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Some Fixed Point Theorems in Generalized Dislocated Metric Spaces

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Abstract: The notion of a generalized dislocated metric space is introduced, its topological properties are established and an analogue of seghals fixed point theorem is obtained from which existence and uniqueness of fixed points for self maps that satisfy the metric analogues of contractive conditions mentioned and it can be derived.
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1. Introduction: Pascal Hitzler presented variants of Banach's Contraction principle for various modified forms of a metric space including dislocated metric space and applied them to semantic analysis of logic programs. In this context Hitzler raised some related questions on the topological aspects of dislocated metrics.

In this paper we present results that establish existence of a topology induced by a generalized dislocated metric and show that this topology is metrizable , by actually showing a metric that induces the topology.

Rhoades collected a large number of variants of Banach's Contractive conditions on self maps on a metric space and proved various implications or otherwise among them. We pick up a good number of these conditions which ultimately imply Seghal's condition .We prove that these implications hold good for self maps on a generalized dislocated metric space and prove the generalized dislocated metric version of Seghal's result then by deriving the generalized dislocated analogue's of fixed point theorems of Banach, Kannan ,Bianchini, Reich and others.

In what follows, R^+ the set of all positive real numbers.

1.1:Let binary operation $\Diamond : R^+ \times R^+ \to R^+$ satisfies the following conditions:

(I) $\Diamond$ is Associative and Commutative,

(II) $\Diamond$ is continuous w.r.t to the usual metric R^+

A few typical examples are $a \Diamond b = \max\{a, b\}$, $a \Diamond b = a + b$, $a \Diamond b = ab$, $a \Diamond b = ab + a + b$ and $a \Diamond b = \dfrac{ab}{\max\{a,b,1\}}$ for each $a, b \in R^+$

In what follows we fix a binary operation $\Diamond$ that satisfies (l) and (ll)

Definition 1.2:A binary operation $\Diamond$ on R^+ is said to satisfy β-property if

(lll)there exists a positive real number β such that $a \Diamond b \le \beta \max\{a, b\}$ for every $a, b \in R^+$.

Definition 1.3: Let X be a non empty set. A generalized dislocated (simply gd) metric (or d^* metric) on X is a function $d^* : X^2 \to R^+$ that satisfies the following conditions:

(1) $d^*(x,y) \ge 0$,

(2) $d^*(x,y) = 0$ Implies $x = y$

(3) $d^*(x,y) = d^*(y,x)$

(4) $d^*(x,z) \le d^*(x,y) \lozenge d^*(y,z)$ for each $x, y, z \in X$.

The pair (X, d^*) is called a generalized dislocated (or simply d^*) metric space.

Fix a d^* metric space (X, d^*). For $r > 0$ write $B_{d^*}(x,r) = \{ y \in X : d^*(x,y) < r \}$.

Definition 1.4: We say that a net $\{ x_\alpha / \alpha \in \Delta \}$ in X converges to x in (X, d^*) and write $\lim_\alpha \{ x_\alpha / \alpha \in \Delta\} = x$ if $\lim d^*(x_\alpha, x) = 0$ i.e. for each $\in > 0$ there exists $\alpha_o \in \Delta$ such that for all $\alpha \ge \alpha_0 \Rightarrow d^*(x, x_\alpha) < \in$.

Remark: If $\lozenge$ satisfies β-property with $\beta > 0$.then the limit of a net in (X, d^*) is unique. Now on we assume that(X, d^*) has β-property with $0 < \beta \le 1$

Notation: For $A \subset X$ we write $D(A) = \{ x \in X \,/\, x$ is a limit of a net in $(X, d^*)\}$

Proposition 1.5: Let A, $B \subseteq$ X. Then

I. $D(A) = \phi$ if $A = \phi$

II. $D(A) \subseteq D(B)$ if $A \subseteq B$

III. $D(A \cup B) = D(A) \cup D(B)$ and

IV. $D(D(A)) \subseteq D(A)$

Proof:(i) and (ii) are clear. That $D(A) \cup D(B) \subseteq D(A \cup B)$ follows from (ii). To prove the reverse inclusion,

Let $x \in D(A \cup B)$ and $x = \lim_{x \in \Delta} (x_\alpha)$ where $(x_\alpha / \alpha \in \Delta)$ is a net in $A \cup B$. If $\exists$ $\lambda \in \Delta$ such that $x_\alpha \in A$ for $\alpha \in \Delta$ and $\alpha \ge \lambda$ then $(x_\alpha / \alpha \ge \lambda, \alpha \in \Delta)$ is a cofinal subnet of $(x_\alpha / \alpha \in \Delta)$ and is in A and $\lim_{\alpha \ge \lambda} d^*(x, x_\alpha) = \lim_{\alpha \in \Delta} d^*(x, x_\alpha) = 0$ so that $x \in$ D(A).

