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First Order Linear Non Homogeneous Ordinary Differential Equation in Fuzzy Environment

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Abstract

In this paper, the solution procedure of a first order linear non homogeneous ordinary differential equation in fuzzy environment is described. It is discussed for three different cases. They are i) Ordinary Differential Equation with initial value as a fuzzy number, ii) Ordinary Differential Equation with coefficient as a fuzzy number and iii) Ordinary Differential Equation with initial value and coefficient are fuzzy numbers. Here fuzzy numbers are taken as Generalized Triangular Fuzzy Numbers (GTFNs). An elementary application of population dynamics model is illustrated with numerical example.

Keywords: Fuzzy Ordinary Differential Equation (FODE), Generalized Triangular fuzzy number (GTFN), strong solution.

1. Introduction: The idea of fuzzy number and fuzzy arithmetic were first introduced by Zadeh [11] and Dubois and Parade [5]. The term "Fuzzy Differential Equation (FDE)" was conceptualized in 1978 by Kandel and Byatt [1] and right after two years, a larger version was published [2]. Kaleva [16] and Seikkala [17] are the first persons who formulated FDE. Kaleva showed the Cauchy problem of fuzzy sets in which the Peano theorem is valid. The Generalization of the Hukuhara derivative which is based on fuzzy derivative was defined by Seikkala, and brought that the fuzzy initial value problem (FIVP) $x'(t) = f(t, x(t)), x(0) = x_0$ which has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which confirms a unique solution of the deterministic initial value problem. Fuzzy differential equation and initial value problem were extensively treated by other researchers (see [4,18,19,13,8,9,10]). Recently FDE has also used in many models such as HIV model [7], decay model [6], predator-prey model [15], population models [12] ,civil engineering [14], modeling hydraulic [3] etc.

In this paper we have considered 1st order linear non homogeneous fuzzy ordinary differential equation and have described its solution procedure in section-3. In section-4 we have applied it in a bio-mathematical model.

2. Preliminary concept:

Definition 2.1: Fuzzy Set: Let X be a universal set. The fuzzy set $\tilde{A} \subseteq X$ is defined by the set of tuples as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : \mu_{\tilde{A}} : X \to [0,1]\}$. The membership function $\mu_{\tilde{A}}(x)$ of a fuzzy set $\tilde{A}$ is a function with mapping $\mu_{\tilde{A}} : X \to [0,1]$. So every element x in X has membership degree $\mu_{\tilde{A}}(x)$ in [0,1] which is a real number. As closer the value of $\mu_{\tilde{A}}(x)$ is to 1, so much x belongs to $\tilde{A}$. $\mu_{\tilde{A}}(x_1) > \mu_{\tilde{A}}(x_2)$ implies relevance of x_1 in $\tilde{A}$ is greater than the relevance of x_2 in $\tilde{A}$. If $\mu_{\tilde{A}}(x_0)$= 1, then we say $x_0$exactly belongs to $\tilde{A}$, if $\mu_{\tilde{A}}(x_1) = 0$ we say x_1 does not belong to $\tilde{A}$, and if $\mu_{\tilde{A}}(x_2)$ = a where 0 < a < 1. We say the membership value of x_2 in $\tilde{A}$ is a. When $\mu_{\tilde{A}}(x)$ is always equal to 1 or 0 we get a crisp (classical) subset of X. Here the term "crisp" means not fuzzy. A crisp set is a classical set. A crisp number is a real number.

Definition 2.2: **α-Level or α-cut of a fuzzy set:** Let X be an universal set. Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}(\subseteq X)$ be a fuzzy set. α -cut of the fuzzy set $\tilde{A}$ is a crisp set. It is denoted by A_α. It is defined as $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha \; \forall x \in X\}$

Note: A_α is a crisp set with its characteristic function χ_{A_α}(x) defined as $\chi_{A_\alpha}(x) = 1 \; \mu_{\tilde{A}}(x) \geq \alpha \; \forall x \in X$
= 0 otherwise.

