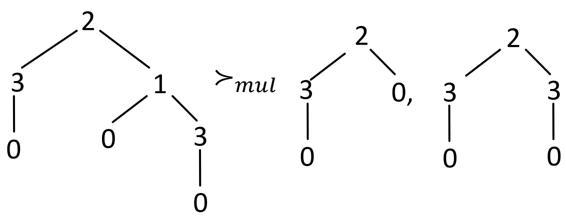
we have



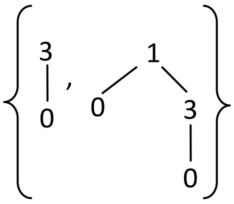
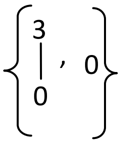
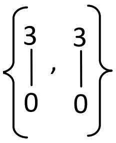
This is an *rpo* over set of terms of natural numbers, $\mathcal{T}(\mathbb{N})$, with the operators ordered by $>$. Since the two terms have the same outermost operator, using the definition of $\succ_{rpo}$, we only need to compare the multisets of their subterms, viz:



Since 2 > 1, in order that the former be greater than the latter, we must have:

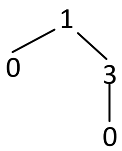


Since 2 = 2, we must now compare

 with  and 

in the multiset ordering $\succ_{mul}$.

Finally, since



is greater than both

0 and ,

we indeed have $s \succ_{mpo} t$.

For ordering operators, we take D to be greater than all other operators, as shown below.

Example 5 (Dershowitz and Manna, 1979).

The following rewrite rules symbolically differentiates an expression with respect to x.

$$Dx \longrightarrow 1$$

$$Dy \longrightarrow 0$$

$$D(\alpha + \beta) \longrightarrow (D\alpha + D\beta)$$

$$D(\alpha.\beta) \longrightarrow ((\beta.D\alpha) + (\alpha.D\beta))$$

$$D(-\alpha) \longrightarrow (-D\alpha)$$

$$D(\alpha - \beta) \longrightarrow (D\alpha - D\beta)$$

$$D(\alpha/\beta) \longrightarrow ((D\alpha/\beta) - ((\alpha.D\beta)/\beta \uparrow 2)))$$

$$D(\ln \alpha) \longrightarrow (D\alpha/\alpha)$$

$$D(\alpha \uparrow \beta) \longrightarrow (\Big(D\alpha.\big(\beta.\big(\alpha \uparrow (\beta - 1)\big)\big)\Big) + \Big(\big((ln\alpha).D\beta\big).(\alpha \uparrow \beta)\Big))$$

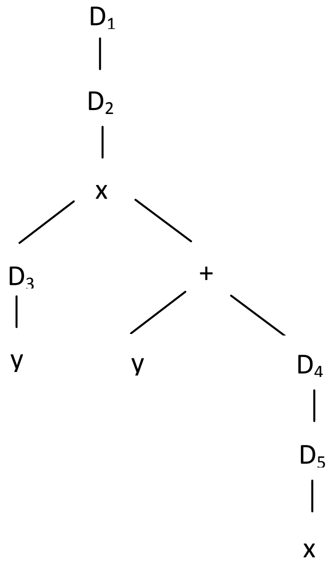
To prove termination, we use the multiset over sequences of natural numbers. The termination function is

$\mathcal{T}(e) = \{(d_1(x), d_2(x),. . .): x$ is an occurrence of a symbol in $e\}$,where $d_i(x)$ is the number of operators between x and the ith position enclosing D.

For example, consider the expression

$$t = D_x D_x\big(D_x y \times (y + D_x D_x x)\big) \quad \text{or} \quad t = D_1 D_2\big(D_3 y \times (y + D_4 D_5 x)\big)$$

with the $D's$ numbered (for expository purposes) in tree form:



There are three atoms y, y and x, which constitutes three paths. The leftmost atom y contributes the element $(0,2,3)$ to the multiset. This is because there are no operators between D_3 and y, there are two operators between D_2 and y, and there are three operators between D_1 and y. Similarly, the other two remaining atoms y and x contribute elements $(2, 3)$ and $(0, 1, 4, 5)$ respectively to the multiset. Thus,
$\mathcal{T}(t) = \{(0,2,3), (2,3), (0,1,4,5)\}$, or,

$$\mathcal{T}(t) = \{(D_1, D_2, \times, D_3, y), (D_1, D_2, \times, +, y), (D_1, D_2, \times, +, D_4, D_5, x)\}.$$

Applying the rewrite rule

$$D_x(\alpha \times \beta) \longrightarrow \beta \times D_x\alpha + \alpha \times D_x\beta$$