If no such λ exists in Δ then for every $\alpha \in \Delta$, choose $\beta(\alpha) \in \Delta$ such that $\beta(\alpha) \ge \alpha$ and $x_{\beta(\alpha)} \in B$ then $\{x_{\beta(\alpha)} / \alpha \in \Delta\}$ is a cofinal subset in B of $(x_\alpha / \alpha \in \Delta)$ and $\lim_{\alpha \in \Delta} d^*(x_{\beta(\alpha)}, x)$ $= \lim_{\alpha \in \Delta} d^*(x_\alpha, x) = 0$ so that

$x \in D(B)$. It now follows that $D(A \cup B) \subseteq D(A) \cup D(B)$ and hence (iii) holds. To prove (iv) let $x \in D(D(A))$, $x = \lim_{\alpha \in \Delta} x_\alpha$, $x_\alpha \in D(A)$ for $\alpha \in \Delta$, and $\forall$ $\alpha \in \Delta$, let $(x_{\alpha_\beta} / \beta \in \Delta(\alpha))$ be a net in A $\ni$

$x_\alpha = \lim_{\beta \in \Delta(x)} x_{\alpha_\beta}$. For each positive integer i $\exists$ $\alpha_i \in \Delta$ such that $d^*(x_{\alpha_i}, x) < \frac{1}{i}$.and $\beta_i \in \Delta(\alpha_i) \ni$

$d^*(x_{\alpha_{i_{\beta_i}}}, x_{\alpha_i}) < \frac{1}{i}$ write $\alpha_{i_{\beta_i}} = \gamma_i$ $\forall$ i, then $\{\gamma_1, \gamma_2, ...\}$ is directed set with $\gamma_i < \gamma_j$

If $i < j$, and $d^*(x_{\gamma_i}, x) \le d^*(x_{\gamma_i}, x_{\alpha_i}) \lozenge d^*(x_{\alpha_i}, x)$

$$\le \beta \max\{ d^*(x_{\gamma_i}, x_{\alpha_i}), d^*(x_{\alpha_i}, x)\}$$

$$< d^*(x_{\gamma_i}, x_{\alpha_i}) + d^*(x_{\alpha_i}, x)$$

$< \frac{2}{i}$. This implies that $x \in D(A)$.

As a corollary, we have the following

Theorem 1.6: If for $A \subset$ X and $\overline{A} = A \cup D(A)$, then the operation $A \to \overline{A}$ on P(X) satisfies Kuratowski's Closure axioms:

i. $\overline{\phi} = \phi$
ii. $A \subset \overline{A}$
iii. $\overline{A} = \overline{\overline{A}}$ and
iv. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Consequently we have the following

Theorem 1.7 : Let $\Im$ be the set of family of all subsets A of X for which $\overline{A} = A$ and τ be the complements of members of $\Im$. Then the $\Im$ is a topology for X and the $\Im$-closure of a subset A of X is $\overline{A}$.

Definition 1.8: The topology $\Im$ obtained in Theorem 1.7 is called the topology induced by d^* and simply referred to as the d-topology of X and is denoted by (X , $d^*, \Im$).

Proposition 1.9: Let $A \subseteq X$. Then $x \in D(A)$ iff for every $\delta > 0$, $B_\delta(x) \cap A \neq \phi$

Proof: if $x \in$ D(A),there exist a net $(x_\alpha / \alpha \in \Delta)$ in A such that lim $x_\alpha = x$, if $\delta > 0$ $\exists$ $\alpha_\delta \in \Delta$ such that

$d^*(x_\alpha, x) < \delta$ .If $\alpha \in \Delta$ and $\alpha \geq \alpha_\delta$. Hence $x_\alpha \in B_\delta(x) \cap A$ for $\alpha \geq \alpha_\delta$.

Conversely if for every $\delta > 0$, $B_\delta(x) \cap A \neq \phi$,we choose one x_n in $B_{\frac{1}{n}}(x) \cap A$ for every integer $n \geq 1$ in (0,1). Clearly ($x_n / n \geq 1$) is a net in A and let $\in > 0$ and $N_0 > \frac{1}{\in}$ if $n > N_0$,

$d^*(x_n, x) < \frac{1}{n} < \frac{1}{N_0} < \in$

Hence $x \in D(A)$.

