Journal of Information Engineering and Applications www.iiste.org
ISSN 2224-5782 (print) ISSN 2225-0506 (online)
Vol.4, No.11, 2014

On Quadruple Random Fixed Point Theorems in Partially Ordered Metric Spaces

Rajesh Shrivastava [1],Ramakant Bhardwaj [2], Manish Sharma [3]
Professor, Department of Mathematics, Institute for Excellence in Higher Education, Bhopal
[2]Department of Mathematics,TIT Group of Institutes, (TIT-Excellence) Bhopal
[3]Truba Institute of Engg. and Information Technology Bhopal, M.P., India

1. Introduction

Bhaskar and Lakshmikantham in [15] introduced the concept of coupled fixed point of a mapping $F: X \times X \to X$ and investigated the existence and uniquencess of a coupled fixed point theorem in partially ordered complete metric space. Lakshmikantham and Ciric [16] defined mixed g-monotone property and coincidence point in partially ordered metric space. V. Berinde and M. Borcut[18] introduced the concept of triple fixed point and proved some related theorems. Following this trand, Karapinar[19] introduced the nation of quadruple fixed point. The object of this note is to prove quadruple random fixed point theorem in partially ordered metric spaces.

2. Preliminaries

Definition 2.1[19]. Let $(X, \leq)$ be a partially ordered set and $F: X^4 \to X$. The map F has the mixed monotone property if $F(x, y, z, t)$ is monotone nondecreasing in x and z and is monotone nonincreasing in y, t; that is, for any $x, y, z, t \in X$,

$$x_1, x_2 \in X, \qquad x_1 \leq x_2 \Rightarrow F(x_1, y, z, t) \leq F(x_2, y, z, t)$$
$$y_1, y_2 \in X, \qquad y_1 \leq y_2 \Rightarrow F(x, y_1, z, t) \geq F(x, y_2, z, t)$$
$$z_1, z_2 \in X, \qquad z_1 \leq z_2 \Rightarrow F(x, y, z_1, t) \leq F(x, y, z_2, t)$$
$$t_1, t_2 \in X, \qquad t_1 \leq t_2 \Rightarrow F(x, y, z, t_1) \geq F(x, y, z, t_2)$$

Definition 2.2[19]. An element $(x, y, z) \in X^4$ is called a quadruple fixed point of a mapping $F: X^4 \to X$ if

$$F(x, y, z, t) = x, \qquad F(y, z, t, x) = y,$$
$$F(z, t, x, y) = z \qquad F(t, x, y, z) = t$$

Definition 2.3[20]. Let $(X, \leq)$ be a partially ordered set and $F: X^4 \to X$ and $g: X \to X$. Then the map F has the mixed g-monotone property if $F(x, y, z, t)$ is monotone g-non-decreasing in x and z and is monotone g-non-increasing in y and t; that is, for any $x, y \in X$.

$$x_1, x_2 \in X, \qquad g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z, t) \leq F(x_2, y, z, t)$$
$$y_1, y_2 \in X, \qquad g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z, t) \geq F(x, y_2, z, t)$$
$$z_1, z_2 \in X, \qquad g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1, t) \leq F(x, y, z_2, t)$$
$$t_1, t_2 \in X, \qquad g(t_1) \leq g(t_2) \Rightarrow F(x, y, z, t_1) \geq F(x, y, z, t_2)$$

Definition 4[20]. An element $(x, y, z, t) \in X^4$ is called a quadruple coincidence point of a mappings $F: X^4 \to X$ and $g: X \to X$ if

$$F(x, y, z, t) = g(x), \qquad F(y, z, t, x) = g(y),$$
$$F(z, t, x, y) = g(z), \qquad F(t, x, y, z) = g(t)$$

Definition 5[20]. Let $F: X^4 \to X$ and $g: X \to X$ be mappings. We say F and g are commutative if

$$g\big(F(x, y, z, t)\big) = F\big(g(x), g(y), g(z), g(t)\big) \qquad \text{for all } x, y, z, t \in X.$$

Let Φ denote the all functions $\varphi: [0, \infty) \to [0, \infty)$ which are continuous and satisfy that

(i) $\varphi(t) < t$,

(ii) $\lim_{r \to t^+} \varphi(t) < t$ for each $t > 0$.