Definition 2.3: Convex fuzzy set: A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \subseteq X$ is called convex fuzzy set if all A_α are convex sets i.e. for every element $x_1 \in A_\alpha$ and $x_2 \in A_\alpha$ and for every $\alpha \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha \quad \forall \lambda \in [0,1]$. Otherwise the fuzzy set is called non convex fuzzy set.

Definition 2.4: Fuzzy Number: $\tilde{A} \in \mathcal{F}(R)$ is called a fuzzy number where R denotes the set of whole real numbers if

i. $\tilde{A}$ is normal i.e. $x_0 \in R$ exists such that $\mu_{\tilde{A}}(x_0) = 1$.
ii. $\forall \boldsymbol{\alpha} \in (0,1]$ A_α is a closed interval.

If $\tilde{A}$ is a fuzzy number then $\tilde{A}$ is a convex fuzzy set and if $\mu_{\tilde{A}}(x_0) = 1$ then $\mu_{\tilde{A}}(x)$ is non decreasing for $x \leq x_0$ and non increasing for $x \geq x_0$.
The membership function of a fuzzy number $\tilde{A}$ (a_1, a_2, a_3, a_4) is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \in [a_2, a_3] \neq \phi \\ L(x), & a_1 \leq x \leq a_2 \\ R(x), & a_3 \leq x \leq a_4 \end{cases}$$

Where L(x) denotes an increasing function and $0 < L(x) \leq 1$ and R(x) denotes a decreasing function and

$0 \leq R(x) < 1$.

Definition 2.5: Generalized Fuzzy number (GFN): Generalized Fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; \omega)$ where $0 < \omega \leq 1$, and a_1, a_2, a_3, a_4 $(a_1 < a_2 < a_3 < a_4)$ are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}}(x)$: R → [0, 1]
2) $\mu_{\tilde{A}}(x) = 0$ for $x \leq a_1$
3) $\mu_{\tilde{A}}(x)$is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = \omega$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$

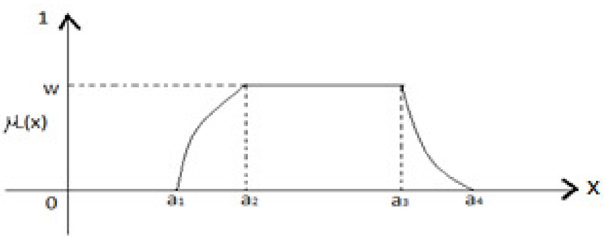

Fig-2.1: Membership function of a GFN

Definition 2.6: Generalized triangular fuzzy number (GTFN) : A Generalized Fuzzy number is called a Generalized Triangular Fuzzy Number if it is defined by $\tilde{A} = (a_1, a_2, a_3; \omega)$its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0\,, & x \le a_1 \\ \omega\frac{x-a_1}{a_2-a_1}\,, & a_1 \le x \le a_2 \\ \omega\,, & x = a_2 \\ \omega\frac{a_3-x}{a_3-a_2}\,, & a_2 \le x \le a_3 \\ 0\,, & x \ge a_3 \end{cases}$$

or, $\mu_{\tilde{A}}(x) = max\left(min\left(\omega\frac{x-a_1}{a_2-a_1}, \omega, \omega\frac{a_3-x}{a_3-a_2}\right), 0\right)$

Definition 2.7: Fuzzy ordinary differential equation (FODE):

Consider a simple 1st Order Linear non-homogeneous Ordinary Differential Equation (ODE) as follows:

$$\frac{dx}{dt} = kx + x_0 \text{ with initial condition } x(t_0) = \gamma$$

The above ODE is called FODE if any one of the following three cases holds:

(i) Only γ is a generalized fuzzy number (Type-I).
(ii) Only k is a generalized fuzzy number (Type-II).
(iii) Both k and γ are generalized fuzzy numbers (Type-III).

Definition 2.8: Strong and Weak solution of FODE:

Consider the 1st order linear non homogeneous fuzzy ordinary differential equation $\frac{dx}{dt} = kx + x_0$ with $(t_0) = x_0$. Here k or (and) x_0 be generalized fuzzy number(s).