Corollary 1.1O: $x \in \overline{A} \Leftrightarrow x \in A$ or $B_\delta(x) \cap A \neq \phi$ $\forall \delta > 0$.

Corollary 1.11: A is open in $(X, d^*, \Im)$ if and only if for every $x \in A$ $\exists$ $\delta > 0$ such that $\{x\} \cup B_\delta(x) \subseteq A$

Proposition 1.12: If $x \in X$ and $\delta > 0$ then $\{x\} \cup B_\delta(x)$ is an open set in(X , $d^*, \Im$).

Proof: Let $A = \{x\} \cup B_\delta(x)$, $y \in B_\delta(x)$ and $0 < r < \delta - d^*(x, y)$

Then $B_r(y) \subseteq B_\delta(x) \subset A$

Since $z \in B_r(y)$

$\Rightarrow d^*(y, z) < r < \delta - d^*(x, y)$

$\Rightarrow d^*(x, y) + d^*(y, z) < \delta$

Now $d^*(x, z) \leq d^*(x, y) \lozenge d^*(y, z)$

$\leq \beta \max\{ d^*(x, y), d^*(y, z)\}$

$\leq d^*(x, y) + d^*(y, z) < \delta$

$\Rightarrow z \in B_\delta(x)$.

Proposition 1.13: If $x \in X$ and $V_r(x) = B_r(x) \cup \{x\}$ for $r > 0$ then the collection $\{V_r(x) / x \in X\}$ is an open base at x in $(X, d^*, \Im)$.

Proof: The first condition for a basis is trivial, before checking the second condition for a basis we show that if 'y' is a basis element of $V_r(x)$.Then there is a basis element $V_\delta(y)$ that is contained in $V_r(x)$.

Define $\delta = r - d^*(x, y)$ then $V_\delta(y) \subset V_r(x)$.

For if $z \in V_\delta(y)$ then $d^*(y, z) < \delta = r - d^*(x, y)$

$\Rightarrow d^*(x, y) + d^*(y, z) < r$

Now $d^*(x, z) \leq d^*(x, y) \lozenge d^*(y, z)$

$$\leq \beta \max\{d^*(x, y), d^*(y, z)\}$$

$$\leq d^*(x, y) + d^*(y, z)$$

$$< r$$

Let V_1 and V_2 be two basis elements and let $y \in V_1 \cap V_2$ then there exist δ_1, δ_2 so that $V_{\delta_1}(y) \subset V_1$ and $V_{\delta_2}(y) \subset V_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$,then $V_\delta(y) \subset V_1 \cap V_2$.

Proposition 1.14: $(X, d^*, \Im)$ is a Hausdorff space and first countable.

Proof: If $x, y \in X$, and $d^*(x, y) > 0$ then $V_{\frac{\delta}{2}}(x) \cap V_{\frac{\delta}{2}}(y) = \phi$.

Hence (X, d^*) is a Hausdorff space. If $x \in X$ the collection $\{B_{\frac{1}{n}}(x)\}$ is base at X. Hence (X, d^*) is first countable.

Remark: Proposition 1.14 enables us to deal with sequence instead of nets.

Definition 1.15: A sequence $\{x_n\}$ is called a Cauchy sequence in (X, d^*), if for each $\in > 0$, there exists $n_0 \in N$ such that $d^*(x_n, x_m) < \in$ for each $n, m \geq n_0$. (X, d^*) is said to be complete if every Cauchy sequence is convergent.

Definition 1.16: Let (X, d^*) be a d^* metric space. If there is a number $0 < \gamma < 1$ such that $d^*(f(x), f(y)) \leq \gamma d^*(x, y)$ $\forall x, y \in X$ then f is called a contraction.

Let (X, d^*) be a d^*-metric space and $f : X \to X$ be a mapping write $V(x) = d^*(x, f(x))$ and $z(f) = \{x / V(x) = 0\}$

Clearly every point of Z(f) is a fixed point of f but the converse is not necessarily true. We call points of Z(f) as coincidence points of f .The set Z(f) is a closed subset of X .Mathew's theorem states that a contraction on a complete dislocated metric space has a unique fixed point . The same theorem has been justified by an alternate proof by Pascal Hitzler.We present the generalized dislocated metric version of this theorem for coincidence points.

2 MAIN RESULTS:

Theorem 2.1: If (X, d^*) is a complete and $f : X \to X$ is a contraction,then there is a unique coincidence point for f .

