

Applications of Homotopy Analysis Method in Science and Engineering Research Problems

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Abstract
In this Paper we have discussed Homotopy Analysis Method (Homotopy perturbation method) with an example of boundary problems (Blasius Flow), which is having more importance in research of science and engineering.
Keywords: Perturbation method, Homotopy analysis Method, Boundary value Problems, Non-Linear equations

1. INTRODUCTION

We know that Perturbation methods are based on an assumption that there may be a small parameter exist in the equation this small parameter assumption restrict the application of perturbation technique, the determination of small parameter seems to be a special technique. But here an appropriate choice of small parameter leads to a good result. Homotopy Perturbation method does not depend upon a small parameter in the equation. This Homotopy method is a combination technique and, which provides us a convenient way to obtain analytic or approximate solution to a wide variety of problems arising in different fields of science and engineering. Hence this method was introduced as a powerful to solve various kinds of non-linear problems.

Nayfeh [1] discussed in detail about perturbation technique for solving various types of equations. He [2-3] discussed the Homotopy Perturbation technique identical, for solving boundary value problems.

Ganji et.al [4-5] also discussed some new application of Homotopy perturbation method .he studied solution of non linear Blasius equation. Liao [6] discussed another type of Homotopy analysis method called optimal Homotopy analysis method equation; he defined a new kind of average residual error. He showed this method greatly accelerate the convergence of series solution. This method can be used to get fast converge series solution of different types with strong non-linearity. Nandeppanavar et.al [9] studied the solution of flow due to non-linear stretching sheet, The solution is obtained by Optimal Homotopy analysis method.

In this paper we have obtained the Solution of Blasius problem with Homotopy perturbation method. The solution is also compared with its numerical solution.

2. Homotopy Perturbation Method

In recent years, the Homotopy Perturbation Method has been successfully applied to solve many types of differential equation. It was proposed by "Ji-Huan He" in 1990. In the Homotopy perturbation technique we will first propose a new perturbation technique coupled with the Homotopy technique. In topology two continuous functions from one topological space to another is called "homo-topic". Formally a Homotopy between two continuous function f and g from a topological space Y is defined to be a continuous function

$$H : X \times [0,1] \rightarrow Y$$

Such that

$$H(x,0) = f(x) \quad and \; H(x,1) = g(x), \forall \; x \in X$$

The Homotopy perturbation method does not depend upon a small parameter in the equation. By the Homotopy technique in topology, a Homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a small parameter.

2.1 Basic Idea of HPM:

Let us consider the non-linear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

$$B\left(u, \frac{\partial u}{\partial \eta}\right) \;, \quad r \in \Gamma \tag{2}$$

Where A is a general differential operator, B is a boundary operator. Γ Is the boundary of domain Ω . $f(\tau)$ Is a known as analytic function. Now, the operator A can be divided into two parts L and N, where L is linear and non linear. Equation (1) can be written as follows

$$L(u)+N(u)-F(r)=0 \tag{3}$$

By Homotopy technique, we construct a Homotopy

$$\upsilon(r,p):\Omega\times[0,1]\to R,$$

Which

$$H(\upsilon,p)=(1-p)\left[L(\upsilon)-L(u_0)\right]+p\left[A(\upsilon)-f(r)\right]=0, p\in\Omega$$

Or

$$H(\upsilon,p)=L(\upsilon)-L\left(u_0+p\left[N(\upsilon)-f(r)\right]\right)=0 \tag{4}$$

Where, u_0 is an initial approximation of equation (1), which satisfies the boundary condition. From equation (4)

$$H(\upsilon,o)=L(\upsilon)-L(u_0)=0 \tag{5}$$

$$H(\upsilon,1)=A(\upsilon)-f(r)=0 \tag{6}$$

The changing process of p from zero to unity is just that of $\upsilon(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and is called Homotopy.

In this paper, we will use the embedding parameter p as a small parameter and assume that the solution of equ^n (4) can be written as a power series of p.

$$\upsilon=\upsilon_0+p\upsilon_1+p^2\upsilon_2+.... \tag{7}$$

Setting p=1, result then approximate solution of equ^n (1)

$$u=\lim_{p\to 1}\upsilon=\upsilon_0+\upsilon_1+\upsilon_2+.... \tag{8}$$

The series (8) is convergent for most cases, however the convergent rate depends upon the non-linear operator $A(\upsilon)$.