Let the solution of the above FODE be $\tilde{x}(t)$ and its α-cut be $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$.
If $x_1(t,\alpha) \le x_2(t,\alpha) \forall\, \alpha \in [0,\omega]$ where $0 < \omega \le 1$ then $\tilde{x}(t)$ is called strong solution otherwise $\tilde{x}(t)$ is called weak solution and in that case the α-cut of the solution is given by

$$x(t,\alpha) = [\min\{x_1(t,\alpha), x_2(t,\alpha)\}, \max\{x_1(t,\alpha), x_2(t,\alpha)\}].$$

3. Solution Procedure of 1st Order Linear Non Homogeneous FODE

The solution procedure of 1st order linear non homogeneous FODE of Type-I, Type-II and Type-III are described. Here fuzzy numbers are taken as GTFNs.

3.1. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-I

Consider the initial value problem $\frac{dx}{dt} = Kx + x_0$ …………(3.1.1)

with Fuzzy Initial Condition (FIC) $\tilde{x}(t_0) = \tilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$

Let $\tilde{x}(t)$ be a solution of FODE (3.1.1) .

Let $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$ be the α-cut of $\tilde{x}(t)$

and $(\tilde{\gamma_0})_\alpha = [x_1(t_0,\alpha), x_2(t_0,\alpha)] = \left[\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}, \gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right] \quad \forall\, \alpha \,\epsilon\, [0,\omega], \quad 0 < \omega \le 1$

where $l_{\gamma_0} = \gamma_2 - \gamma_1$ and $r_{\gamma_0} = \gamma_3 - \gamma_2$

Here we solve the given problem for $k > 0$ and $k < 0$ respecively.

Case 3.1.1. When $k > 0$

The FODE (3.1.1) becomes a system of linear ODE

$$\frac{dx_i(t,\propto)}{dt} = kx_i(t,\alpha) + x_0 \quad \text{for } i = 1,2 \qquad \text{.............(3.1.2)}$$

with initial condition $x_1(t_0,\alpha) = \gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}$ and $x_2(t_0,\alpha) = \gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}$

The solution of (3.1.2) is

$$x_1(t,\alpha) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + (\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega})\right\} e^{k(t-t_0)} \qquad \text{.............(3.1.3)}$$

and $$x_2(t,\alpha) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + (\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega})\right\} e^{k(t-t_0)}\ . \qquad \text{............(3.1.4)}$$

Now $\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{l_{\gamma_0}}{\omega} e^{k(t-t_0)} > 0$, $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = -\frac{r_{\gamma_0}}{\omega} e^{k(t-t_0)} < 0$

and $x_1(t,\omega) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + \gamma_2\right\} e^{k(t-t_0)} = x_2(t,\omega)$.

So the solution of FODE (3.1.1) is a generalized fuzzy number $\tilde{x}$. The α-cut of the solution is

$$x(t,\alpha) = -\frac{x_0}{k} + \left[\frac{x_0}{k} + \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right), \frac{x_0}{k} + (\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega})\right] e^{k(t-t_0)}.$$

Case 3.1.2. when $k < 0$

Let $k = -m$ where m is a positive real number.

Then the FODE (3.1.1) becomes a system of ODE as follows

$$\left.\begin{matrix} \frac{dx_1(t,\propto)}{dt} = -mx_2(t,\alpha) + x_0 \\ \frac{dx_1(t,\propto)}{dt} = -mx_2(t,\alpha) + x_0 \end{matrix}\right\} \qquad \text{........... ...(3.1.5)}$$

with initial condition $x_1(t_0,\alpha) = \gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}$ and $x_2(t_0,\alpha) = \gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}$.

The solution of (3.1.5) is

$$x_1(t,\alpha) = \frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\} e^{-m(t-t_0)} + \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right) e^{m(t-t_0)}$$

and

$$x_2(t,\alpha) = \frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\} e^{-m(t-t_0)} - \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right) e^{m(t-t_0)}\ .$$

Here

$$\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right) e^{-m(t-t_0)} + \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right) e^{m(t-t_0)} \quad ,$$

$$\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right) e^{-m(t-t_0)} - \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right) e^{m(t-t_0)}$$

and $x_1(t,\omega) = \frac{x_0}{m} + \left(-\frac{x_0}{m} + \gamma_2\right) e^{-m(t-t_0)} = x_2(t,\omega)$

Here three cases arise.

