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A Current Perspectives of Corrected Operator Splitting (OS) for Systems

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Abstract
In this paper we studied mathematical models for fluid flow often involve systems of convection-diffusion equations as a main ingredient. Operator splitting - one splits the time evolution into partial steps to separate the effects of convection and diffusion. In the present paper it has been shown that that the temporal splitting error can be significant when there is a shock present in the solution, is well-understood for scalar convection – diffusion equation.
Keywords: Convection diffusion, convection solver, front tracking method, nonlinear error mechanism, operator splitting,

1. Introduction

This paper deals the motivation for operator splitting methods is that it is easy to combine efficient methods for solving the convection step with efficient methods for the diffusive step. In the case for convection dominated systems, it is a major advantage to be able to use an accurate and efficient hyperbolic solver developed for tracking discontinuous solutions. Furthermore, combining this with efficient methods for the diffusive-step, we obtain a powerful and efficient numerical method which is well suited for solving parabolic problems with sharp gradients.

Moreover, the obvious disadvantage of operator splitting methods is the temporal splitting errors. The temporal splitting error in OS methods can be significant in regions containing viscous shocks. (Dahle H., Norway,(1988); Karlsen, K, Brusdal, K., Dahle, H., Evje, S. and Lie, K. A., (1998); Karlsen, K. & Risebro, N. (1997)) . To resolve viscous shock profiles correctly, we must resort to very small splitting steps. But, this imposes a time step restriction that is not present in the underlying numerical methods for the convective and diffusive step. I addition, to reduce the influence of temporal splitting errors in OS methods, to allow for the use of large splitting steps, the *corrected operator splitting* (COS) method was introduced (Espedal, M. & Ewing, R (1987). It was found that a recent mathematical description of the method was demonstrated (Karlsen, K,. & Risebro, N (2000). In the present paper our main concern behind the scalar COS method is to take into account the unphysical entropy loss (due to Olenik's convexification) Oleinik, O (1963), produced by the hyperbolic solver in the convective step. The COS approach uses the wave structure from the convective step to identify where the (nonlinear) splitting errors occur. The purpose of the paper demonstrated here is to derive through understanding of the nonlinear mechanism behind the viscous splitting error typically appearing in operator splitting methods for systems of convection-diffusion equations.

2.Operator Splitting

We consider the one-dimensional Cauchy problem for $\ell \times \ell$ $(\ell \geq 1)$ systems of convection-diffusion equations

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=D\frac{\partial^2 U}{\partial x^2}\,; \qquad U(x,0)=U_0(x) \qquad (1)$$

where $x \in \mathbb{R}$ and t > 0, now $U=(u_1,u_2,....u_\ell)^T$ denotes the unknown state vector, $F(U)=(f_1(U),\ ....f_\ell(U))^T$ represents a vector of flux functions for each variable and D = diag (ε_1,…….

$\varepsilon_\ell) > 0$ is a constant diagonal matrix. Again, S_t denotes the solution operator which takes the initial data $V_0(x)$ to a weak solution at time t of the purely hyperbolic problem

$$\frac{\partial V}{\partial t}+\frac{\partial F(V)}{\partial x}=0 \qquad V(x,t)=V_0(x) \tag{2}$$

Moreover, we denote $V(x, t) = S_t V_0(x)$ for the weak solution. Next, we denote H_t the operator which takes the initial data $W_0(x)$ as a weak solution at a time of the purely parabolic problem

$$\frac{\partial W}{\partial t}=D\frac{\partial^2 W}{\partial x^2}=0 \qquad W(x,t)=W_0(x) \tag{3}$$

In this case, we consider a fixed final computing time $T > 0$. But, we also choose a fixed splitting step $\Delta t > 0$ and an integer N_t such that $N_t \Delta t = T$. We define the semi-discrete OS algorithm by

$U_{\Delta t}(.\,, n\Delta t) := [H_{\Delta t} \circ S_{\Delta t}]^n \quad U_0(.), \quad n = 0,\ldots..N_t$ (4)

In addition in applications, the exact solution operators S_t , H_t in Eq. (4) are replaced by numerical methods. We use front tracking, demonstrated by Risebro (Risebro, N. (1993); Risebro, N. Tveito, A. (1991)
Risebro, N. Tveito, A. (1992)) as an approximate solution operator for the hyperbolic part. Now, for the parabolic part, we use a simple explicit central difference method.

