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An Order-seven Implicit Symmetric Scheme Applied to Second Order Initial Value Problems of Differential Equations

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Abstract

In this paper, a five-step predictor-corrector method of algebraic order seven is presented for solving second order initial value problems of ordinary differential equations directly without reduction to first order systems. Analysis of the basic properties of the method is considered and found to be consistent, zero-stable and symmetric. Some sample linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution well when compared with the two existing schemes that solved the same set of problems.

Keywords: zero-stability, convergence, consistent, predictor-corrector, error constant, symmetric

1. Introduction

Much research has been done in the last decade to show interest in the approximate solution of ordinary differential equation of the form

$$y^{(m)} = f(t, y, y', y'',..., y^{(m-1)}), y^{(m-1)}(t_0) = \eta_{m-1}, m = 1,2,..., f(t, y), f : \Re\times\Re \to \Re, \quad (1)$$

the differential equations for which the function **f** is independent from the derivatives of **y.** It is assumed that the solution of (1) exist and unique.

The symmetric multistep methods were first proposed in the joint effort of Lambert & Watson (1976) to solve problem of equation (1) by reduction to the form

$$y' = f(t, y), \ y(t_0) = \eta_0, t \in [a,b], \quad (2)$$

it was discovered in their letter that the interval of periodicity of the symmetric multistep methods is non-vanishing to ascertain the existence of periodic solutions. Based on the approach of Lambert, Quinlan & Tremaine (1990) developed high order symmetric methods whose solution exhibits a pronounced oscillatory character, this type of ordinary differential equation problems often arise in different fields of applied sciences such as astrophysics, electronics, celestial mechanics, molecular dynamics, radio-active and transverse motion to mention a few. One way to obtain a more efficient integration process is to construct numerical methods with an increased algebraic order, although the implementation of high algebraic order method meets several challenges but its accuracy is quite enormous.

The empirical problems leading to second order differential equations is obtained by letting m=2 in equation (1) to have

$$y'' = f(t, y, y') \quad y(t_0) = \eta_0, \quad y'(t_0) = \eta_1, \ t \in [a,b]. \quad (3)$$

Numerical methods adopted for such higher order differential equations in literature are only capable of handling first order equations of the type (2), see for example (Abhulimen & Otunta 2006, Chan & Tsai 2004, Juan 2001, Fatunla 1988). The approach of reducing such equation to a system of first order equations leads to serious computational burden and computer time wastage (Awoyemi 2001, 2005).

Many attempts have been made to formulate numerical algorithms capable of solving special type of equation (1) without reduction to first order systems (Bun & Vasil'Yel 1992, Jacques & Judd 1987), to avoid these shortcomings, researchers (Badmus & Yahaya 2009, Awoyemi & Kayode 2005 and the references therein) were provoked to solve second order equation (1) directly, in their approach, sufficient attentions were not paid to the property of zero stability (Kayode & Awoyemi 2005, Aruchunan & Sulaiman 2010, Parand & Hojjati 2008) an essential ingredient to guarantee convergence. In this paper, we shall develop a five-step implicit formula of order-seven, numerical results are given from the application of the new method to some linear and nonlinear problems.

2. Derivation of the Method

We consider second derivative implicit methods of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} - h^2\sum_{j=0}^{k}\beta_j f_{n+j} = 0, \qquad (4)$$

operationally defines as

$$\rho(E)y_n = h^2\delta(E)f_n \qquad (5)$$

by Fatunla (1988), where $\rho(E), \delta(E)$, are the first and second characteristic polynomial of equation (4), α_j and β_j are real constants with constraints

$$\alpha_k \neq 0, |\alpha_0| + |\beta_0| \neq 0 \qquad (6)$$

The values of the coefficients are determined by the local truncation error (lte).

Definition 1: *The truncation error is the quantity T which must be added to the true value representation of the computed quantity in order that the result be exactly equal to the quantity we are seeking to generate.*

Y(true representation) + T = Y(exact)

Our local truncation error in (4) is defined as

$$T_{n+k} = y_{n+k} - \sum_{j=0}^{k-1}\alpha_j y_{n+j} + h^2\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (7)$$