Case1: When $l_{\gamma_0} = r_{\gamma_0}$ i.e., $\widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$ is a symmetric GTFN

$\therefore \frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{1}{2\omega}(l_{\gamma_0} + r_{\gamma_0})e^{m(t-t_0)} > 0$, $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = -\frac{1}{2\omega}(l_{\gamma_0} + r_{\gamma_0})e^{m(t-t_0)} < 0$

and $x_1(t,\omega) = x_2(t,\omega)$

So the solution of the FODE (3.1.1) is a strong solution.

Case2: When $l_{\gamma_0} < r_{\gamma_0}$ i.e., $\widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$ is a non symmetric GTFN

Here $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] < 0$ and $x_1(t,\omega) = x_2(t,\omega)$

but $\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] > 0$ implies $t > t_0 + \frac{1}{2m}\log\left[\frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$.

So the solution of the FODE (3.1.1) is a strong solution if $t > t_0 + \frac{1}{2m}\log\left[\frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$.

Case3: When $l_{\gamma_0} > r_{\gamma_0}$ i.e., $\widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$ is a non symmetric GTFN

Here $\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] < 0$ and $x_1(t,\omega) = x_2(t,\omega)$

but $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] < 0$ implies $t > t_0 + \frac{1}{2m}\log\left[\frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$.

So the solution of the FODE (3.1.1) is a strong solution if $t > t_0 + \frac{1}{2m}\log\left[\frac{l_{\gamma_0} - r_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$.

3.2. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-II

Consider the initial value problem $\frac{dx}{dt} = \tilde{k}x + x_0$ ………….(3.2.1)

with IC $x(t_0) = \gamma$. Here $\tilde{k} = (\beta_1, \beta_2, \beta_3; \lambda)$.

Let $\tilde{x}(t)$ be the solution of FODE (3.2.1)

Let $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$ be the α-cut of the solution and the α-cut of $\tilde{k}$ be

$(\tilde{k})_\alpha = [k_1(\alpha), k_2(\alpha)] = \left[\beta_1 + \frac{\alpha l_k}{\lambda}, \beta_3 - \frac{\alpha r_k}{\lambda}\right] \quad \forall\, \alpha \in [0,\lambda], \quad 0 < \lambda \le 1$

where $l_k = \beta_2 - \beta_1$ and $r_k = \beta_3 - \beta_2$.

Here we solve the given problem for $\tilde{k} > 0$ and $\tilde{k} < 0$ respecively.

Case 3.2.1: when $\tilde{k} > 0$

The FODE (3.2.1) becomes a system of linear ODE

$\frac{dx_i(t,\propto)}{dt} = k_i(\alpha)x_i(t,\propto) + x_0$ for $i = 1,2$ ....(3.2.2)

with IC $x(t_0) = \gamma$.

The solution of (3.2.1)

$$x_1(t,\alpha) = -\frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})} + \left\{\gamma + \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})}\right\} e^{(\beta_1+\frac{\alpha l_k}{\lambda})(t-t_0)}$$

and

$$x_2(t,\alpha) = -\frac{x_0}{(\beta_3-\frac{\alpha r_k}{\lambda})} + \left\{\gamma + \frac{x_0}{(\beta_3-\frac{\alpha r_k}{\lambda})}\right\} e^{(\beta_3-\frac{\alpha r_k}{\lambda})(t-t_0)} .$$

Case 3.2.2: when $\tilde{K} < 0$

Let $\tilde{k} = -\tilde{m}$, where $\tilde{m} = (\beta_1, \beta_2, \beta_3; \lambda)$ is a positive GTFN.