2.2 Application Of Homotopy Perturbation Method:

2.2.1 Derivation of Blasius Equation

For a two-dimensional flow, steady state, incompressible flow with zero pressure gradients over a flat plate, governing equation is simplied to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{9}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2} \tag{10}$$

Subjected to boundary conditions

$$\left.\begin{array}{ll} y=0, & u=0 \\ y=\infty, & u=U_\infty, \dfrac{\partial u}{\partial y}=0 \end{array}\right\} \tag{12}$$

Take $x^*=\dfrac{x}{L},\ y^*=\dfrac{y}{\delta},\ u^*=\dfrac{u}{U_\infty},\ \upsilon^*=\dfrac{L\upsilon}{\delta U_\infty},\ p^*=\dfrac{p}{\rho U_\infty^2}$

Take the stream function $\psi=\sqrt{\upsilon x U_\infty}\,f(\eta)$ (13)

Is a dimensionless function of the similarity variable η.

$$\eta=\frac{y}{\sqrt{\upsilon x/U_\infty}} \tag{14}$$

Now

$$u=\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$=\sqrt{\upsilon xU_\infty}\,f'(\eta)\frac{1}{\sqrt{\upsilon x/U_\infty}} \tag{15}$$

$$=U_\infty\frac{df}{d\eta}$$

$$v=-\frac{\partial\psi}{\partial y}=-\left[\frac{\partial}{\partial x}\sqrt{\upsilon xU_\infty}\,f(\eta)+\sqrt{\upsilon xU_\infty}\frac{\partial}{\partial x}f(\eta)\right]$$

Similarly $$=-\left[\frac{1}{2}\sqrt{\frac{\upsilon U_\infty}{x}}+\sqrt{\upsilon xU_\infty}\frac{\partial f}{\partial\eta}\left(\frac{-1}{2}\right)\frac{yx^{-\frac{3}{2}}}{\sqrt{\upsilon/U_\infty}}\right]\sqrt{\upsilon xU_\infty}\,f'(\eta)\frac{1}{\sqrt{\upsilon x/U_\infty}}$$

$$=-\left[\frac{1}{2}f(\eta)\sqrt{\frac{\upsilon U_\infty}{x}}-\frac{1}{2}\frac{U_\infty y}{x}\frac{df(\eta)}{d\eta}\right]$$

$$=\frac{1}{2}\sqrt{\frac{\upsilon U_\infty}{x}}\left[\eta\frac{df}{d\eta}-f\right]$$

Now

$$\frac{\partial u}{\partial y}=U_\infty\frac{d^2f}{d\eta^2}\frac{1}{\sqrt{\upsilon x/U_\infty}}$$

$$=\frac{d^2f}{d\eta^2}\frac{U_\infty}{\sqrt{\upsilon x/U_\infty}}$$

$$\frac{\partial^2u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{U_\infty}{\sqrt{\upsilon x/U_\infty}}\frac{d^2f}{d\eta^2}\right)U_\infty\frac{d^2f}{d\eta^2}\frac{1}{\sqrt{\upsilon x/U_\infty}}$$

$$=\left(\frac{d^3f}{d\eta^3}\frac{1}{\sqrt{\upsilon x/U_\infty}}\right)\frac{U_\infty}{\sqrt{\upsilon x/U_\infty}}$$

$$=\frac{U_{\infty^2}}{\upsilon x}\frac{d^3f}{d\eta^3}$$

Putting these values in following equation, we get

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}=\upsilon\frac{\partial^2u}{\partial y^2}$$

$$\Rightarrow U_\infty\frac{df}{d\eta}\left[\frac{-U_\infty}{2x}\eta\frac{d^2f}{d\eta^2}\right]+\frac{1}{2}\sqrt{\frac{\upsilon U_\infty}{x}}\left[\eta\frac{df}{d\eta}-f\right]\frac{U_\infty}{\sqrt{\upsilon x/U_\infty}}\frac{d^2f}{d\eta^2}=\upsilon\frac{U_{\infty^2}}{\upsilon x}\frac{d^3f}{d\eta^3}$$

$$\Rightarrow-\frac{U_\infty^2}{2x}\eta\frac{df}{d\eta}\frac{d^2f}{d\eta^2}+\frac{1}{2}\frac{U_\infty^2}{2x}\left[\eta\frac{df}{d\eta}-f\right]\frac{d^2f}{d\eta^2}=\frac{U_\infty^2}{x}\frac{d^3f}{d\eta^3}$$

$$\Rightarrow\frac{-\eta}{2}\frac{df}{d\eta}\frac{d^2f}{d\eta^2}+\frac{\eta}{2}\frac{df}{d\eta}\frac{d^2f}{d\eta^2}-\frac{1}{2}f\frac{d^2f}{d\eta^2}=\frac{d^3f}{d\eta^3} \tag{16}$$

$$\Rightarrow\frac{d^3f}{d\eta^3}+\frac{1}{2}f\frac{d^2f}{d\eta^2}$$

With boundary condition,

$$\left.\begin{aligned}\eta = 0 \quad , \quad f = \frac{df}{d\eta} = 0 \\ \eta \to \infty \quad , \frac{df}{d\eta} = 1\end{aligned}\right\} \tag{17}$$

