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Coincidence Points for Mappings under Generalized Contraction

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Abstract

In this paper we establish some results on the existence of coincidence and fixed points for multi-valued and single valued mappings extending the result of *Feng and Liu [2]* and *Liu et.al [5]*. It is also proved with counter example that our results generalize and extend some well known results.

Key Words: common fixed point, Coincidence point, multi-valued mappings.

1. Introduction and Preliminaries

Generalizing Banach Contraction Principle, *Nadler [3]* introduced the concept of multivalued contraction mapping. Let *(X, d)* be a metric space. Following *Nadler [3]* and *Liu et.al [5]* we follow following notations throughout this paper.

CB(X) (resp.*CL(X)*) denote the family of all closed and bounded (resp. closed) subsets of *X*. *C(X)* represents set of all compact subsets of *X*. The Hausdorff distance for two subsets *A*, *B* of *X* is defined as:

 $H(A, B) = max (\{ sup d(a, B): a \in A \}, \{ sup d(A, b): b \in B \})$

where $d(a, B) = \inf \{ d(a, b) : b \in B \}$ *,*

It is well known that *CB(X)* (resp*. CL(X)*) is a metric space with Hausdorff distance function.

Let *T:X →CL(X).* Using the concept of Hausdorff distance, *Nadler [3]* defined multivalued contraction as following,

 $H(Tx, Ty) \leq \alpha d(x, y)$ $\forall x, y \in X$ and $\alpha < 1$.

Nadler proved that for a multivalued contraction in a complete metric space there exists a fixed point. Recently *Feng and Liu [2]* and *Liu et.al [5]* generalized the Nadler's result. *Feng and Liu [2]* gave an example to establish that if the mapping *T* does not satisfy the above contractive condition even then it has a fixed point. *Feng and Liu [2]* generalized the above contractive condition by considering the point $y \in T(x)$ for any $x \in X$ in place of *x, y ∈ X* and proved the following result.

Theorem 1 [4]. Let (X, d) be a complete metric space, and let *T* be a multivalued mapping from *X* to $CL(X)$. If there exist constant *b, c* \in (0, 1), *c* < *b*, such that for any *x* \in *X* there is $y \in T(x)$ satisfying,

$$
bd(x, y) \le f(x), \quad f(y) \le cd(x, y) \tag{1.1}
$$

then *T* has a fixed point in *X* provided the function $f(x) = d(x, T(x))$, $x \in X$ is lower semi continuous.

Generalizing above result and the result of Ciric [1], *Liu et.al [5]* relax the contractive condition by taking $\alpha(f(x))$ and $\mathbf{B}(\mathbf{f}(\mathbf{x}))$ in place of constant *b* and *c*, where

$$
\boldsymbol{\alpha}: B \to (0,1], \ \boldsymbol{\beta}: B \to [0,1) \text{ and } B = \begin{cases} [0, \text{supf}(\boldsymbol{x})], & \text{if } \text{sup } f(\boldsymbol{x}) < +\infty \\ [0, +\infty), & \text{if } \text{supp}(\boldsymbol{x}) = +\infty \end{cases} \tag{A}
$$

In this paper we extend the result of [2] and [5] for the existence of coincidence points.

2. **Main Result**

Let (X, d) be a metric space, *T:* $X \rightarrow CL(X)$ and *f:* $X \rightarrow X$. An orbit of the multivalued map *T* at a point x_0 in *X* is a

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sequence $\{x_n, x_n \in Tx_{n-1}, n =1, 2, 3, ...\}$. The space X is T –orbitally complete if every Cauchy sequence of the from $\{x_n, x_n \in Tx_{n-1}\}$ converges in *X*. If for a point x_0 in *X*, there exists a sequence $\{x_n\} \subset X$ such that $f(x_{n+1}) \in Tx_n$ $n = 0, 1, 2, \ldots$, then $O_f(x_0) = \{fx_n : n = 1, 2, \ldots\}$ is an orbit of *(T, f)* at x_0 . A space *X* is *(T, f)* –orbitally complete if every Cauchy sequence of the form $\{fx_{n_i}: fx_{n_i} \in Tx_{n_i-1}\}$ converges in *X*. A function $\mathcal{D}: X \to R$ such that $\mathcal{D}(x) = d$ (*fx, Tx*) is called *(T, f)* -orbitally lower semi continuous if for any point $z \in X \exists$ an orbit $\{f(x_n)\}\$ of *(T, f)* with $\lim f x_n = f z$ implying that $\mathcal{O}(z) \leq \lim_{n \to \infty} \mathcal{O}(x_n)$.