So $(\tilde{m})_\alpha = [m_1(\alpha), m_2(\alpha)] = \left[\beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda}\right] \forall\, \alpha \in [0,\lambda], 0 < \lambda \le 1$

where $l_m = \beta_2 - \beta_1$ and $r_m = \beta_3 - \beta_2$

Then the FODE (3.2.1) becomes a system of ODE as follows

$$\left.\begin{aligned}\frac{dx_1(t,\propto)}{dt} &= -m_2(\alpha)x_2(t,\alpha) + x_0\\ \frac{dx_2(t,\propto)}{dt} &= -m_1(\alpha)x_1(t,\alpha) + x_0\end{aligned}\right\} \qquad ....(3.2.3)$$

with IC $x(t_0) = \gamma$

Thus the solution is

$x_1(t,\alpha)$

$$= \frac{1}{2}\left\{\gamma\left(1 - \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} - \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}(t-t_0)}$$

$$+\frac{1}{2}\left\{\gamma\left(1 + \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{-\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}(t-t_0)} + \frac{x_0}{\beta_1 + \frac{\alpha l_m}{\lambda}}$$

$x_2(t,\alpha)$

$$= -\frac{1}{2}\sqrt{\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}}\left\{\gamma\left(1 - \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} - \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda}(\beta_3-\frac{\alpha r_m}{\lambda})}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{\beta_1+\frac{\alpha l_m}{\lambda}}{\beta_3-\frac{\alpha r_m}{\lambda}}}\left\{\gamma\left(1+\sqrt{\frac{\beta_3-\frac{\alpha r_m}{\lambda}}{\beta_1+\frac{\alpha l_m}{\lambda}}}\right)-x_0\left(\frac{1}{\beta_1+\frac{\alpha l_m}{\lambda}}+\frac{1}{\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}}\right)\right\}e^{-\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}(t-t_0)}+\frac{x_0}{\beta_3-\frac{\alpha r_m}{\lambda}}$$

3.3. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-III

Consider the initial value problem $\frac{dx}{dt}=\widetilde{K}x+x_0$ …………(3.3.1)

With fuzzy IC $\tilde{x}(t_0)=\widetilde{\gamma_0}=(\gamma_1,\gamma_2,\gamma_3;\omega)$, where $\tilde{k}=(\beta_1,\beta_2,\beta_3;\lambda)$

Let $\tilde{x}(t)$ be the solution of FODE (3.3.1) .

Let $x(t,\alpha)=[x_1(t,\alpha),x_2(t,\alpha)]$ be the α-cut of the solution.

Also $\left(\tilde{k}\right)_\alpha=\left[\beta_1+\frac{\alpha l_k}{\lambda},\beta_3-\frac{\alpha r_k}{\lambda}\right]\forall\,\alpha\,\epsilon\,[0,\lambda],0<\lambda\leq 1$

where $l_k=\beta_2-\beta_1$ and $r_k=\beta_3-\beta_2$

and $(\widetilde{\gamma_0})_\alpha=[x_1(t_0,\alpha),x_2(t_0,\alpha)]=\left[\gamma_1+\frac{\alpha l_{\gamma_0}}{\omega},\gamma_3-\frac{\alpha r_{\gamma_0}}{\omega}\right]\forall\,\alpha\,\epsilon\,[0,\omega],0<\omega\leq 1$

where $l_{\gamma_0}=\gamma_2-\gamma_1$ and $r_{\gamma_0}=\gamma_3-\gamma_2$

Let $\eta=\min(\lambda,\omega)$

Here we solve the given problem for $\tilde{k}>0$ and $\tilde{k}<0$ respecively.

