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Homotopy Perturbation and Elzaki Transform for Solving Nonlinear Partial Differential Equations

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Abstract

In this work, we present a reliable combination of homotopy perturbation method and Elzaki transform to investigate some nonlinear partial differential equations. The nonlinear terms can be handled by the use of homotopy perturbation method. The proposed homotopy perturbation method is applied to the reformulated first and second order initial value problem which leads the solution in terms of transformed variables, and the series solution is obtained by making use of the inverse transformation. The results show the efficiency of this method.

Keywords: Homotopy perturbation methods, Elzaki transform nonlinear partial differential equations.

1. Introduction

Linear and nonlinear partial differential equations are of fundamental importance in science and engineering. Some integral transform method such as Laplace and Fourier and Sumudu transforms methods see [Kilicman and E.Gadain. (2009), (2010) lslam, Yasir Khan, Naeem Faraz and Francis Austin (2010)], are used to solve linear partial differential equations and use fullness of these integral transform lies in their ability to transform differential equations into algebraic equations which allows simple and systematic solution procedures.

However, using integral transform in nonlinear differential equations may increase its complexity. In recent years, many research workers have paid attention to find the solutions of nonlinear differential equations by using various methods. Among these are the Adomian decomposition method [Hashim, Noorani, Ahmed. Bakar. Ismail and Zakaria, (2006)], the tanh method, the homotopy perturbation method [Sweilam, Khader (2009), Sharma and Giriraj Methi (2011), Jafari, Aminataei (2010), (2011)], the differential transform method [(2008)], and the variational iteration method.

Elzaki transform [Tarig and Salih, (2011), (2012)] is totally incapable of handling the nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities, one of these combinations of homotopy perturbation method and Elzaki transform which is studies in this paper.

The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear partial differential equations. This article considers the effectiveness of the homotopy perturbation Elzaki transform method in solving nonlinear partial differential equations both homogeneous and non-homogeneous.

1.1. Elzaki Transform

The basic definitions of modified of Sumudu transform or Elzaki transform is defined as follows, Elzaki transform of the function $f(t)$ is

$$E[f(t)] = v\int_0^\infty f(t)\, e^{-\frac{t}{v}} dt, \qquad t > 0 \tag{1}$$

Tarig M. Elzaki and Sailh M. Elzaki in [(2011), (2012)], showed the modified of Sumudu transform [(2007), (2010)] or Elzaki transform was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations. Elzaki transform is a powerful tool for solving some differential equations which can not solve by Sumudu transform in [(2012)]. In this paper, we combined Elzaki transform and homotopy perturbation to solve nonlinear partial differential equations.

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - vf(x,0) \qquad E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}$$

$$E\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}[T(x,v)] \qquad E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$$

Proof:

To obtain ELzaki transform of partial derivatives we use integration by parts as follows:

$$\mathrm{E}\left[\frac{\partial f}{\partial t}(x,t)\right] = \int_0^\infty v\frac{\partial f}{\partial t}e^{\frac{-t}{v}}dt = \lim_{p\to\infty}\int_0^p ve^{\frac{-t}{v}}\frac{\partial f}{\partial t}dt = \lim_{p\to\infty}\left\{\left[ve^{\frac{-t}{v}}f(x,t)\right]_0^p - \int_0^p e^{\frac{-t}{v}}f(x,t)dt\right\}$$

$$= \frac{T(x,v)}{v} - vf(x,0)$$

We assume that f is piecewise continuous and it is of exponential order.

Now $\mathrm{E}\left[\frac{\partial f}{\partial x}\right] = \int_0^\infty ve^{\frac{-t}{v}}\frac{\partial f(x,t)}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty ve^{\frac{-t}{v}}f(x,t)dt$, using the Leibnitz rule to find:

$$\mathrm{E}\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}[T(x,v)]$$

By the same method we find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$

To find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$

Let $\frac{\partial f}{\partial t} = g$, *then* we have:

$$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] = \mathrm{E}\left[\frac{\partial g(x,t)}{\partial t}\right] = \mathrm{E}\frac{[g(x,t)]}{v} - vg(x,0)$$

$$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f}{\partial t}(x,0)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

1.2. Homotopy Perturbation Method:

Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y , it is said that f is homotopic to g , if there is continuous map $F : X \times [0,1] \rightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for each $x \in X$, then the map is called homotopy between f and g .

