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On the Stability and Accuracy of Finite Difference Method for Options Pricing

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Abstract

This paper presents finite difference methods for options pricing. These methods are useful to solve partial differential equations and provide a general numerical solution to the valuation problems, as well as an optimal early exercise strategy and other physical sciences. The methods considered are the basic implicit and Crank Nicolson finite difference methods. The stability and accuracy of each of the methods were considered. Crank Nicolson method is more accurate and converges faster than implicit method.

Key words: Convergence, Crank Nicolson Method, European Option, Finite Difference Method, Implicit Method, Option, Stability.

1.0 Introduction

Options have been considered to be the most dynamic segments of the security markets since the inception of the Chicago Board Options Exchange (CBOE) in April 1973, with more than one million contracts per day, CBOE is the largest and business option exchange in the world. After that, several other option exchanges such as London International Financial Futures and Options Exchange (LIFFE) had been set up.

Black, Scholes and Merton approached the problem of pricing an option in a physicist's way by assuming a reasonable model for the price of a risky asset and since then option valuation problem has gained a lot of attention. In Black and Scholes (1973) [2] seminar paper titled ``The pricing of options and corporate liabilities", the assumption of log-normality was obtained and its application for valuing various range of financial instruments and derivatives is considered essential.

One of the major contributors to the world of finance was Black and Scholes [2]. They ushered in the modern era of derivative securities with a seminar paper titled ``Pricing and Hedging of European call and put options". In this paper, the famous Black-Scholes formula made its debut and the Ito calculus was applied to finance. Later Merton (1976) proposed a jump diffusion model. Boyle [3] introduced a Monte Carlo approach for pricing options. Twenty years later, Boyle, Brodie and Glasserman [4] describe research advances that had improved efficiency and broadened the types of problem where simulation can be applied. Brennan and Schwarz [5] considered finite difference methods for pricing American options for the Black-Scholes leading to one dimensional parabolic partial differential inequality. Cox, Ross and Rubenstein [6] derived the tree methods of pricing options based on risk-neutral valuation, the binomial option pricing European option prices under various alternatives, including the absolute diffusion, pre-jump and square root constant elasticity of variance methods [5] just to mention few. The complexity of option pricing formula and the demand of speed in financial trading market require fast ways to process these calculations; as a result, the development of computational methods for option pricing models can be the only solution.

In this paper, we shall consider only implicit and Crank Nicolson finite difference methods for pricing European options and the stability and accuracy of the methods.

2.0 Finite Difference Methods

Many option contract values can be obtained by solving partial differential equations with certain initial and boundary conditions. The finite difference approach is one of the premier mathematical tools employed to solve partial differential equations. These methods were pioneered for valuing derivative securities by Brennan and Schwarz [5]. The most common finite difference methods for solving the Black-Scholes partial differential equations are the

- Explicit Method.
- Implicit Method.
- Crank Nicolson method.

These schemes are closely related but differ in stability, accuracy and execution speed, but we shall only consider implicit and Crank Nicolson schemes. In the formulation of a partial differential equation problem, there are three components to be considered.

- The partial differential equation.
- The region of space time on which the partial differential is required to be satisfied.
- The ancillary boundary and initial conditions to be met.

2.1 Discretization of the Equation

The finite difference method consists of discretizing the partial differential equation and the boundary conditions using a forward or a backward difference approximation. The Black-Scholes partial differential equation is given by

$$f_t(t,S_t) + rS_t f_{S_t} + \frac{\sigma^2 S^2{}_t}{2} f_{S_t S_t} = rf(t,S_t) \quad (1)$$

We discretize (1) with respect to time and to the underlying price of the asset. Divide the (t,S_t) plane into a sufficiently dense grid or mesh and approximate the infinitesimal steps Δ_t and Δ_{S_t} by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0,\ldots,t_N$ to discretize the time derivative with $t_{n+1} - t_n = \frac{T}{N} = \Delta_t$. Using the same procedures, we obtain for the underlying price of the asset as follows:

$S_{M+1} - S_M = \frac{S_{\max}}{M} = \Delta_{S_t}$. This gives us a rectangular region on the (t,S_t) plane with sides $(0,S_{\max})$ and $(0,T)$. The grid coordinates (n,m) enables us to compute the solution at discrete points. We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as

$$f_{m,n} = f(n\Delta t, m\Delta S) = f(t_n, S_m) = f(t, S_t) \tag{2}$$

Where n and m are the numbers of discrete increments in the time to maturity and stock price respectively.