Let (Ω, Σ) be a measurable space with Σ, a sigma algebra of subsets of Ω and let (X, d) be a metric space. A mapping $T: \Omega \to X$ is called measurable if for open subset U of X, $T^{-1}(U) = \{\omega: T(\omega) \in U\} \in \Sigma$. A mapping $T: \Omega \times X \to X$ is said to be random mapping if for each fixed $x \in X$, the mapping $T(., x): \Omega \to X$ is measurable. A measurable mapping $\xi: \Omega \to X$ is called a random fixed point of the random mapping $T: \Omega \times X \to X$ if $T\big(\omega, \xi(\omega)\big) = \xi(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi: \Omega \to X$ is called a random coincidence of $T: \Omega \times X \to X$ and $g: \Omega \times X \to X$ if $T\big(\omega, \xi(\omega)\big) = g\big(\omega, \xi(\omega)\big)$ foe each $\omega \in \Omega$.

3. Main Result

Theorem: Let (X, d) be a complete separable metric space, and let (Ω, Σ) be a measurable space and $\varphi \in \Phi$. Let $F: \Omega \times X^4 \to X$ and $g: \Omega \times X \to X$ be mappings such that

(1) $F(\omega, .), g(\omega, .)$ are continuous for all $\omega \in \Omega$,

(2) $F(.,u), g(.,v)$ are measurable for all $u \in X^4$ and $v \in X$ respectively,
(3) $F: \Omega \times X^4 \to X$ and $g: \Omega \times X \to X$ are such that F has the mixed g-monotone property and

$$d\left(F(\omega,(x,y,z,s)), F(\omega,(u,v,r,t))\right) \le \varphi\left[\max\begin{Bmatrix} d(g(\omega,x),g(\omega,u)), d(g(\omega,y),g(\omega,v)), \\ d(g(\omega,z),g(\omega,r)), d(g(\omega,s),g(\omega,t)) \end{Bmatrix}\right] \quad (1)$$

For all $x,y,z,s,u,v,r,t \in X$ for which $g(\omega,x) \le g(\omega,u), g(\omega,y) \ge g(\omega,v), g(\omega,z) \le g(\omega,r)$ and $g(\omega,s) \ge g(\omega,t)$ for all $\omega \in \Omega$. Suppose $g(\omega \times X) = X$ for each $\omega \in \Omega$ And g is continuous and commutes with F and also suppose either

(a) F is continuous or
(b) X has the following property:
  (i) If a non decreasing sequence $\{x_n\} \to x$ then $x_n \le x$ for all n,
  (ii) If a non increasing sequence $\{y_n\} \to y$ then $y_n \ge y$ for all n.

If there exist measurable mappings $\xi_0, \eta_0, \zeta_0, \rho_0 : \Omega \to X$ such that

$$\left.\begin{aligned} g(\omega,\xi_0(\omega)) &\le F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right),\\ g(\omega,\eta_0(\omega)) &\ge F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right),\\ g(\omega,\zeta_0(\omega)) &\le F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right),\\ g(\omega,\rho_0(\omega)) &\ge F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{aligned}\right\} \quad \text{For all } \omega \in \Omega.$$

Then there are measurable mappings $\xi, \eta, \zeta, \rho : \Omega \to X$ such that

$$\left.\begin{aligned} F\left(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\right) &= g(\omega,\xi(\omega)),\\ F\left(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))\right) &= g(\omega,\eta(\omega)),\\ F\left(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))\right) &= g(\omega,\zeta(\omega)),\\ F\left(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))\right) &= g(\omega,\rho(\omega)) \end{aligned}\right\} \quad \text{For all } \omega \in \Omega.$$

that is, F and g have a quadruple random coincidence point .

Proof . Let $\Theta = \{\xi : \Omega \to X\}$ be a family of measurable mappings. Define a function $h : \Omega \times X \to R^+$ as follows: $h(\omega,x) = d(x, g(\omega,x))$. Since $x \to g(\omega,x)$ is continuous for all $\omega \in \Omega$, we conclude that $h(\omega,.)$ is continuous for all $\omega \in \Omega$. Also, since $\omega \to g(\omega,x)$ is measurable for all $x \in \Omega$, we conclude that $h(\omega,.)$ is measurable for all $\omega \in \Omega$ (see Wagner [11],page 868).Thus, $h(\omega,x)$ is the Caratheodory function. Therefore, if $\xi : \Omega \to X$ is a measurable mapping, then $\omega \to (h(\omega,\xi(\omega))$ is also measurable (see [9]). Also, for each $\xi \in \Theta$, the function $\eta : \Omega \to X$ defined by $\eta(\omega) = g(\omega,\xi(\omega))$ is measurable; that is, $\eta \in \Theta$.