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0 \quad (2)$$

Where L is any differential operator, we define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u) \quad (3)$$

Where $F(u)$ is a functional operator with known solution v_0 which can be obtained easily. It is clear that, for

$$H(u,p) = 0 \quad (4)$$

We have: $H(u,0) = F(u)$, $H(u,1) = L(u)$

In topology this show that $H(u,p)$ continuously traces an implicitly defined carves from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The HPM uses the embed ling parameter p as a small parameter and write the solution as a power series

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + ...$$

(5)

If $p \to 1$, then (5) corresponds to (3) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i \tag{6}$$

We assume that (6) has a unique solution. The comparisons of like powers of p give solutions of various orders.

2. Homotopy Perturbation Elzaki Transform Method:

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{7}$$

$$u(x,0) = h(x) \quad , \quad u_t(x,0) = f(x)$$

Where D is linear differential operator of order two, R is linear differential operator of less order than D, N is the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking Elzaki transform on both sides of equation (7), to get:

$$E[Du(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[g(x,t)] \tag{8}$$

Using the differentiation property of Elzaki transforms and above initial conditions, we have:

$$E[u(x,t)] = v^2E[g(x,t)] + v^2h(x) + v^3f(x) - v^2E[Ru(x,t) + Nu(x,t)] \tag{9}$$

Appling the inverse Elzaki transform on both sides of equation (9), to find:

$$u(x,t) = G(x,t) - E^{-1}\left\{v^2E\left[Ru(x,t) + Nu(x,t)\right]\right\} \tag{10}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the homotopy perturbation method.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{11}$$

And the nonlinear term can be decomposed as

$$N[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n(u) \tag{12}$$

Where $H_n(u)$ are given by:

$$H_n(u_0,u_1,....u_n)=\frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0}, \quad n=0,1,2,......$$

(13)

Substituting equations (11) and (12) in equation (10), we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left\{E^{-1}\left[v^2E\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right\} \tag{14}$$

This is the coupling of the Elzaki transform and the homotopy perturbation method.

Comparing the coefficient of like powers of p , the following approximations are obtained.

$$p^0: u_0(x,t)=G(x,t)$$
$$p^1: u_1(x,t)=-E^{-1}\left\{v^2E\left[Ru_0(x,t)+H_0(u)\right]\right\}$$
$$p^2: u_2(x,t)=-E^{-1}\left\{v^2E\left[Ru_1(x,t)+H_1(u)\right]\right\}$$
$$p^3: u_3(x,t)=-E^{-1}\left\{v^2E\left[Ru_2(x,t)+H_2(u)\right]\right\} \tag{15}$$

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Then the solution is $u(x,t)=u_0(x,t),+u_1(x,t)+u_2(x,t)+....$

3. Applications:

In this section we apply the homotopy perturbation Elzaki transform method for solving nonlinear partial differential equations.

Example 3.1:

Consider the following homogenous nonlinear partial differential equations

$$u_t+uu_x=0 \quad , \quad u(x,0)=-x \tag{16}$$

Taking Elzaki transform of equation (16) subject to the initial condition, we have:

$$E\left[u(x,t)\right]=xv^2-vE\left[uu_x\right] \tag{17}$$

The inverse Elzaki transform implies that:

$$u(x,t)=-x-E^{-1}\left\{vE\left[uu_x\right]\right\} \tag{18}$$

Now applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=-x-p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right\} \tag{19}$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms.

Or

$$p\left[uu_x\right]=0 \quad \Rightarrow p(u_0+pu_1+p^2u_2+....)(u_{0x}+pu_{1x}+p^2u_{2x}+....)$$
$$u=u_0+pu_1+p^2u_2+....$$

The first few components of He's polynomials, are given by

$$H_0(u)=u_0u_{0x}$$
$$H_1(u)=u_0u_{1x}+u_1u_{0x}$$
$$H_2(u)=u_0u_{2x}+u_1u_{1x}+u_2u_{0x}$$
$$\vdots$$