Taylors series expansions of y_{n+k}, y_{n+j} and f_{n+j} about the point (t_n, y_n) with the terms collected in powers of **h** gives

$$T_{n+k} = (1 - \sum_{j=0}^{k-1}\alpha_j)y_n + (k - \sum_{j=0}^{k-1} j\alpha_j)hy_n^{(1)} + (\frac{k^2}{2!} - \sum_{j=0}^{k-1}\frac{j^2}{2!}\alpha_j - \sum_{j=0}^{k}\beta_j)h^2 y_n^{(2)} + \ldots + (\frac{k^p}{p!} - \sum_{j=0}^{k-1}\frac{j^p}{p!}\alpha_j - \sum_{j=0}^{k}\frac{j^{p-2}}{(p-2)!}\beta_j)h^p y_n^{(p)} + O(h^{p+1}) \qquad (8)$$

compactly written as

$$T_{n+k} = C_0 y_n + C_1 h y_n^{(1)} + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + \ldots + C_p h^p y_n^{(p)} + C_{P+1}h^{p+1}y_n^{(p+1)} + O(h^{p+2}) \qquad (9)$$

With k=5, equation (4) yields

$$y_{n+5} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + h^2\{\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_4 f_{n+4} + \beta_5 f_{n+5}\} \qquad (10)$$

Imposing accuracy of order-seven on T_{n+5}, note $\alpha_k = \alpha_5 = 1$ we have

$$1 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$5 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$$

$$\frac{25}{2} = \frac{1}{2!}(\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5)$$

$$\frac{125}{6} = \frac{1}{3!}(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4) + (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5)$$

$$\frac{625}{24} = \frac{1}{4!}(\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4) + \frac{1}{2!}(\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + 25\beta_5)$$

$$\frac{3125}{120} = \frac{1}{5!}(\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4) + \frac{1}{3!}(\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + 125\beta_5)$$

$$\frac{15625}{720} = \frac{1}{6!}(\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4) + \frac{1}{4!}(\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4 + 625\beta_5)$$

$$\frac{78125}{5040} = \frac{1}{7!}(\alpha_1 + 128\alpha_2 + 2817\alpha_3 + 16384\alpha_4) + \frac{1}{5!}(\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4 + 3125\beta_5)$$

$$\frac{390625}{40320} = \frac{1}{8!}(\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4) + \frac{1}{6!}(\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4 + 15625\beta_5)$$

$$\frac{1953125}{362880} = \frac{1}{9!}(\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4) + \frac{1}{7!}(\beta_1 + 128\beta_2 + 2187\beta_3 + 16384\beta_4 + 78125\beta_5) \quad (11)$$

By adopting the technique used in Owolabi (2011a & 2011b), equation (11) is written in the form **AX=B** and solved with **MATLAB** package to have

$$\alpha_0 = -1,\ \alpha_1 = -5,\ \alpha_2 = 10,\ \alpha_3 = -10,\ \alpha_4 = 5$$
$$\beta_0 = -\frac{1}{12},\ \beta_1 = -\frac{7}{12},\ \beta_2 = \frac{26}{12},\ \beta_3 = -\frac{26}{12},\ \beta_4 = \frac{7}{12},\ \beta_5 = \frac{1}{12} \quad (12)$$

Substituting (12) into (4), we have a symmetric five-step scheme

$$y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n + \frac{h}{12}\{f_{n+5} + 7f_{n+4} - 26f_{n+3} + 26f_{n+2} - 7f_{n+1} - f_n\} \quad (13)$$

In order to use formula (13) for the integration of initial value problem (3), four important factors are considered

(a) the need to generate the starting values $y_{n+j},\ j = 0(1)5$ and their corresponding derivatives $y''_{n+j},\ j = 0(1)5$, this is achieved by the adoption of predictor-corrector mode denoted by **PEC** meaning Predict, Evaluate and Correct. The mode is described follows

$$P: y_{n+j}\ ,\ j = 0(1)5$$
$$E: y''_{n+j} = y''(t_{n+j}, y_{n+j}),\ j = 0(1)5$$
$$C: y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n + \frac{h}{12}\{f_{n+5} + 7f_{n+4} - 26f_{n+3} + 26f_{n+2} - 7f_{n+1} - f_n\}$$

The error estimate is obtained from

$$Error = \frac{y^{(s+1)}_{n+5} - y^{(s)}_{n+5}}{y^{(s)}_{n+5} - y^{(s-1)}_{n+5}} \quad (14)$$

the iteration terminated whenever $Error < tolerance$

(b) the choice of appropriate step-size **h**

(c) the need to solve implicit equation (13), now

$$y_{n+5} = A + \frac{h^2}{12}G(y_{n+5}) \quad (15)$$

where

$$A = y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n$$
$$G = y''_{n+5} + 7y''_{n+4} - 26y''_{n+3} + 26y''_{n+2} - 7y''_{n+1} - y''_n$$