Now, we will construct four sequences of measurable mappings $\{\xi_n\}, \{\eta_n\}, \{\zeta_n\}$ and $\{\rho_n\}$ *in* Θ and four sequences $\{g(\omega,\xi_n(\omega))\}, \{g(\omega,\eta_n(\omega))\}, \{g(\omega,\zeta_n(\omega))\}$, and $\{g(\omega,\rho_n(\omega))\}$ in X as follows:

Let $\xi_0, \eta_0, \zeta_0, \rho_0 \in \Theta$ such that

$$\left.\begin{aligned} g(\omega,\xi_0(\omega)) &\le F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right)\\ g(\omega,\eta_0(\omega)) &\ge F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right)\\ g(\omega,\zeta_0(\omega)) &\le F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right)\\ g(\omega,\rho_0(\omega)) &\ge F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{aligned}\right\} \quad \text{for all } \omega \in \Omega. \quad (2)$$

Since $F(\omega \times X^4) \in X = g(\omega \times X)$, then by a sort of filippov measurable implicit function theorem [1,5,6,24] , we can choose $\xi_1, \eta_1, \zeta_1, \rho_1 \in \Theta$ such that

$$\left.\begin{aligned} g(\omega,\xi_1(\omega)) &= F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right)\\ g(\omega,\eta_1(\omega)) &= F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right)\\ g(\omega,\zeta_1(\omega)) &= F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right)\\ g(\omega,\rho_1(\omega)) &= F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{aligned}\right\} \quad \text{for all } \omega \in \Omega. \quad (3)$$

Again taking into account that $F(\omega \times X^4) \in X = g(\omega \times X)$ and continuing this process, we can construct sequences $\{\xi_n\}, \{\eta_n\}, \{\zeta_n\}$ and $\{\rho_n\}$ in X such that

$$\left.\begin{array}{l} g(\omega, \xi_{n+1}(\omega)) = F\left(\omega, (\xi_n(\omega), \eta_n(\omega), \zeta_n(\omega), \rho_n(\omega))\right), \\ g(\omega, \eta_{n+1}(\omega)) = F\left(\omega, (\eta_n(\omega), \zeta_n(\omega), \rho_n(\omega), \xi_n(\omega))\right), \\ g(\omega, \zeta_{n+1}(\omega)) = F\left(\omega, (\zeta_n(\omega), \rho_n(\omega), \xi_n(\omega), \eta_n(\omega))\right), \\ g(\omega, \rho_{n+1}(\omega)) = F\left(\omega, (\rho_n(\omega), \xi_n(\omega), \eta_n(\omega), \zeta_n(\omega))\right) \end{array}\right\} \quad \text{for all } \omega \in \Omega. \qquad (4)$$

We shall show that

$$\left.\begin{array}{l} g(\omega, \xi_n(\omega)) \leq g(\omega, \xi_{n+1}(\omega)), g(\omega, \eta_{n+1}(\omega)) \leq g(\omega, \eta_n(\omega)), \\ g(\omega, \zeta_n(\omega)) \leq g(\omega, \zeta_{n+1}(\omega)), g(\omega, \rho_{n+1}(\omega)) \leq (\omega, \rho_n(\omega)) \end{array}\right\} \text{ for } n = 0,1,2,.. \qquad (5)$$

For this purpose, we will use mathematical induction. By using (2) and (3), we obtain

$$g(\omega, \xi_0(\omega)) \leq F\left(\omega, (\xi_0(\omega), \eta_0(\omega), \zeta_0(\omega), \rho_0(\omega))\right) = g(\omega, \xi_1(\omega))$$
$$g(\omega, \eta_0(\omega)) \geq F\left(\omega, (\eta_0(\omega), \zeta_0(\omega), \rho_0(\omega), \xi_0(\omega))\right) = g(\omega, \eta_1(\omega))$$
$$g(\omega, \zeta_0(\omega)) \leq F\left(\omega, (\zeta_0(\omega), \rho_0(\omega), \xi_0(\omega), \eta_0(\omega))\right) = g(\omega, \zeta_1(\omega))$$
$$g(\omega, \rho_0(\omega)) \geq F\left(\omega, (\rho_0(\omega), \xi_0(\omega), \eta_0(\omega), \zeta_0(\omega))\right) = g(\omega, \rho_1(\omega))$$