to t, yields

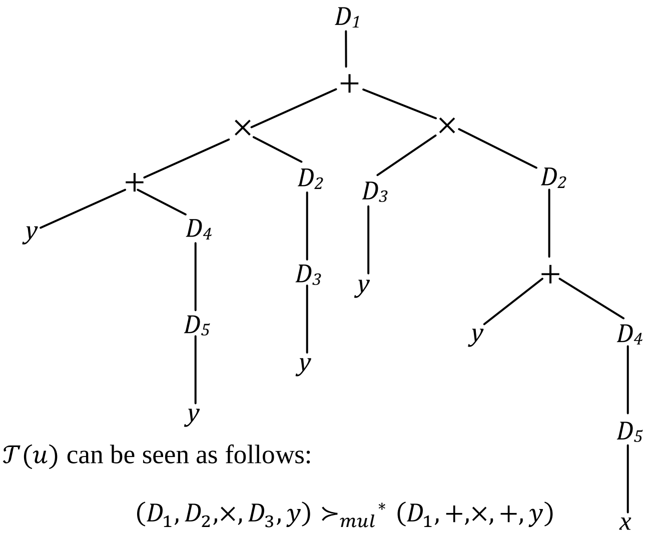
$$u = D_1(((y + D_4D_5x) \times D_2D_3y) + (D_3y \times D_2(y + D_4D_5x)))$$

with the labeling of the $D_x's$ retained, and thereby,

$\mathcal{T}(u) =$
$\{(D_1, +, \times, +, y), (D_1, +, \times, +, D_4, D_5, x), (D_1, +, \times, D_2, D_3, y), \quad (D_1, +, \times, D_3, y), (D_1, +, \times, D_2, +, y), (D_1, +, \times, D_2, +, D_4, D_5, x)\}$

or, $\mathcal{T}(u) = \{(3), (0, 1, 5), (0, 1, 4), (0, 3), (1, 4), (0, 1, 3, 6)\}$.

Representing in the form of a tree, we have



$\mathcal{T}(t) \succ_{mul}{}^{*} \mathcal{T}(u)$ can be seen as follows:

$$(D_1, D_2, \times, D_3, y) \succ_{mul}{}^{*} (D_1, +, \times, +, y)$$

$$(D_1, +, \times, D_2, +, D_4, D_5, x) \succ_{mul}{}^{*} (D_1, +, \times, +, D_4, D_5, x)$$

$$(D_1, D_2, \times, D_3, y) \succ_{mul}{}^{*} \quad (D_1, D_2, \times, D_3, y) \succ_{mul}{}^{*} (D_1, +, \times, D_2, y)$$

$$(D_1, D_2, \times, D_3, y) \succ_{mul}{}^{*} (D_1, +, \times, D_2, +, y)$$

$$(D_1, +, \times, D_2, +, D_4, D_5, x) \succ_{mul}{}^{*} (D_1, +, \times, D_2, +, D_4, D_5, x).$$

Remark 7.

The *mpo* defined here is only for strict precedences. This is because in multiset difference for quasi-precedences, equivalent terms have to be removed. This is known to be problematic because when the constraints for *mpo* are encoded in propositional logic, the equivalence relation on terms is still unknown. A foreseable way out would be to encode all possible multiset differences (exponential number) if that is computationally feasible which is an open question at the moment (Zankl, 2006).

4. Concluding remarks and some further research directions

The paper illustrates that, for a class of *trs*$_s$, *mpo*$_s$ are easy to handle and amenable to mechanization (Leclerc, 1995, for details). However, an *mpo*, being a simplification ordering, can handle only *simply terminating systems* (Borralleras and Rubio, 2001). Several techniques have been developed for overcoming this weakness of *rpo*$_s$. A very recent such technique is the *monotonic semantic path ordering (mspo)*, a simple and easily automatable ordering which generalizes other simplification methods (Borralleras and Rubio, 2001). Application of the techniques of *size-change* to *mpo*$_s$ is another promising area of research. The Size-change principle when compared with classical simplification orders can simulate a certain form of lexicographic and multiset comparison. Hence, the size-change principle in connection with *mpo* can often prove termination of *trs*$_s$ where one would otherwise need more complex orders (Lee, Jone and Ben-Amram, 2001).

It is known (Hofbauer, 1992) that termination proofs using multiset path orderings yield a *primitive recursive upper bound on the length of derivations measured in the size of the starting term*. This is as well true for many other path orderings as long as *status* is restricted to *multiset status* only. In general, the use of *mpo* or simplification orders is critically limited as shown by the analysis of induced derivational complexity (Hofbauer, 1992). In recent years, a less restricted variant of *mpo* has been developed (Avanzini and Moser, 2008). It has been shown in (Avanzini and Moser, 2008) that *Polynomial path order (POP^*) on terms* induces polynomial derivation height for innermost rewriting. POP^* is closely related to the *light multiset path order (lmpo, for short)* introduced by Marion (Marion, 2003). Besides, greater challenges lie in the area of *automation* of *termination techniques*. The current direction of research is largely concerned with resolving problems related to automation of termination analysis for *trs*$_s$ and developing thereby competing termination tools. The increasing interest in automated termination analysis of *trs*$_s$ has led to an annual *International Competition of Termination Tools* initiated in 2004 (Marché and Zantema, 2007, for details). It aims at identifying most talented competitors who could obtain an assigned task by applying appropriate choices of termination proving techniques within a time limit of 60 seconds. Using automata techniques is a relatively new and elegant approach of automatically proving the termination of rewrite systems. Initially proposed for string rewriting by Geser, Hofbauer, and Waldmann (2004), the method has recently been extended to left-linear trs_s. Variations and improvements are in progress (Endrullis, 2006; Geser, Hofbauer, and Waldmann, 2004, for details). The fact that the method has been implemented in several different termination provers (Waldmann, 2004; Zantema, 2005) is a clear indication of the success of the approach.

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