Case I: when $\tilde{k}>0$

The FODE (3.1.1) becomes a system of linear ODE

$\frac{dx_i(t,\propto)}{dt}=k_ix_i(t,\alpha)+x_0$ for $i=1,2$ …………..(3.3.2)

with initial condition $x_1(t_0,\alpha)=\gamma_1+\frac{\alpha l_{\gamma_0}}{\omega}$ and $x_2(t_0,\alpha)=\gamma_3-\frac{\alpha r_{\gamma_0}}{\omega}$

Therefore the solution is of (3.3.1)

$$x_1(t,\alpha)=-\frac{x_0}{(\beta_1+\frac{\alpha l_k}{\eta})}+\left\{\left(\gamma_1+\frac{\alpha l_{\gamma_0}}{\eta}\right)+\frac{x_0}{(\beta_1+\frac{\alpha l_k}{\eta})}\right\}e^{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)(t-t_0)}$$

And

$$x_2(t,\alpha)=-\frac{x_0}{(\beta_3-\frac{\alpha r_k}{\eta})}+\left\{\left(\gamma_3-\frac{\alpha r_{\gamma_0}}{\eta}\right)+\frac{x_0}{(\beta_3-\frac{\alpha r_k}{\eta})}\right\}e^{\left(\beta_3-\frac{\alpha r_k}{\eta}\right)(t-t_0)}$$

Case II: when $\tilde{k}<0$

Let $\tilde{k}=-\widetilde{m}$ where $\widetilde{m}=(\beta_1,\beta_2,\beta_3;\lambda)$ is a positive GTFN.

Then $(\widetilde{m})_\alpha=\left[\beta_1+\frac{\alpha l_m}{\lambda},\beta_3-\frac{\alpha r_m}{\lambda}\right]\quad\forall\,\alpha\,\epsilon\,[0,\lambda],\quad 0<\lambda\leq 1$

Let $\eta = \min(\lambda, \omega)$

Then the FODE (3.3.1) becomes a system of ODE as follows

$$\left.\begin{aligned}\frac{dx_1(t,\alpha)}{dt} &= -m_2(\alpha)x_2(t,\alpha)+x_0\\ \frac{dx_2(t,\alpha)}{dt} &= -m_1(\alpha)x_1(t,\alpha)+x_0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.3.3)$$

with IC $x_1(t_0,\alpha) = \gamma_1 + \frac{\alpha l_{\gamma 0}}{\omega}$ and $x_2(t_0,\alpha) = \gamma_3 - \frac{\alpha r_{\gamma 0}}{\omega}$.

Therefore the solution of (3.3.1) is

$x_1(t,\alpha) =$

$$\frac{1}{2}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} - \sqrt{\frac{\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)}}(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta})\right) - \left(\frac{1}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)} - \frac{1}{\sqrt{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}}\right)x_0\right\} e^{\sqrt{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}(t-t_0)}$$

0)

$$+\frac{1}{2}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} + \sqrt{\frac{\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)}}(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega})\right) - \left(\frac{1}{(\beta_1 + \frac{\alpha l_m}{\eta})} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}}\right)x_0\right\} e^{-\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}(t-t_0)} + \frac{x_0}{(\beta_1 + \frac{\alpha l_m}{\eta})}$$

and

$x_2(t,\alpha) =$

$$-\frac{1}{2}\sqrt{\frac{(\beta_1 + \frac{\alpha l_m}{\eta})}{(\beta_3 - \frac{\alpha r_m}{\eta})}}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} - \sqrt{\frac{(\beta_3 - \frac{\alpha r_m}{\eta})}{(\beta_1 + \frac{\alpha l_m}{\eta})}}(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta})\right) - \left(\frac{1}{(\beta_1 + \frac{\alpha l_m}{\eta})} - \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}}\right)x_0\right\} e^{\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{(\beta_1 + \frac{\alpha l_m}{\eta})}{(\beta_3 - \frac{\alpha r_m}{\eta})}}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} + \sqrt{\frac{(\beta_3 - \frac{\alpha r_m}{\eta})}{(\beta_1 + \frac{\alpha l_m}{\eta})}}(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta})\right) - \left(\frac{1}{(\beta_1 + \frac{\alpha l_m}{\eta})} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}}\right)x_0\right\} e^{-\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}(t-t_0)}$$

$$+\frac{x_0}{(\beta_3 - \frac{\alpha r_m}{\eta})} \quad .$$

4. Application: Population Dynamics Model

Bacteria are being cultured for the production of medication. Without har-vesting the bacteria, the rate of change of the population is proportional to its current population, with a proportionality constant k per hour. Also, the bacteria are being harvested at a rate of N per hour. If there are initially P_0 bacteria in the culture, solve the initial value problem: $\frac{dP}{dt} = k\,P - N$, $P(0) = P_0$ when