2.2 Finite Difference Approximations

In finite difference method, we replace the partial derivative occurring in the partial differential equation by approximations based on Taylor series expansions of function near the points of interest [9]. Expanding $f(t, \Delta S + S)$ and $f(t, S - \Delta S)$ in Taylor series we have the forward and backward difference respectively with $f(t,S)$ represented in the grid by $f_{n,m}$ [1]:

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m}}{\Delta S_t} \tag{3}$$

$$f_{S_t} \approx \frac{f_{n,m} - f_{n,m-1}}{\Delta S_t} \tag{4}$$

Also the first order partial derivative results in the central difference given by

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S_t} \tag{5}$$

And the second order partial derivative gives symmetric central difference approximation of the form

$$f_{S_t S_t} \approx \frac{f_{n,m+1} - 2f_{n,m} - f_{n,m-1}}{\Delta S_t^{\ 2}} \tag{6}$$

Similarly, we obtained forward difference approximation for the maturity time given by

$$f_t \approx \frac{f_{n+1,m} - f_{n,m}}{\Delta S_t} \tag{7}$$

Substituting equations (5), (6) and (7) into (1), we have

$$\rho_{1m} f_{n,m-1} + \rho_{2m} f_{n,m} + \rho_{3m} f_{n,m+1} = f_{n+1,m} \tag{8}$$

Where

$\rho_{1m} = \frac{1}{2} rm\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t$, $\rho_{2m} = 1 + r\Delta t + \sigma^2 m^2 \Delta t$, $\rho_{3m} = -\frac{1}{2} rm\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t$,

(8) is called a finite difference equation which gives equation that we use to approximate the solution of $f(t,S)$ [4].

Similarly, we obtained for the explicit, implicit and Crank Nicolson finite difference method as follows [7]:
Explicit case:

$$\frac{1}{1+r\delta_t}(\alpha_{1m}f_{n+1,m-1}+\alpha_{2m}f_{n+1,m}+\alpha_{3m}f_{n+1,m+1})=f_{n,m} \qquad (9)$$

Where $\alpha_{1m}=\frac{\sigma^2m^2\delta_t}{2}-\frac{rm\delta_t}{2},\alpha_{2m}=1-\sigma^2m^2\delta_t$ and $\alpha_{3m}=\frac{\sigma^2m^2\delta_t}{2}+\frac{rm\delta_t}{2}$. This method is accurate to $O(\delta_{t,}\delta^2s)$.

For implicit case we have,

$$\frac{1}{1-r\delta_t}(\beta_{1m}f_{n,m-1}+\beta_{2m}f_{n,m}+\beta_{3m}f_{n,m+1})=f_{n+1,m} \qquad (10)$$

Where the parameters in (10) are given by $\beta_{1m}=-\frac{\sigma^2m^2\delta_t}{2}+\frac{rm\delta_t}{2},\beta_{2m}=1+\sigma^2m^2\delta_t$ and $\beta_{3m}=-\frac{\sigma^2m^2\delta_t}{2}-\frac{rm\delta_t}{2}$. Similar to the explicit method, implicit method is accurate to $O(\delta_{t,}\delta^2s)$.

Crank Nicolson method is obtained by taking the average of the explicit and implicit methods in (9) and (10) respectively. Then we have

$$v_{1m}f_{n,m-1}+v_{2m}f_{n,m}+v_{3m}f_{n,m+1}=\varphi_{1m}f_{n+1,m+1}+\varphi_{2m}f_{n+1,m}+\varphi_{3m}f_{n+1,m+1} \qquad (11)$$

Then the parameters are given by

$$v_{1m}=rm\Delta t/4-\sigma^2\ m^2\Delta t/4,\ v_{2m}=1+r\Delta t/2+\sigma^2\ m^2\Delta t/2, v_{3m}=-rm\Delta t/4-\sigma^2\ m^2\Delta t/4$$

$$\varphi_{1m}=-rm\Delta t/4+\sigma^2\ m^2\Delta t/4, \varphi_{21m}=1-r\Delta t/2-\sigma^2\ m^2\Delta t/2,\ \varphi_{3m}=rm\Delta t/4+\sigma^2\ m^2\Delta t/4.$$

$n=0,1,2,...,N-1$ and $m=1,2,...,M-1$ [8].

2.3 Stability Analysis

The two fundamental sources of error are the truncation error in the stock price discretization and in the time discretization. The importance of truncation error is that the numerical scheme solves a problem that is not exactly the same as the problem we are trying to solve.

The three fundamental factors that characterize a numerical scheme are consistency, stability and convergence [8].

- Consistency: A finite difference of a partial differential equation is consistent, if the difference between partial differential equation and finite differential equation vanishes as the interval and time step size approach zero. Consistency deals with how well the finite difference equation approximates the partial differential equation and it is the necessary condition for convergence.

- Stability: For a stable numerical scheme, the errors from any source will not grow unboundedly with time.
- Convergence: It means that the solution to a finite difference equation approaches the true solution to the partial differential equation as both grid interval and time step sizes are reduced. The necessary and sufficient conditions for convergent are consistency and stability.

These three factors that characterize a numerical scheme are linked together by Lax equivalence theorem [9] which states that given a well posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence.

In general, a problem is said to be well posed if:

- A solution to the problem exists.
- The solution is unique when it exists.
- The solution depends continuously on the problem data.