3. Nonlinear Error Mechanism

The Operator splitting approximations can be too diffusive near viscous shock when the splitting step Δt is large; we study first the *scalar* case. The entropy condition introduces a local linearization of $f(.)$ once a shock is formed in the convection step. Thus this linearization represents the entropy loss associated with the formation of a shock in the hyperbolic solution. Thus, the evolution of the hyperbolic is governed locally by some convex / concave envelope f_c of f between the left and right shock values shown in Figure 1. Furthermore, a similar linearization can be introduced locally for the parabolic problem. Hence, the flux function f can be decomposed into a convective part f_c and a shelf-sharpening part $f - f_c$ that tends to counteract the diffusive forces. We study that f_c governs the local translation and $f - f_c$ the shape (or structure) of the viscous front. In the present paper the OS algorithm, the local residual flux $f - f_c$ is disregarded in the hyperbolic step and the corresponding self-sharpening effects are therefore not taken into account in the splitting, resulting in a splitting error. Figure 1, shows an illustration of f, f_c and the residual flux $f_{res} = f - f_c$ in the scalar case. Now, we study and consider the propagation of a single viscous shock. Assume that the splitting step is sufficiently large so that a shock has developed in the hyperbolic sudstep Eq. (2) *i.e* the solution $V(.,t=\bar{t})$ consists a single discontinuity at $x = \bar{x}$

With left and right shock values $V^l(v_1^l,.....v_1^l)^T$ and $V^r(v_1^r,.....v_1^r)^T$. Then the behaviour (forward and backward in time) of $V(x, t)$ locally around $(\bar{x}, \bar{t})$ is governed by the linearized hyperbolic problem

$$\frac{\partial V}{\partial t}+\frac{\partial}{\partial x}(\bar{\sigma} V)=0, \quad V(x,\bar{t})=\begin{cases} V^l \text{ for } x<\bar{x}; \\ V^r \text{ for } x>\bar{x}, \end{cases} \tag{5}$$

where $\bar{\sigma}$ denotes the Rankine-Hugoniot shock speed satisfying

$$F(V^l) - F(V^r) = \bar{\sigma}(V^l - V^r),$$

We *claim* that a large part of the splitting error that occurs locally around $(\bar{x}, \bar{t})$ in the standard OS algorithm can be understood in terms of the difference between the nonlinear system in Eq. (1) and the linearized system in Eq.(5) with right-hand side $D\frac{\partial^2 V}{\partial x^2}$. In the operator splitting (OS) strategy, the self-sharpening disappears once a shock develops because $F(U)$ is in effects replaced by $\bar{\sigma} U$ locally. Thus, one step in OS effectively amounts to solving $\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(\bar{\sigma} U) = D\frac{\partial^2 U}{\partial x^2}$ and not Eq.(1).