(i) $\tilde{P}_0 = (7800{,}8000{,}8150; 0.8)$ and $k = 0.2$, $N = 1000$,
(ii) $P_0 = 8000$ and $\tilde{k} = (0.17{,}0.2{,}0.24; 0.6)$, $N = 1000$,
(iii) $\widetilde{P_0} = (7900{,}8000{,}8200; 0.7)$ and $\tilde{k} = (0.16{,}0.2{,}0.23; 0.8)$, $N = 1000$.

Solution: (i) Here $(\tilde{P}_0)_\alpha = [7800 + 250\alpha, 8150 - 187.5\alpha]$.

Therefore solution of the model

$P_1(t,\alpha) = 5000 + (2800 + 250\alpha)e^{0.2t}$ and $P_2(t,\alpha) = 5000 + (3150 - 187.5\alpha)e^{0.2t}$.

Table-1: Value of $P_1(t,\alpha)$ and $P_2(t,\alpha)$ for different α and t=3

α	$P_1(t,\alpha)$	$P_2(t,\alpha)$
0	10101.9326	10739.6742
0.1	10147.4856	10705.5095
0.2	10193.0386	10671.3448
0.3	10238.5916	10637.1800
0.4	10284.1445	10603.0153
0.5	10329.6975	10568.8506
0.6	10375.2505	10534.6859
0.7	10420.8034	10500.5211
0.8	10466.3564	10466.3564

From above table-1 we see that for this particular value of t=3, $P_1(t,\alpha)$ is an increasing function, $P_2(t,\alpha)$ is a decreasing function and $P_1(t,0.8) = P_2(t,0.8) = 10466.3564$. So Solution of above model for particular value of t is a strong solution.

(ii) Here $(\tilde{k})_\alpha = [0.17 + 0.050\alpha, 0.26 - 0.066\alpha]$.

Therefore solution of the model is

$$P_1(t,\alpha) = \frac{1000}{(0.17+0.050\alpha)} + \left\{8000 - \frac{1000}{(0.17+0.050\alpha)}\right\} e^{(0.17+0.050\alpha)t}$$

and $$P_2(t,\alpha) = \frac{1000}{(0.24-0.066\alpha)} + \left\{8000 - \frac{1000}{(0.24-0.066\alpha)}\right\} e^{(0.24-0.066\alpha)t}.$$

Table-2: Value of $P_1(t,\alpha)$ and $P_2(t,\alpha)$ for different α and t=3

α	$P_1(t,\alpha)$	$P_2(t,\alpha)$
0	9408.8519	12041.9940
0.1	9578.1917	11765.8600
0.2	9750.2390	11495.6109
0.3	9925.0361	11231.1346
0.4	10102.6256	10972.3224
0.5	10283.0510	10719.0687
0.6	10466.3564	10471.2714

From above table-1 we see that for this particular value of t=3, $P_1(t,\alpha)$ is an increasing function, $P_2(t,\alpha)$ is a decreasing function and $P_1(t,0.8) < P_2(t,0.8)$. So Solution of above model for particular value of t is a strong solution.

(iii) Here

$(\tilde{k})_\alpha = [0.16 + 0.057\alpha, 0.23 - 0.042\alpha]$ and $(\widetilde{P_0})_\alpha = [7900 + 142.8\alpha, 8200 - 285.7\alpha]$

Therefore solution of the model is

$$P_1(t,\alpha) = \frac{1000}{(0.16+0.057\alpha)} + \left\{(7900 + 142.8\alpha) - \frac{1000}{(0.16+0.057\alpha)}\right\} e^{(0.16+0.057\alpha)t}$$

and $P_2(t,\alpha) = \frac{1000}{(0.23-0.042\alpha)} + \left\{(8200 - 285.7\alpha) - \frac{1000}{(0.23-0.042\alpha)}\right\} e^{(0.23-0.042\alpha)t}$