2.3.1 A Necessary and Sufficient Condition for Stability

Let $f_{n+1} = Af_n$ be a system of equations, where A and f_{n+1} are matrix and column vectors respectively. Then

$$\begin{aligned} f_n &= Af_{n-1} \\ &= A^2 f_{n-2} \\ &\vdots \\ &= A^n f_0 \end{aligned} \tag{12}$$

For $n = 1,2,..., N$ and f_0 is the vector of initial value. We are concerned with stability and we also perturbed the vector of the initial value f_0 to t_0. The exact solution at the n^{th} row will then be

$$t_n = A^n t_0 \tag{13}$$

Let the perturbation or error vector e be denoted by

$$e = t - f$$

and using the perturbation vectors (12) and (13), we have

$$\begin{aligned} e_n &= t_n - f_n \\ &= A^n f_0 - A^n t_0 \end{aligned}$$

$$= A^n (t_0 - f_0)$$

Therefore,

$$e_n = A^n e_0 \quad (14)$$

Hence for compatible matrix and vector norms [9]

$$\| e_n \| \leq \| A^n \| \|e_0\|$$

Lax and Richmyer defined the difference scheme to be stable when there exists a positive number L which is independent of n, δ_t and δ_s, then $\|A\| \leq L$. This limits the amplification of any initial perturbation and therefore of any arbitrary initial rounding errors, since $\| e_n \| \leq L \|e_0\|$ and $\| A^n \| \leq \| A \|^n$, then the Lax-Richmyer definition of stability is satisfied when

$$\| A \| \leq 1 \quad (15)$$

Hence (15) is the necessary and sufficient condition for the finite difference equations to be stable [9]. Since the spectral radius $\rho(A)$ satisfies $\rho(A) \leq \| A \|$, it follows from (15) that $\rho(A) \leq 1$.

By Lax equivalence theorem, the three finite difference methods are consistent and convergent but in the analysis of their stability, explicit method is quite stable, while the implicit and Crank Nicolson methods are conditionally and unconditionally stable finite difference methods respectively because they calculate small change in the option value for a small change of the initial conditions, converge to the solution of the partial differential equation and calculation error decreases when number of time and price partitions increase.

3.0 Numerical Examples

Now, we present here some numerical examples. We consider the convergence of the fully implicit and the Crank Nicolson method with relation to the Black-Scholes value of the option.

We price the European call option on a non-dividend paying stock with the following parameters:

$$S = 50, K = 60, r = 0.05, \sigma = 0.2, T = 1$$

The Black-Scholes price for the call option is 1.6237.

The result obtained is shown in the Table 1 below and the illustrative result for the performance of the Implicit method and Crank Nicolson method when N and M are different is shown in Table 2 below.

4.0 Discussion of Results

Table 1 shows that the Crank Nicolson method in (11) converges faster than implicit method in (10) as $N \to \infty$, $\delta_t \to 0$ and $M \to \infty$, $\delta_s \to 0$. The multi-period binomial is closer to the solution for small values of N than the two finite difference methods.

Table 2 shows that when N and M are different, the finite difference methods converges faster than when N and

M the same. For the implicit and Crank Nicolson schemes, the number of time steps N initially set at 10 and doubled with each grid M refinement. The number of time steps in explicit case cannot be determined in this way. We conclude that the Crank Nicolson method has a higher accuracy than the implicit method and therefore it converges faster. The above results highlight that the two methods are stable.

5.0 Conclusion

Options come in many different flavours such as path dependent or non-path dependent, fixed exercise time or early exercise options and so on. Each of the available numerical methods is suited to dealing with only some of these option flavours. Finite difference methods are useful for pricing most especially vanilla options.

In general, finite difference methods require sophisticated algorithms for solving large sparse linear systems of equations but cannot be used in high dimensions. They are flexible in handling different processes for the underlying state variables and relatively difficult to code but these methods are somewhat problematic for path dependent options.

From Tables 1 and 2, we conclude that Crank Nicolson finite difference method is more stable, accurate and converges faster than its counterpart implicit method.

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Table of Results

Table 1: The comparison of the convergence of the Implicit method and the Crank Nicolson method as we increase M and N

$M = N$	Implicit Method	Crank Nicolson Method
10	1.3113	1.4782
20	1.4957	1.5739
30	1.5423	1.6010
40	1.5603	1.6110
50	1.5692	1.6156
60	1.5743	1.6181
70	1.5776	1.6196
80	1.5798	1.6205
90	1.5814	1.6212
100	1.5826	1.6216

Table 2: Illustrative results for the performance of the Implicit method and Crank Nicolson method M and N when are different.

M	N	Implicit Method	Crank Nicolson Method
10	20	1.4781	1.5731
20	40	1.5505	1.6108
30	60	1.5677	1.6180
40	80	1.5748	1.6205
50	100	1.5786	1.6216
60	120	1.5808	1.6222
70	140	1.5808	1.6225
80	160	1.5824	1.6227
90	180	1.5844	1.6229
100	200	1.5849	1.6230