Proof: For any $x \in X$ the sequence of iterates satisfies

$d^*(f^n(x), f^{n+1}(x)) \leq \gamma^n d^*(x, f(x))$ where γ is any contractive constant.

Consequently if $n < m$,

$d^*(f^n(x), f^m(x)) \le d^*(f^n(x), f^{n+1}(x)) \lozenge\, d^*(f^{n+1}(x), f^{n+2}(x)) \lozenge \ldots\ldots d^*(f^{m-1}(x), f^m(x))$

$\le \beta \max\{d^*(f^n(x), f^{n+1}(x)), d^*(f^{n+1}(x), f^{n+2}(x)) \ldots\ldots d^*(f^{m-1}(x), f^m(x))\}$

$\le d^*(f^n(x), f^{n+1}(x)) + d^*(f^{n+1}(x), f^{n+2}(x)) + \ldots\ldots d^*(f^{m-1}(x), f^m(x))$

$$\le \gamma^n \left(1+\gamma + \gamma^2 + \ldots\ldots + \gamma^{m-n-1}\right) d^*(x, f(x))$$

$$\le \frac{\gamma^n}{1-\gamma}\, d^*(x, f(x))$$

Hence $\{f^n(x)\}$ is Cauchy sequence in X.

if $\xi = \lim_n f^n(x)$

then $f(\xi) = \lim_n f^{n+1}(x)$ so

$d^*(\xi, f(\xi)) = \lim_n d^*(f^n(x), f^{n+1}(x))$. Since $d^*(f^n(x), f^{n+1}(x)) < \gamma^n\, d^*(x, f(x))$

Since $0<\gamma<1$; $\lim \gamma^n\, d^*(x, f(x))=0$ Hence $d^*(\xi, f(\xi))=0$

Uniqueness: If $d^*(\xi, f(\xi)) = d^*(\eta, f(\eta))=0$, then $f(\xi)=\xi$ and $f(\eta)=\eta$ so that

$d^*(\xi, \eta) \le d^*(\xi, f(\xi)) \lozenge\, d^*(f(\xi), f(\eta)) \lozenge\, d^*(f(\eta), \eta)$

$\le \beta \max\{d^*(\xi, f(\xi)), d^*(f(\xi), f(\eta)), d^*(f(\eta), \eta)\}$

$\le d^*(\xi, f(\xi)) + d^*(f(\xi), f(\eta)) + d^*(f(\eta), \eta)$

$\le \gamma\, d^*(\xi, \eta)$ so that

$d^*(\xi, \eta)=0$, hence $\xi=\eta$.

Theorem 2.2: let (X, d^*) be any d^*-metric space and $f : X \to X$ be continuous .Assume that $d^*(f(x), f(y)) < \max\{d^*(x, f(x)), d^*(y, f(y)), d^*(x, y)\}$ whenever $d^*(x, y) \ne 0$.Then f has a unique coincidence point whenever cl O(x) is nonempty for some $x \in X$.

Proof: Write V(x)= $d^*(x, f(x))$, $z = \{x / \mathrm{V}(x)=0\}$; O(x)= $\{f^n(x) / n \ge 0\}$

Since f is continuous, V is continuous. If $x \notin$ Z , then

V $(f(x))$= $d^*(f(x), f^2(x)) < \max\{d^*(x, f(x)), d^*(f(x), f^2(x)), d^*(x, f(x))\}$

=max { V(x) , V ($f(x)$) }

$\Rightarrow$ V ($f(x)$) < V(x) , whenever V(x) $\ne 0$ i.e $x \notin$ Z ----------------------------(1)

If O(x) $\cap z = \phi$ then V($f^{k+1}(x)$) < V ($f^k(x)$) $\forall k$

Hence V($f^n(x)$) is convergent. ------------------------------- (2)

let ξ be a cluster point of O(x). $\exists (n_i)\uparrow \ni \xi = \lim f^{n_i}(x)$

$\Rightarrow f^k(\xi) = \lim f^{n_i+k}(x)$

$\Rightarrow$ O(ξ) $\le cl$ 0(x) , since V is continuous

V($f^{k}(\xi)$) = lim V($f^{n_i+k}(x)$)

Since O(x) $\cap$ z = ϕ by (2) { V ($f^{n}(x)$) } is convergent.