Table3: Value of $P_1(t,\alpha)$ and $P_2(t,\alpha)$ for different α and t=3

α	$P_1(t,\alpha)$	$P_2(t,\alpha)$
0	8916.5228	12027.9651
0.1	9124.4346	11797.2103
0.2	9336.5274	11570.1611
0.3	9552.8820	11346.7614
0.4	9773.5808	11126.9561
0.5	9998.7076	10910.6908
0.6	10228.3479	10697.9120
0.7	10462.5889	10488.5669

From above table-1 we see that for this particular value of t=3, $P_1(t,\alpha)$ is an increasing function, $P_2(t,\alpha)$ is a decreasing function and $P_1(t,0.8) < P_2(t,0.8)$. So Solution of above model for particular value of t is a strong solution.

5. Conclusion: In this paper we have solved a first order linear non homogeneous ordinary differential equation in fuzzy environment. Here fuzzy numbers are taken as GTFNs. We have also discussed three possible cases. For further work the same problem can be solved by Generalized L-R type Fuzzy Number. This process can be applied for any economical or bio-mathematical model and problems in engineering and physical sciences.

References:

[1] A. Kandel and W. J. Byatt, "Fuzzy differential equations," in Proceedings of the International Conference on Cybernetics and Society, pp. 1213–1216, Tokyo, Japan, 1978.

[2] A. Kandel and W. J. Byatt, "Fuzzy processes," Fuzzy Sets and Systems, vol. 4, no. 2, pp. 117–152, 1980.

[3] A. Bencsik, B. Bede, J. Tar, J. Fodor, Fuzzy differential equations in modeling hydraulic differential servo cylinders, in: Third Romanian_Hungarian Joint Symposium on Applied Computational Intelligence, SACI, Timisoara, Romania, 2006.

[4] B. Bede, I.J. Rudas, A.L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences 177 (2007) 1648–1662.

[5] Dubois, D and H.Parade 1978, Operation on Fuzzy Number. International Journal of Fuzzy system, 9:613-626

[6] G.L. Diniz, J.F.R. Fernandes, J.F.C.A. Meyer, L.C. Barros, A fuzzy Cauchy problem modeling the decay of the biochemical oxygen demand in water,2001 IEEE.

[7] Hassan Zarei, Ali Vahidian Kamyad, and Ali Akbar Heydari, Fuzzy Modeling and Control of HIV Infection, Computational and Mathematical Methods in Medicine Volume 2012, Article ID 893474, 17 pages.

[8] J. J. Buckley and T. Feuring, "Fuzzy differential equations," Fuzzy Sets and Systems, vol. 110, no. 1, pp.43–54, 2000.

[9] James J. Buckley, Thomas Feuring, Fuzzy initial value problem for Nth-order linear differential equations, Fuzzy Sets and Systems 121 (2001) 247–255.

[10] J.J. Buckley, T. Feuring, Y. Hayashi, Linear System of first order ordinary differential equations: fuzzy initial condition, soft computing6 (2002)415-421.

[11] L. A. Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.

[12] L.C. Barros, R.C. Bassanezi, P.A. Tonelli, Fuzzy modelling in population dynamics, Ecol. Model. 128 (2000) 27-33.

[13] L.J. Jowers, J.J. Buckley, K.D. Reilly, Simulating continuous fuzzy systems, Information Sciences 177 (2007) 436–448.

[14] M. Oberguggenberger, S. Pittschmann, Differential equations with fuzzy parameters, Math. Modelling Syst. 5 (1999) 181-202.

[15] Muhammad Zaini Ahmad, Bernard De Baets, A Predator-Prey Model with Fuzzy Initial Populations, IFSA-EUSFLAT 2009.

[16] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[17] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.

[18] W. Congxin, S. Shiji, Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions, Information Sciences 108 (1998) 123–134.

[19] Z. Ding, M. Ma, A. Kandel, Existence of the solutions of fuzzy differential equations with parameters, Information Sciences 99 (1997) 205–217.