3. Solution of Blasius Equation By Homotopy Perturbation Method

So, to get a solution of equation (16) by the Homotopy technique, we construct a Homotopy

$\upsilon(r,p):\Omega\times[0,1]\to R,$

This satisfies,

$$H(\upsilon,p)=(1-p)\left[L(\upsilon)-L(u_0)\right]+p\left[A(\upsilon)-f(r)\right]=0, \quad p\in[0,1], r\in\Omega$$

$$or \tag{18}$$

$$H(\upsilon,p)=L(\upsilon)-L(u_0)+pL(u_0)+p\left[N(\upsilon)-f(r)\right]=0$$

Where, u_0 is an initial approximation of equation (12), which satisfies the boundary condition .

Now, from equation (16),

$$(1-p)\left(\frac{\partial^3 F}{\partial\eta^3}-\frac{\partial^3 f_0}{\partial\eta^3}\right)+p\left(\frac{\partial^3 F}{\partial\eta^3}+\frac{F}{2}+\frac{\partial^2 F}{\partial\eta^2}\right)=0$$

$$0r \tag{19}$$

$$\left(\frac{\partial^3 F}{\partial\eta^3}-\frac{\partial^3 f_0}{\partial\eta^3}\right)+p\left(\frac{\partial^3 F}{\partial\eta^3}+\frac{F}{2}+\frac{\partial^2 F}{\partial\eta^2}\right)=0$$

Suppose that the solution of the equation (16) to be in the following form

$$F=F_0+pF_1+p^2F_2+\ldots \tag{20}$$

Substituting equ^n (20) in (16) we get,

$$\frac{\partial^3 F_0}{\partial\eta^3}+p\frac{\partial^3 F_1}{\partial\eta^3}+p^2\frac{\partial^3 F_2}{\partial\eta^3}-\frac{\partial^3 f_0}{\partial\eta^3}+p\frac{\partial^3 f_0}{\partial\eta^3}+p\left[\frac{F_0}{2}\left(\frac{\partial^2 F_0}{\partial\eta^2}+p\frac{\partial^2 F_1}{\partial\eta^2}\right)+p\frac{F_1}{2}\left(\frac{\partial^2 F_0}{\partial\eta^2}+p\frac{\partial^2 F_1}{\partial\eta^2}\right)+\ldots\right]=0$$

Re-arranging the co-efficient of the terms with identical powers of p, we have

$$\frac{\partial^3 F_0}{\partial\eta^3}+p\frac{\partial^3 F_1}{\partial\eta^3}+p^2\frac{\partial^3 F_2}{\partial\eta^3}-\frac{\partial^3 f_0}{\partial\eta^3}+p\frac{\partial^3 f_0}{\partial\eta^3}+p\left[\frac{F_0}{2}\left(\frac{\partial^2 F_0}{\partial\eta^2}+p\frac{\partial^2 F_1}{\partial\eta^2}\right)+p\frac{F_1}{2}\left(\frac{\partial^2 F_0}{\partial\eta^2}+p\frac{\partial^2 F_1}{\partial\eta^2}\right)+\ldots\right]=0$$

$$p^0=\frac{\partial^3 F_0}{\partial\eta^3}-\frac{\partial^3 f_0}{\partial\eta^3}=0$$

$$p^1=\frac{\partial^3 F_1}{\partial\eta^3}+\frac{\partial^3 f_0}{\partial\eta^3}+\frac{F_0}{2}\frac{\partial^2 F_0}{\partial\eta^2}=0 \tag{21}$$

$$p^2:\frac{\partial^3 F_2}{\partial\eta^3}+\frac{F_1}{2}\frac{\partial^2 F_0}{\partial\eta^2}+\frac{F_0}{2}\frac{\partial^2 F_1}{\partial\eta^2}=0$$

$$p^3:\frac{\partial^3 F_3}{\partial\eta^3}+\frac{F_1}{2}\frac{\partial^2 F_1}{\partial\eta^2}+\frac{F_2}{2}\frac{\partial^2 F_0}{\partial\eta^2}+\frac{F_0}{2}\frac{\partial^2 F_2}{\partial\eta^2}=0$$

$$. \quad \vdots$$

First we take $F_0 = f_{0.}$ we start iteration by defining $f_{0.}$ as a Taylor series of order two near $\eta = 0,$ so that it could be accurate near $\eta = 0.$