3.1 The Cos Strategy

To compensate for the loss of self-sharpening effects, the *scalar* COS approach proposes to include the residual flux F_{res} in the diffusion step of the splitting. The COS method therefore replaces the purely parabolic split problem by Eq.(3), we obtain

$$\frac{\partial W}{\partial t} + \frac{\partial F_{res}(x, W)}{\partial x} = D\frac{\partial^2 W}{\partial x^2}, \quad W(x,0) = W_0(x) \tag{6}$$

where F_{res} (x, W) represents residual flux in the point of x. Letting $\wp_t$ denotes the solution operator associated with Eq.(6), the COS solution is obtained

$$U_{\Delta t}(., n\Delta t) := [\wp_{\Delta t} \circ S_{\Delta t}]^n U_0(.) \tag{7}$$

In addition, we have replaced the convection-diffusion equation in Eq.(6), where the flux term in Eq.(6) is seemingly more complicated than the one in Eq.(1). However, we see that while F contains convective *and* self-sharpening effects, F_{res} only contributes to self-sharpening effects. Hence, viscous shock fronts are moved to the correct location in the convective step and given a correct shape in the diffusive step.

When applied to systems of parabolic equations, the correction algorithm needs to be reformulated, since one cannot simply write down the solution of hyperbolic step in terms of convex / concave envelopes. Instead, we identify the following term as

$$\frac{\partial}{\partial x} F_{res}(U) = \frac{\partial}{\partial x}\left(F(U) - \bar{\sigma} U\right) \tag{8}$$

for each discontinuity in the solution from the hyperbolic step. Then the parabolic subproblem Eq.(3) is modified locally by adding F_{res} (U), which gives the new split problem Eq.(6). Now by integrating Eq.(8) with respect to x we obtain the *residual* flux

$$F_{res}(U) = (F(U) - F(V')) - \bar{\sigma}(U - V') \tag{9}$$

Again we have chosen the constant of integration such that

$$F_{res}(V') = F_{res}(V^r) = 0$$

3.2 A COS Method

The OS methods introduced above result in two different sub-problems that each must be solved numerically. We have solved convection *part* by a front tracking method while the *diffusion* part is solved by an explicit central finite difference scheme.

3.2.1 Convection Solver

We have demonstrated front tracking method [7, 8, 9] for solving systems of conservation laws Eq.(2)

$$\frac{\partial V}{\partial t}+\frac{\partial F(V)}{\partial x}=0, \qquad V(x,0)=V_0(x)$$

The advantage of front tracking method is an algorithm for computing a piecewise constant approximation to $V(x, t)$, it directly identifies the correct physical envelop, given that the Riemann solver is correct. We will not study Riemann solver in the present paper but just finding good approximative Riemann solvers for systems in general a difficult problem.

First, V_0 is approximated by a step function so that a Riemann problem can be associated with each jump in the approximate initial data. The solution of each Riemann problem is approximated by step functions. Furthermore, in front tracking approximation, rarefaction waves are approximated by step functions sampled along the wave curves (according to a pre-set, parameter δ), while the rest of the Riemann solution is left intact. By doing this each Riemann problem produces a sequence of jump discontinues (fronts) that travel with a *finite* wave speed. On the other hand, the Riemann solution is represented by a list of fronts, sorted according to increasing wave speeds. Thus a global solution (in x) is formed by connecting the local Riemann solution – it consists constant states separated by a space-time rays *i.e.*, a list of fronts sorted from left to right. But, there will be first time at which two or more space-time rays intersect, i.e., two or more fronts collide. This collision describes a new Riemann problem which can be solved and inserted into the list. In addition, the algorithm proceeds in this manner from collision to collision. The numerical method is unconditionally stable and very first.

3.2.2 Diffusion Solver

The parabolic step in Cauchy problem of the form

$$\frac{\partial W}{\partial t}+\frac{\partial G(W)}{\partial x}=D\frac{\partial^2 W}{\partial x^2}, \quad W(x,0)=W_0(x) \qquad (10)$$

In application G denotes residual flux term in Eq.(8), D is still the diagonal matrix diag($\varepsilon_1,\ldots.\varepsilon_m$) > 0. To solve this system, one can instance use the explicit we have central finite difference method such that,

$$\frac{W_j^{n+1}-W_j^n}{\tau}-\frac{G(W_{j+1}^n)-G(W_{j-1}^n)}{2\Delta x}=D\frac{(W_{j+1}^n)-2(W_j^n)+(W_{j-1}^n)}{(\Delta x)^2} \qquad (11)$$