Let γ = lim V($f^{n_i}(x)$) = V (ξ)

Also γ = lim V($f^{n_i+1}(x)$) = V (f (ξ)) ; $\forall k$

$\Rightarrow$ V (f (ξ)) = V (ξ) ---------------------------------- (3)

From (1) and (3) it follows that V (ξ) =0

Uniqueness: If V(ξ) =V(η)= 0 then

$\xi = f(\xi)$, $\eta = f(\eta)$ if $d^*(\xi, \eta) \neq 0$

$d^*(\xi, \eta) = d^*(f(\xi), f(\eta)) <$ max {V(ξ), V(η), $d^*(\xi, \eta)$ }

$= d^*(\xi, \eta)$ which is a contradiction.

Hence $d^*(\xi, \eta)$=0.

B.E Rhodes presented a list of definitions of contractive type conditions for a self map on a metric space (X , d) and established implications and nonimplications among them ,there by facilitating to check the implication of any new contractive condition through any one of the condition mentioned so as to derive a fixed point theorem. Among the conditions in seghal's condition is significant as a good number of contractive conditions imply seghal's condition .These implications also hold good in the present context as well. In fact the usual metric on R is a d^*metric, if we write $a \lozenge b = a + b$ .We state the d^*metric version of some of the contractive conditions mentioned in and derive various implications and non implications and deduce fixed point theorems for d^*metrics from 2.2

Let (X, d^*) be a d^*metric space with $a \lozenge b = a + b$ for each $a, b \in R^+$ and f : X $\to$ X be a mapping and x, y be any elements of X.

1. (Banach) : there exists a number α , $0 \leq \alpha \leq 1$ such that for each $x, y \in$ X
   $d^*(f(x), f(y)) \leq \alpha\, d^*(x, y)$
2. (Rakotch) : there exists a monotone decreasing function $\alpha : (0,\infty) \to [0,1)$ such that
   $d^*(x, y) \leq \alpha\, d^*(x, y)$ whenever $d^*(x, y) \neq 0$
3. (Edelstein) : $d^*(f(x), f(y)) < d^*(x, y)$ whenever $d^*(x, y) \neq 0$
4. (Kannan) :there exists a number α , $0 < \alpha < \frac{1}{2}$ such that
   $d^*(f(x), f(y)) < \alpha\, [d^*(x, f(x)) \lozenge d^*(y, f(y))]$
5. (Bianchini): there exists a number h ,$0 \leq h < 1$ such that
   $d^*(f(x), f(y)) \leq h$ max{ $d^*(x, f(x)), d^*(y, f(y))$
6. $d^*(f(x), f(y)) <$ max $d^*(x, f(x)), d^*(y, f(y))$ whenever $d^*(x, y) \neq 0$
7. (Reich) : there exist nonnegative numbers a, b, c satisfying a + b + c < 1 such that
   $d^*(f(x), f(y)) \leq$ a $d^*(x, f(x)) \lozenge$ b $d^*(y, d^*(y)) \lozenge$ c $d^*(x, y)$
8. (Reich) : there exist monotonically decreasing functions a, b, c from (0 , ∞) to [0 ,1) satisfying a(t) + b(t) + c(t) < 1 such that ,

$d^*(f(x), f(y)) <$ a(s)$d^*(x, f(x))$ ◊ b(s) $d^*(y, f(y))$ ◊ c(s)s where t= $d^*(x, y) \neq 0$

9. there exist nonnegative functions a, b, c satisfying $\sup_{x,y \in X}$ a(x, y)+b(x, y)+c(x, y) < 1 such that

   $d^*(f(x), f(y)) \leq$ a(t)$d^*(x, f(x))$◊b(t)$d^*(y, f(y))$◊ c(t)t where t=(x, y)

10. (sehgal): $d^*(f(x), f(y))$<max{ $d^*(x, f(x))$, $d^*(y, f(y))$, $d^*(x, y)$ } if $d^*(x, y) \neq 0$

Theorem 2.3: if f is a self map on a generalized dislocated metric space (X, d^*) and f satisfies any of the conditions (1) through (9) then f has a unique coincidence point provided cl O(x) is nonempty for some $x \in X$.

Proof:B.E Rhodes proved that when d is a metric
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (10)
(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (10)
(4) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (10)
(5) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (10)

These implications hold good in a d^*- metric space as well since $x \neq y \Rightarrow d^*(x, y) \neq 0$ in a d^*- metric space .It now follows from theorem 2.2 that f has a fixed point which is unique when O(x) has a cluster point for some x.

Remark: Every coincidence point in a generalized dislocated metric space is a fixed point but the converse is not true.

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