Theorem 2.1

Let (X, d) be a metric space. *T:* $X \rightarrow CL(X)$ and $f: X \rightarrow X$ such that $T(X) \subseteq f(X)$ and $f(X)$ is (T, f) -orbitally complete. If for any $x \in X$ there exists $y \in X$ such that $f(y) \in T(x)$ and

$$
\alpha(\mathbf{D}(\mathbf{x}))d(fx, fy) \le \mathbf{D}(\mathbf{x}) \text{ and } \mathbf{D}(\mathbf{y}) \le \mathbf{B}((\mathbf{D}(\mathbf{x}))d(fx, fy))
$$

 where and are defined as (A) *satisfying*

$$
\lim_{r \to 0^+} \alpha(r) > 0, \lim_{r \to 0^+} \frac{\sup \beta(r)}{\alpha(r)} < 1 \ \ \forall \ t \in [0, \sup \emptyset(x)), \tag{2.1}
$$

and the function \boldsymbol{a} is (T, f) -orbitally lower semi continuous at *z*. Then there exist a coincidence point *z* of *f* and *T*.

Proof: Let
$$
\mathbf{y}(t) = \frac{\mathbf{B}(t)}{\alpha(t)}, \forall t \in [0, \text{supp}(\mathbf{x}))
$$
 (2.2)

Let $x_0 \in X$, since $T(x) \subseteq f(X)$ we choose $x_1 \in X$ so that $fx_1 \in Tx_0$ and $\alpha(\mathbf{C}(x_0)) d(fx_0, fx_1) \leq \mathbf{C}(x_0) = d(fx_0, Tx_0)$, $\mathbf{Z}(x_1) = d(fx_1, Tx_1) \leq \mathbf{B}$ ($\mathbf{Z}(x_0)$) $d(fx_0, fx_1)$ *implies*

$$
\Phi(x_1) \leq \beta \left(\Phi(x_0) \right) \frac{\Phi(x_0)}{\alpha(\Phi(x_0))} .
$$

Using (2.2) we get

$$
\boldsymbol{\varnothing}(x_1) \leq \boldsymbol{\beta} \; (\boldsymbol{\varnothing}(x_0)) \; \frac{\boldsymbol{\varnothing}(\boldsymbol{x}_0)}{\boldsymbol{\alpha}(\boldsymbol{\varnothing}(x_0))} = \boldsymbol{\gamma} \; (\boldsymbol{\varnothing}(x_0)) \; (\boldsymbol{\varnothing}(x_0))
$$

continuing the process we get an orbit $\{fx_n\}_{n\geq 0}$ of *T* satisfying

 $\alpha(\mathbf{D}(x_n)) d(fx_n, fx_{n+1}) \leq \mathbf{D}(x_n) = d(fx_n, Tx_n)$ and $\mathbf{B}(x_{n+1}) = d(fx_{n+1}, Tx_{n+1}) \leq \mathbf{B}(\mathbf{B}(x_n)) d(fx_n, fx_{n+1}), \quad \forall n \geq 0.$ (2.3) Using (2.2) we get

$$
\mathbf{Q}(x_{n+1}) \leq \mathbf{B} \left(\mathbf{Q}(x_n) \right) \frac{\mathbf{Q}(x_n)}{\mathbf{a}(\mathbf{Q}(x_n))} = \mathbf{\gamma} \left(\mathbf{Q}(x_n) \right) \left(\mathbf{Q}(x_n) \right). \tag{2.4}
$$

Since $0 \le \gamma$ *(t) < 1 and by (2.4) it is clear that* $\langle \mathbf{a}_{x_n} \rangle_{n \ge 0}$ *is a nonnegative and decreasing sequence. Hence* $\mathbf{a}(x_n)$ *is* convergent.