(d) the accuracy of the approximation y_{n+5} requires the solution of implicit equation (15) rewritten as

$$F(y_{n+5}) = 0 \quad (16)$$

This can be achieved by the adoption of quasi Newton iteration scheme

$$[y^{s+1}_{n+5} - y^{s}_{n+5}] - G[y^{s}_{n+5}]/[I - \frac{h^2}{12}\zeta]$$
$$\zeta = \frac{\partial G}{\partial y_{n+5}}(y^{s}_{n+5}),\ s = 0,1,2$$

The convergence condition is that

$$\theta = \frac{y^{(s+1)}_{n+5} - y^{(s)}_{n+5}}{y^{(s)}_{n+5} - y^{(s-1)}_{n+5}} \le Tolerance \quad (17)$$

2. Basic properties of the method

In order to ascertain the accuracy and suitability of the method (13), analysis of its basic properties such as consistency, order of accuracy and error-constant, symmetry, convergence and zeo-stability are undertaken in this section.

3.1 Order of Accuracy and Error-constant

The local truncation error (9) when k=5 can be written as

$$T_{n+5} = C_0 y_n + C_1 h y_n^{(1)} + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + ... + C_8 h^8 y_n^{(8)} + C_9 h^9 y_n^{(9)} + O(h^{10})$$

Using the values of α_j and β_j as contained in equation (12) in above (11), we have

$$C_0 = 1 - 5 + 10 - 10 + 5 - 1 = 0$$
$$C_1 = 5 - 20 + 30 - 20 + 5 = 0$$
$$C_2 = \frac{1}{2}(25 - 80 + 90 - 40 + 5) - \frac{1}{12}(1 + 7 - 26 + 26 - 7 - 1) = 0$$
$$C_3 = \frac{1}{6}(125 - 320 + 270 - 80 + 5) - \frac{1}{12}(5 + 28 - 78 + 52 - 7) = 0$$
$$C_4 = \frac{1}{24}(625 - 1280 + 810 - 106 + 5 - 25 - 112 + 234 - 104 + 7) = 0$$
$$C_5 = \frac{1}{120}(3125 - 510 + 2430 - 320 + 5) - \frac{1}{72}(125 + 448 - 702 + 208 - 7) = 0$$
$$C_6 = \frac{1}{720}(15625 - 20480 + 7290 - 640 + 5) - \frac{1}{288}(625 + 1792 - 2106 + 416 - 7) = 0$$
$$C_7 = \frac{1}{5040}(78125 - 81920 + 21870 - 1280 + 5) - \frac{1}{1440}(3125 + 7168 - 6318 + 832 - 7) = 0$$
$$C_8 = \frac{1}{40320}(390625 - 327680 + 65610 - 2560) - \frac{1}{8640}(15625 + 28672 - 18954 + 1664 - 7) = 0$$
$$C_9 = \frac{1}{362880}(195125 - 1310720 + 196830 - 5120 + 5) - \frac{1}{60480}(78125 + 114688 - 56862 + 3328 - 7) = \frac{756}{181440} \qquad (18)$$

Thus, $C_9 \neq 0$, which by Lambert (1973) implies that

$$C_0 = C_1 = C_2 = C_3 = C_4 =, \ldots, = C_8 = 0, \text{ but } C_9 = C_{p+2} \neq 0$$

Hence, method (13) is of order **P = 7** with principal error-constant

$$C_{p+2} = -\frac{1}{240}$$

3.2 Symmetry

A linear multistep method (13) is said to be symmetric (Lambert & Watson 1976, Fatunla 1988, Owolabi 2011) if the parameters α_j and β_j satisfy the following conditions

$$\alpha_j = \alpha_{k-j},\ \beta_j = \beta_{k-j},\ j = 0(1)k$$
$$\alpha_j = -\alpha_{k-j},\ \beta_j = -\beta_{k-j},\ j = 0(1)k$$

for even and odd step-numbers respectively.

Now, for k=5

$$\alpha_0 = -\alpha_5 = -1 \qquad \beta_0 = \beta_5 = -1$$
$$\alpha_1 = -\alpha_4 = 5 \qquad \beta_1 = \beta_4 = -7$$
$$\alpha_2 = -\alpha_3 = -10 \qquad \beta_2 = \beta_3 = 26$$

Hence, method (13) is symmetric.