Now, this scheme is stable provided the discretization parameters τ and Δx satisfy the under mentioned conditions

$$\tau \le 0.5\Delta x^2/\max\varepsilon_i; \qquad \Delta x \max|\lambda_G| \le 2\max\varepsilon_i$$

Where λ_G represents the eigenvalues of G', the derivative of G (Strikwerda, J. (1989)). Hence for Cauchy problem Eq.(10), which has linear diffusion, convergence and error estimates for this is shown in (Hoff, D. Smoller, J. (1985)). The stability conditions above may severe restrictions on the discretizations parameters – especially on Δx for small values of ε.However, both these conditions can be weakened or removed by using a more sophisticated process. In concrete, to keep the technical details at a minimal level, we

chose the simple explicit scheme.

4. Construction Of The Residual Flux

In this section we will construct residual flux F_{res} $(x, \quad .)$ appearing in Eq.(6). Now we assume that the discontinuities of $U^{n+1/2}(x)$ are located at the point $\{x_i\}$. Hence given a piecewise constant front tracking solution, $U^{n+1/2}$ of the hyperbolic equation in Eq.(15). Again, we consider $U_i = \left(u_1^i \ldots\ldots u_m^i ..\right)^T$ and $U_{i+1} = \left(u_1^{i+1} \ldots\ldots u_m^{i+1} ..\right)^T$ denotes the values of $U^{n+1/2}(x)$ in the intervals $[x_{i-1}, \quad x_i]$ and $[x_i \ , \ x_{i+1}]$, respectively. Locally, around the ith discontinuity emerging from (x_i, t_o) the nonlinear problem Eq.(2) is governed by the linearized problem

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}(\sigma_i V) = 0 \,, V(x_i, t_o) = \{U_i \ for \quad x < x_i \,; \qquad U_{i+1} \ for \ x > x_i \tag{12}$$

As σ_i denotes the Rankine-Hugoniot shock speed satisfying $F(U_i) - F(U_{i+1}) = \sigma_i (U_i - U_{i+1})$ and we have residual flux F_{res}^i associated with the ith discontinuity as

$$F_{res}^i(U) = \{(F(U) - F(U_i)) - \sigma_i (U - U_i); U \in (u_1^i, u_1^{i+1}) \times \ldots (u_m^i, u_m^{i+1}) \tag{13}$$

Otherwise

hence $F_{res}^i(U_i) = F_{res}^i(U_{i+1}) \equiv 0$

Although a residual flux term can be identified for every discontinuity in the front tracking solution, it should not be included fro discontinuities which approximate rarefaction waves or for weak shocks. Hence, we only include residual terms for shock waves with strength exceeding a user-defined threshold parameter γ. Further, the process of identifying relevant residuals can be made more rigorous, and simplified, by tagging fronts in the front tracker according to wave type (shock/rarefaction/ contact) (Datta Gupta, A. King, M.. (1995; Hewett, T. A. Yamada, T. (1997; King, M. Datta Gupta, A., (1998); Chavent, G. & Jaffre, J. (1996); Chen, Z. Ewing, R. (1997)).

For explicit discretization we apply the residual fluxes in state space $(u_1, \ldots\ldots u_m)$. We demonstrate that in each special interval where the solution is monotone in all its components (monotonicity), all residual fluxes are explained on disjoints sets in state space. Therefore, the residual flux is set to zero outside (a subset of) the associated monotonicity interval,

$$F_{res}(x, U) = \sum_i F_{res}^i(U) \chi D_i(x) \tag{14}$$

$$S_t + \mathrm{v}.\nabla F(S) = 0 \tag{15}$$

Where $\chi_I(x)$ represents the indicator function of the interval $I \subset \mathbb{R}$ and D_i is the (subset of) monotonicity interval, S and $F(S)$ denotes vectors of saturations and fluxes respectively. We apply for implicit discretization a simpler approach where – it prescribes the length of the intervals where the correction is applied. However, specifying reasonable length for the correction intervals must be based on experience.