Let
$$
\lim_{n \to \infty} \phi(x_n) = a
$$
 (2.5)

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where $a \ge 0$, suppose $a > 0$, taking limit $n \rightarrow \infty$ in (2.4) and by (2.1), (2.2) and (2.5)

 $a = lim_{n\to\infty} supp\emptyset(x_{n+1}) \leq lim_{n\to\infty}[\gamma(\emptyset(x_n))\emptyset(x_n)]$

 $\leq \lim_{n\to\infty} \sup \gamma(\phi(x_n) \lim_{n\to\infty} \sup \phi(x_n))$

$=$ a $\lim_{n\to\infty} \sup \left(\emptyset(x_n) \right)$ < a

Which is a contradiction hence *a = 0*

i.e.,
$$
\lim_{n \to \infty} \mathcal{C}(x_n) = 0. \tag{2.6}
$$

To prove that $\{fx_n\}$, $n \ge 0$ is a Cauchy sequence.

Let
$$
b = \lim_{n \to \infty} \sup \gamma(\mathcal{D}(x_n), c = \lim_{n \to \infty} \inf \alpha(\mathcal{D}(x_n))
$$
 (2.7)

Then from (2.1), (2.2) and (2.7) $0 \leq b \leq 1, c > 0.$ Let $p \in (0, c), q \in (b, 1)$ then from (2.7)

$\gamma(\emptyset(x_n)) \leq q$, $\alpha(\emptyset(x_n)) \geq p$, $\forall n \geq 0$

which together with (2.3) and (2.4) gives

$$
\mathcal{O}(x_{n+1}) \leq q \mathcal{O}(x_n), \ d(fx_n, fx_{n+1}) \leq \frac{\mathcal{O}(x_n)}{p}
$$

Calculating similar calculation we get

$$
\varphi(x_{n+1}) \leq q^{n+1-n_0} f x_{n_0}, \quad d(f x_n, f x_{n+1}) \leq \frac{\varphi(x_n)}{n} q^{n-n_0}.
$$

which gives

$$
d(fx_n, fx_m) \leq \sum_{k=1}^{n-1} d(fx_k, fx_{k+1}) \leq \frac{\phi(x_{n_0})}{p} \sum_{k=n}^{n-1} q^{k-n_0} \leq \frac{\phi(x_{n_0})}{p(1-q)} q^{n+1-n_0}
$$
\n(2.8)

Since $q < 1$ therefore (2.8) implies that ${f(x_n)}$ is a Cauchy sequence. And since $f(X)$ is (T, f) -orbitally complete

 $\exists z \in X$ such that

$\lim_{n\to\infty} f(x_n) = f$.

Now we will prove that z is coincidence point of *f* and *T*. Since Φ is (*T, f*) -orbitally lower semi continuous therefore $0 \leq d(fz, Tz) = \mathbf{0}(\mathbf{z}) \leq \lim_{n \to \infty} \mathbf{0}(\mathbf{x}_n) = 0$ (by 2.6) $\Rightarrow \varnothing(z) = 0,$

 $f(z) \in T(z)$ i.e. *f* and *T* have a coincidence point.

In Theorem 2.1, taking constants *α* and **β** in place of $\alpha(\mathcal{O}(\bm{x}))$ and $\mathcal{B}((\mathcal{O}(\bm{x}))$ respectively we get following result as a corollary. The following corollary is also serves as a generalization of Singh and Kulsrestha[4].

Corollary 2.1.

Let *(X, d)* be a metric space. *T: X* \rightarrow CL(*X)* and *f: X* \rightarrow *X* are mappings such that $T(X) \subseteq f(X)$ and $f(X)$ is *(T, f)* orbitally complete. If for any $x \in X$ there is $y \in X$ such that $f(y) \in T(x)$ satisfying

$Bd(fx, fy) \leq Q(x)$ And $C(y) \leq ad(fx, fy)$

where α , $\beta \in (0, 1)$ and $\alpha < \beta$ and the function \emptyset is is lower semi continuous. Then *T* and *f* has a coincidence point in *X*.

Proof

Let $x_0 \in X$, since $T(x) \subseteq f(X)$ we choose $x_1 \in X$ such that $fx_1 \in Tx_0$.