3.3 Consistency

Method (13) is consistent, since

- It has order $P \geq 1$
- $\sum_{j=0}^{k} \alpha_j = 0$
- $\rho(r) = \rho'(r) = 0, \ r = 1$
- $\rho''(r) = 2!\delta(r), \ r = 1$

3.4 Zero Stability

Definition 2

(i) *A linear multistep method for a given initial value problem is said to be zero-stable, if no root of its first characteristic polynomial has modulus greater than one and if every root with modulus one is simple.*

That is

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = 0,$$

from (13),

$$\rho(r) = r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1 = 0.$$

Implies that the method (13) is zero-stable since the roots of $\rho(r)$ lie in the unit disk, and those that lie on the unit circle have multiplicity of one

(ii) *A numerical solution to the class of system (1) is stable if the difference between the numerical and the theoretical solution can be made as small as possible, that is, if there exist two positive numbers* ℓ_n *and* C *such that*

$$\|y_n - y(t_n)\| \leq C\|\ell_n\|.$$

3.5 Convergence

Definition 3 *A linear multistep method that is consistent and zero-stable is convergent,* (Ademiluyi 1987, Fatunla 1988, Lambert 1991).

3. Numerical Experiments

Efficiency and applicability of our new method is demonstrated on some initial value problems. The first is an inhomogeneous problem in Simos(1998), the second and third examples are respectively the nonlinear and linear problems taken from Badmus & Yahaya (2009) and Kayode (2010).

4.1 *Inhomogeneous equation*

We consider the following equation:

$$y'' = 100y + 99\sin(x), \ y(0) = 1, \ y'(0) = 11, \qquad (19)$$

whose theoretical solution is

$$y(x) = \cos(10x) + \sin(10x) + \sin(x)$$

In Table 1, equation (19) has been solved numerically for $0 \leq x \leq 1000$ for various step-sizes, using the new method denoted as **[D]** and compared the end-point global errors with Runge-Kutta method of Simos which is indicated as **[A]**.

4.2 *A problem by Badmus and Yahaya*

We consider the second order initial value problem:

$$y''+\frac{6}{x}y'+\frac{4}{x^2}y=0,\ \ y(0)=1,\ \ y'(0)=1,\ x>0,\ \ h=\frac{0.1}{32}, \qquad (20)$$

whose theoretical solution

$$y(x)=\frac{5}{3x}-\frac{2}{3x^4}$$

In Table 2, we present the comparison of the global errors in approximations of the new method (indicated as **[D]**) for the solution of equation (20) at some selected step-sizes given in the first column with both block method (denoted as **[B]**) and zero-stable method (denoted as **[C]**).

4.3 *Nonlinear problem*

Consider the following nonlinear problem:

$$y''=x(y')^2,\ y(0)=1,\ y'(0)=0.5,\ h=0.003125, \qquad (21)$$

whose analytical solution is given by

$$y(x)=1+0.5\ln[\frac{2+x}{2-x}]$$

In Table 3, comparison of the global errors arising from our new method **[D]** was made with the two methods **[B]** and **[C]** at some selected points for solution of equation (21).

5. **Conclusion**

A technique for the construction of an implicit symmetric method for direct integration of second order initial value problem of ordinary differential equations has been developed. Analysis of its basic properties has shown that the new method is consistent, convergent and zero-stable. In order to evaluate the effectiveness of our method, some examples on both linear and nonlinear problems are given. The results shown in Tables 1-3 have depicted that the new method is much more efficient than the three other existing methods on comparison.

References

Abhulimen, C.E. & Otunta, F. O. (2006). "A Sixth-order Multiderivative Multisteps Methods for Systems of differential Equations", *International Journal of numerical Mathematics*, 1, 248-268.

Ademiluyi, R. A. (1987). "New Hybrid methods for Systems of Stiff Equations", *PhD Thesis*, University of Benin, Benin City, Nigeria.

Ademiluyi, R. A. & Owolabi, K. M. (2010). "A Three Sstep Discretization Scheme for Direct numerical Solution of Second-order Differential Equations ", *Journal of Nigerian Association of Mathematical Physics*, 16, 265-276.

Aruchunan, E. & Sulaiman, J. (2010). "Numerical Solution of Second-order Linear Fredholm Integro-differential Equation using Generalized Minimal Residual method", *American Journal of Applied Science*, 7, 780-783.

Awoyemi, D. O. (2001). "A New Sith-order Algorithm for General Second-order Differential Equations", *International Journal of Computational Mathematics*, 77, 117-124.

Awoyemi, D. O. (2005). "Algorithmic Collocation Approach for Direct Solution of Fourth-order Initial Value problems of ODEs", *International Journal of Computational Mathematics*, 82 271-284.

Awoyemi, D. O. & Kayode, S. J. (2005). "An Implicit Collocation Method for Direct Solution of Second-order ODEs", *Journal of Nigerian Association of Mathematical Physics*, 24, 70-78.