$$F_0=f_0=\frac{f''(0)}{2}\eta^2+f'(0)\eta+f(0) \tag{22}$$

Let us take $f''(0)=0.332057,[8]$ and from the given boundary condition $f=0$ and $f'=0$. So,

$$f = \frac{0.332057}{2}\eta^2$$
$$= 0.1660285\eta^2$$

Now, taking value to solve F_1 ;

$$\frac{\partial^3 F_1}{\partial \eta^3} + \frac{\partial^3 f_0}{\partial \eta^3} + \frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2} = 0$$

$$\frac{\partial^3 F_1}{\partial \eta^3} = -\frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2}$$

$$\frac{\partial^3 F_1}{\partial \eta^3} = -\frac{0.1660285}{2}\eta^2 \frac{\partial^2}{\partial \eta^2}(0.1660285)\eta^2$$

$$\frac{\partial^3 F_1}{\partial \eta^3} = -(0.1660285)^2 \eta^2$$

$$F_1 = -(0.1660285)^2 \frac{\eta^5}{3.4.5}$$

$$\Rightarrow F_1 = f_1 = -0.000045942\eta^5$$

Similarly from (4) we can easily calculate the value of $f_2, f_{3,} \ldots$ as

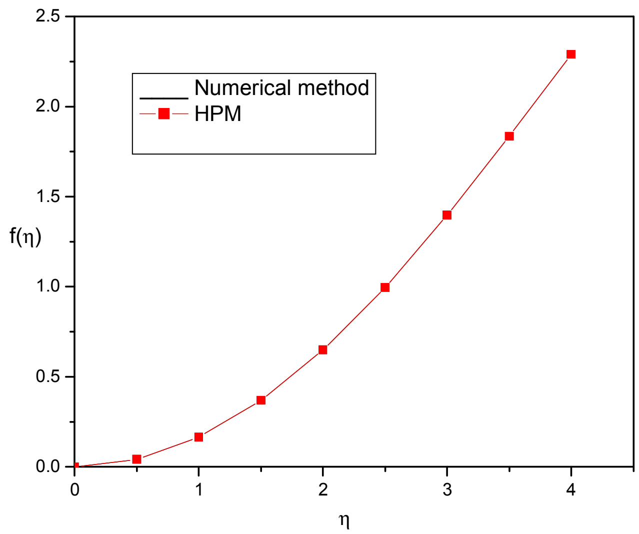
$$f_2 = 0.00000249\eta^8$$
$$f_{3,} = -0.00000001\eta^{11} \tag{23}$$

For the assumption p=1, we get

$$f(\eta) = 0.1660285\eta^2 - 0.0004594\eta^5 + 0.00000249\eta^8 - 0.00000001\eta^{11} \tag{24}$$

4. Result and Discussion:

We have plotted the solution of Blasius equation (both Numerical and HPM) as:



On observing the Plot of flow Profile, we conclude that, the Homotopy perturbation solution is compared with Numerical solution (Runge-Kutta fourth Order method with Efficient shooting Tecnique). Both are overlapped, that shows the both solution are same. As the space variable increases the flow profile enhanced. This Results shows that both Numerical and Analytical methods are not in competition but both are compliment to each other.

References

1. A. H. Nayfeh., Introduction to perturbation technique, Wiley, New York, 1981.
2. J. H. He., Homotopy Perturbation technique. Computer methods in applied mechanics and Engineering, 1999:178:257-262.
3. J. H. He., Homotopy Perturbation method for solving boundary value problems. Physics Letters A, 2006:350:87-88.
4. Ganji, D.D., Soleimani, S., Gorji.M., New application of Homotopy perturbation method, International journal of nonlinear science and numerical simulation VOl.8 (3):2007,(319).
5. Ganji,D.D.,Badazadeh,H.,Noori F.,M.M.,Janipour M.,an application of Homotopy perturbation method for non linear Blasius equation to boundary layer flow over a flat plate ,International Journal of Non-linear Science ,Vol.7,2009,399-404.
6. Shijun Liao, Anotimal Homotopy analysis approach for strong nonlinear different equations CNSNS 15(2010), 2003-2016.
7. E.J.Hinch , Purturbation methods CUP 1991 chapter 1 to 3 5-7.
8. Howarth, L., on the solution of the Laminar Boundary –layer equations, Proceedings of the Royal Society of London, A.164:1983, 547-549.
9. Mahantesh M.Nandeppanavar, K.Vajravelu, P.S.Datti, Optimal Homotopy Asymptotic Solutions for Nonlinear Ordinary Differential Equations Arising in Flow and Heat Transfer due to Nonlinear Stretching Sheet., Heat Transfer-Asian Research,45(1), 2016, 15-29.