By the given contractive condition

<i><i>d (*fx*₀*, fx*₁) \leq *O*(*x*₀) = *d*(*fx*₀*Tx*₀) and $\Phi(x_i) = d(fx_i, Tx_i) \leq \alpha d(fx_0, fx_i).$ In similar way we choose $x_{n+1} \in X$ such that $f(x_{n+1}) \in T(x_n)$ and $d(fx_{n+1}, Tx_{n+1}) \leq \alpha \ d(fx_n, fx_{n+1}), \ d(fx_n, fx_{n+1}) \leq \Phi(x_n) = d(fx_n, Tx_n)$ which implies

$$
d(fx_{n+1}, Tx_{n+1}) \leq \frac{d}{\beta} d(fx_n, Tx_n)
$$

or
$$
d(fx_{n+1}, fx_{n+2}) \leq \frac{a}{\beta} d(fx_{n}, fx_{n+1})
$$

$$
\Longrightarrow d(fx_n, Tx_n) \leq (\frac{a}{B})^{\pi} d(fx_0, Tx_0)
$$

$$
\text{Or } d(fx_n, f x_{n+1}) \leq \left(\frac{a}{a}\right)^n d(fx_0, f x_1). \tag{2.9}
$$

Using (2.9) for *m*, $n \in N$, $m>n$,

$$
d(fx_m, fx_n) \leq d(fx_m, fx_{m-1}) + d(fx_{m-1}, fx_{m-2}) + ... + d(fx_{n+1}, fx_n)
$$

$$
\leq \left(\frac{a}{\beta}\right)^{m-1} d(fx_0, fx_l) + \left(\frac{a}{\beta}\right)^{m-2} d(fx_0, fx_l) + \left(\frac{a}{\beta}\right)^{m-2} d(fx_0, fx_l) + ... + \left(\frac{a}{\beta}\right)^{m} d(fx_0, fx_l)
$$

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 $\leq \frac{\sigma_p^2}{1-\sigma_p^2} d(fx_0, fx_i)$ (2.10)

As
$$
n \to \infty
$$
, $\left(\frac{a}{\beta}\right)^n \to 0$

hence ${f}f{x_n}$ is a Cauchy sequence. Since $f(X)$ is (T, f) -orbitally complete there exists $z \in X$ such that

$(fx_n)_{n=0}^{\infty}$ converges to *fz*.

Now using the condition of (T, f) –orbitally lower semi continuity of ϕ we can be easily prove that *z* is coincidence point of *f* and *T.*

In theorem 2.1 taking $C(X)$ in place of $CL(X)$ we get we get following result as corollary.

Corollary 2.2

Let (X, d) be a metric space. *T: X →C(X)* and *f: X →X* such that *T* and *f* satisfy all conditions as in theorem 2.1 then there exist a coincidence point *z* of *f* and *T*.

Proof.

Proof is same as of theorem 2.1**.**

In corollary 2.1 taking *C(X)* in place of *CL(X)* we get we get following result as corollary.

Corollary 2.3

Let (X, d) be a metric space. *T:* $X \rightarrow C(X)$ and *f:* $X \rightarrow X$ are mappings such that all conditions as in corollary 2.1 are satisfied then *T* and *f* has a coincidence point in *X*.

Proof

Proof is same as of corollary 2.1**.**

Example

Let $X = \{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \cup \{0,1\}$ $2^{\frac{1}{4}}$ $2^{\frac{1}{2}}$ $\left[\frac{1}{n}, \ldots\right] \cup \{0, 1\}, d(x, y) = \left| x - y \right|$, for *x*, $y \in X$; then *X* is complete metric space. Define mapping *T*:

X→ CL(X) as

$$
T(x) = \left\{ \left\{ \frac{1}{2^{2n+2}}, 1 \right\}, x = \frac{1}{2^n}, n = 0, 1, 2, \ldots \right\}
$$

$$
\left\{ 0, \frac{1}{2} \right\}, x = 0
$$

and $f(x) = x^2$, $x \in X$.

Obviously, *T* and *f* does not satisfy hybrid contraction condition [4].

$$
H\left(T\left(\frac{1}{2^{2n}}\right), T(0)\right) = \frac{1}{2} \ge \frac{1}{2^{2n}} = \left|\frac{1}{2^{2n}} - 0\right| = d\left(\frac{1}{2^{2n}}, 0\right), n = 1, 2...
$$

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On the other hand, we have

$$
\varnothing(x) = d(fx, T(x)) = \begin{cases} \frac{1}{2^{2n+2}}, x = \frac{1}{2^n}, n = 1, 2... \\ 0, x = 0, 1 \end{cases}
$$

It shows that \emptyset is continuous,

Further, there exists $y \in X$ for any $x \in X$ such that

$$
\frac{1}{2}d(fx,fy)\leq \varnothing(x)
$$

$$
d(fy, Ty) \leq \frac{1}{2^{\frac{d}{2}}} d(fx, fy).
$$

Then from corollary *2*.1 there exist a coincidence point of *f* and *T*.

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