Badmus, A. M. & Yahaya, Y. A. (2009). "An Accurate Uniform Order 6 Blocks Method for Direct Solution of General Second-order ODEs", *Pacific Journal of Science* , 10, 248-254.

Bun, R. A. Vasil'Yer, Y. D. (1992). "ANumerical method for Solving Differential Equations of any Orders", *Computational Maths Physics*, 32, 317-330.

Chan, R. P. K. & Leon, P. (2004). "Order Conditions and Symmetry for Two-step Hybrid Methods", *International Journal of Computational Mathematics* , 81, 1519-1536.

Fatunla, S.O. (1988), "Numerical methods for IVPs in Ordinary Differential Equations", *Academic Press Inc*. Harcourt Brace Jovanivich publishers, New York.

Jacques, I. & Judd, C. J. (1987). "Numerical Analysis", Chapman and Hall, New York.

Juan, A. (2001). "Numerical methods for Partial differential equations", *http//pde.fusion.tth.se*

Kayode, S. J. & Awoyemi, D. O.(2005). "A 5-step maximal Order for Direct solution of Second-order ODEs ", *Journal of Nigerian Association of Mathematical Physics*, 7, 285-292.

Kayode, S. J. (2010). "A zero-stable Optimal method for Direct Solution of Second-order differential Equation ", *Journal of mathematics & Statistics*, 6, 367-371.

Lambert, J. D. & Watson, A. (1976). "Symmetric Multistep method for periodic Initial Value Problems", *Journal of Inst. Mathematics & Applied*, 18, 189-202

Lambert, J. D. (1991). "Numerical Methods for Ordinary Differential Systems of Initial Value Problems", John Willey & Sons, New York.

Owolabi, K. M. (2011). "An Order Eight Zero-stable Method for direct Integration of Second-order Ordinary Differential Equations ", *Mathematics Applied in Science & Technology*, 3(1), 23-33.

Owolabi, K. M. (2011). "4^{th}-step Implicit formula for Solution of Initial-value problems of Second-order ordinary Differential equations ", *Academic Journal of Mathematics & Computer Science Research*, 4, 270-272.

Parand, K. & hojjati, G. (2008). "Solving Vollterra's Population Model using New Second derivative Multistep Methods ", American Journal of Applied Science, 5, 1019-1022.

Quinlan, G. D. & Tremaine, S. (1990). "Symmetric Multistep methods for the Numerical Integration of Planetry Orbits", Astronomical Journal 100, 1694-1700.

Simos, T. E. (1998). "An Exponentially-fitted Runge-Kutta Method for the Numerical Integration of Initial-value Problems with periodic or Oscillating Solutions", *Computational Physics Communications*, 115, 1-8.

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Table 1: Solution to problem (19)

H	Exact	[D] Computed	Error [A]	Error [B]
1	-0.541621655157926	-0.424800002157926	1.4e-01	1.1e-01
0.5	-0.195836550595709	-0.175498405376949	3.5e-02	2.3e-02
0.25	o.044732487811546	0.044721485379385	1.1e-03	1.1e-05
0.125	1.388981715136082	1.388981253407068	8.4e-05	4.6e-07
0.0625	1.458519710288061	1.458551881193934	5.5e-06	8.9e-08
0.03125	1.290251376613879	1.290251373261517	3.5e-07	3.4e-09

Table 2: Solution to Problem (20)

x	Exact	[D] Computed	Error [B]	Error [C]	Error [D]
1/40	0.04166640620000	0.041665543091291	2.2105e-04	7.7370e-04	8.6315e-07
1/64	0.026041626930237	0.026041035788072	1.5557e-04	5.9320e-06	5.9104e-07
1/80	0.020833317057292	0.020832836894301	1.3548-04	3.6750e-06	4.8017e-07
1/160	0.010416665649414	0.010416603239263	7.5000e-04	1.8600e-07	6.2410e-08
1/320	0.005208333269755	0.0052208200708751	3.8354e-05	1.1040e-07	1.3256e-08

Table 3: Solution to problem (21)

x	Exact	[D] Computed	Error [B]	Error [C]	Error [D]
0.1	1.050041729278491	1.050041728545390	5.8910e-06	6.1254e-08	7.3310e-10
0.2	1.100335347731076	1.100335337249948	8.2400e-05	1.2113e-07	1.0481e-08
0.3	1.151140435936467	1.151140403698324	3.4642e-04	1.8749e-07	3.2238e-08
0.4	1.202732554054082	1.202732503027955	7.5210e-04	2.6159e-07	5.1026e-08
0.5	1.255412811882995	1.255411274317176	1.3803e-03	3.5346e-07	6.8711e-08