Comparing the coefficients of the same powers of p , we get:

$$p^0: u_0(x,t)=-x \quad , \quad H_0(u)=x$$
$$p^1: u_1(x,t)=-E^{-1}\left\{vE\left[H_0(u)\right]\right\}=-xt \quad , \quad H_1(u)=2xt$$
$$p^2: u_2(x,t)=-E^{-1}\left\{vE\left[H_1(u)\right]\right\}=-xt^2 \, , \; H_2(u)=3xt^2$$
$$p^3: u_3(x,t)=-E^{-1}\left\{vE\left[H_2(u)\right]\right\}=-xt^3$$
$$\vdots$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t)=-x(1+t+t^2+t^3+.....)=\frac{x}{t-1}$$

Example 3.2:

Consider the first order nonlinear partial differential equation

$$u_t+uu_x=2t+x+t^3+xt^2 \quad , \quad u(x,0)=-x \tag{20}$$

To find the solution by homotopy perturbation Elzaki transform method, we applying homotopy perturbation method after taking Elzaki and inverse Elzaki transforms of equation (20), we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3}-p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right\} \tag{21}$$

Where

$$H_0(u)=u_0u_{0x}$$
$$H_1(u)=u_0u_{1x}+u_1u_{0x}$$
$$H_2(u)=u_0u_{2x}+u_1u_{1x}+u_2u_{0x}$$
$$\vdots$$

Comparing the coefficients of like powers of p , we have:

$$p^0: u_0(x,t)=t^2+xt+\frac{xt^3}{3}+\frac{t^4}{4}$$
$$p^1: u_1(x,t)=-\frac{t^4}{4}-\frac{xt^3}{3}-\frac{2xt^5}{15}-\frac{7t^6}{72}-\frac{xt^7}{63}-\frac{t^8}{98}$$
$$\vdots$$

The noise terms appear between the components $u_0(x,t), u_1(x,t)$, therefore, the exact solution is given by: $u(x,t)=t^2+xt$

Example 3.3:

Let us consider the second order nonlinear partial differential equation

$$\frac{\partial u}{\partial t}=\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2} \quad , \quad u(x,0)=x^2 \qquad (22)$$

Applying Elzaki transform of equation (22), and making use of the initial condition, to find:

$$E\left[u(x,t)\right]=x^2v^2+vE\left[u_x^2+uu_{xx}\right] \qquad (23)$$

Take the inverse Elzaki transform of equation (23), we get:

$$u(x,t)=x^2+E^{-1}\left\{vE\left[u_x^2+uu_{xx}\right]\right\} \qquad (24)$$

Apply the homotopy perturbation method of (24), to get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=x^2+p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^nH_n(u)\right]\right]\right\} \qquad (25)$$

Or $$p\left[u_x^2+uu_{xx}\right]=0 \quad , \quad u=u_0+pu_1+p^2u_2+\ldots. \qquad (26)$$

Equation (26), can be written in the form;

$$p(u_{0x}+pu_{1x}+p^2u_{2x}+\ldots.)^2+p(u_0+pu_1+p^2u_2+\ldots.)(u_{0xx}+pu_{1xx}+p^2u_{2xx}+\ldots.)=0$$

The first few components of He's polynomials are given by:

$$H_0(u) = u_{0x}^2 + u_0 u_{0xx}$$
$$H_1(u) = 2u_{0x}u_{1x} + u_0 u_{1xx} + u_1 u_{0xx}$$
$$\vdots$$

Therefore

$$p^0 : u_0(x,t) = x^2 \quad , \quad H_0(u) = 6x^2$$
$$p^1 : u_1(x,t) = E^{-1}\{vE[H_0(u)]\} = 6x^2 t \quad , \quad H_1(u) = 72x^2 t$$
$$p^2 : u_2(x,t) = E^{-1}\{vE[H_1(u)]\} = 36x^2 t^2$$
$$\vdots$$

Then the solution of equation (22) is given by: $u(x,t) = x^2(1+6t+36t^2+.....) = \dfrac{x^2}{1-6t}$

4. Conclusion

In this paper, we mixture Elzaki transform and homotopy perturbation method to solve nonlinear partial differential equations. The solution by using Adomian decomposition method is simple, but the calculation of Adomian polynomials is complex. The fact that the developed algorithm solves nonlinear partial differential equations without Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

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Table 1. Elzaki transform of some Functions

$f(t)$	$E[f(t)] = T(u)$
1	v^2
t	v^3
t^n	$n!\,v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sinat$	$\frac{av^3}{1+a^2v^2}$
$cosat$	$\frac{v^2}{1+a^2v^2}$

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