Results And Discussion

The following conclusions can be drawn from the above study

(i) we demonstrate numerically that operator splitting (OS) methods for systems of convection-diffusion equations in one-space dimensions.

(ii) it has a tendency to be too diffusive near viscous shock waves.

(iii) the scalar COS method is to use wave structure from the convection step to identify where the nonlinear splitting error (or *entropy* loss) occurs.

(iv) the potential error is compensated for in the diffusion step (or in a separate correction step).

(v) in case of *scalar* case, the splitting error is closely related to the local linearization introduced implicitly in the convection steps due to the use of an entropy condition.

(vi) a COS method has been proposed.

(vii) the numerical results demonstrate that the COS method is significantly more accurate than the corresponding OS method when the splitting *step* is large and the solution consists of (moving) viscous shock waves.

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Caption to the Figure 1. (a) A single shock solution from a convection step; (b) the corresponding residual flux function f (solid), convex envelop f_c i.e. local linearization (dash), and residual flux f_{res} (dash-dot).

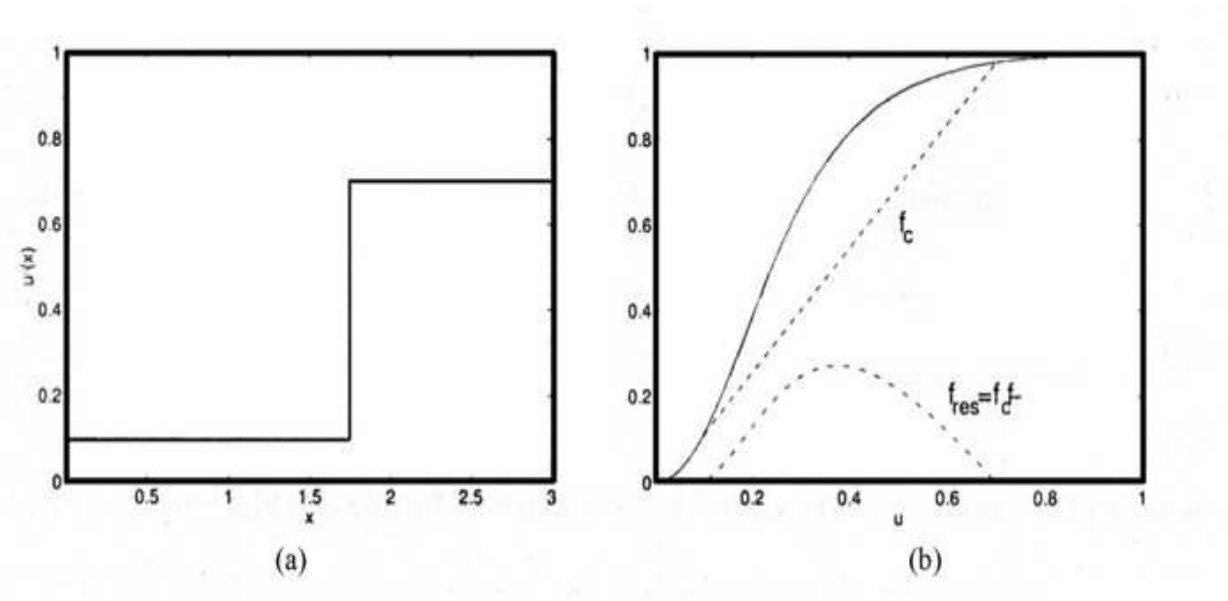

Fig. (1) (a) A single shock solution from a convection step. (b) The corresponding residual flux function; flux function f (solid), convex envelope , i.e., local linearization (dash), and residual flux f_{res} (dash-dot).