For all $\omega \in \Omega$.
Therefore (5) hold for $n = 0$.
Suppose that (5) hold for some $n > 0$. Then since F has the mixed g-monotone property and by (4) we have

$$\begin{aligned} g(\omega, \xi_{n+1}(\omega)) &= F\left(\omega, (\xi_n(\omega), \eta_n(\omega), \zeta_n(\omega), \rho_n(\omega))\right) \\ &\leq F\left(\omega, (\xi_{n+1}(\omega), \eta_n(\omega), \zeta_n(\omega), \rho_n(\omega))\right) \\ &\leq F\left(\omega, (\xi_{n+1}(\omega), \eta_n(\omega), \zeta_{n+1}(\omega), \rho_n(\omega))\right) \\ &\leq F\left(\omega, (\xi_{n+1}(\omega), \eta_{n+1}(\omega), \zeta_{n+1}(\omega), \rho_n(\omega))\right) \\ &\leq F\left(\omega, (\xi_{n+1}(\omega), \eta_{n+1}(\omega), \zeta_{n+1}(\omega), \rho_{n+1}(\omega))\right) = g(\omega, \xi_{n+2}(\omega)) \end{aligned}$$

$$\begin{aligned} g(\omega, \eta_{n+2}(\omega)) &= F\left(\omega, (\eta_{n+1}(\omega), \zeta_{n+1}(\omega), \rho_{n+1}(\omega), \xi_{n+1}(\omega))\right) \\ &\leq F\left(\omega, (\eta_{n+1}(\omega), \zeta_n(\omega), \rho_{n+1}(\omega), \xi_{n+1}(\omega))\right) \\ &\leq F\left(\omega, (\eta_n(\omega), \zeta_n(\omega), \rho_{n+1}(\omega), \xi_{n+1}(\omega))\right) \\ &\leq F\left(\omega, (\eta_n(\omega), \zeta_n(\omega), \rho_{n+1}(\omega), \xi_n(\omega))\right) \\ &\leq F\left(\omega, (\eta_n(\omega), \zeta_n(\omega), \rho_n(\omega), \xi_n(\omega))\right) = g(\omega, \eta_{n+1}(\omega)) \end{aligned}$$

$$\begin{aligned} g(\omega, \zeta_{n+1}(\omega)) &= F\left(\omega, (\zeta_n(\omega), \rho_n(\omega), \xi_n(\omega), \eta_n(\omega))\right) \\ &\leq F\left(\omega, (\zeta_{n+1}(\omega), \rho_n(\omega), \xi_n(\omega), \eta_n(\omega))\right) \\ &\leq F\left(\omega, (\zeta_{n+1}(\omega), \rho_{n+1}(\omega), \xi_n(\omega), \eta_n(\omega))\right) \\ &\leq F\left(\omega, (\zeta_{n+1}(\omega), \rho_{n+1}(\omega), \xi_{n+1}(\omega), \eta_n(\omega))\right) \\ &\leq F\left(\omega, (\zeta_{n+1}(\omega), \rho_{n+1}(\omega), \xi_{n+1}(\omega), \eta_{n+1}(\omega))\right) = g(\omega, \zeta_{n+2}(\omega)) \end{aligned}$$

$$\begin{aligned} g(\omega, \rho_{n+2}(\omega)) &= F\left(\omega, (\rho_{n+1}(\omega), \xi_{n+1}(\omega), \eta_{n+1}(\omega), \zeta_{n+1}(\omega))\right) \\ &\leq F\left(\omega, (\rho_{n+1}(\omega), \xi_n(\omega), \eta_{n+1}(\omega), \zeta_{n+1}(\omega))\right) \end{aligned}$$

$\leq F\big(\omega, (\rho_n(\omega), \xi_n(\omega), \eta_{n+1}(\omega), \zeta_{n+1}(\omega))\big)$
$\leq F\big(\omega, (\rho_n(\omega), \xi_n(\omega), \eta_{n+1}(\omega), \zeta_n(\omega))\big)$
$\leq F\big(\omega, (\rho_n(\omega), \xi_n(\omega), \eta_n(\omega), \zeta_n(\omega))\big) = g(\omega, \rho_{n+1}(\omega))$

Thus (5) holds for all $n \geq 0$.
Assume, for some $n \in N$, that
$g(\omega, \xi_n(\omega)) = g(\omega, \xi_{n+1}(\omega)), \quad g(\omega, \eta_n(\omega)) = g(\omega, \eta_{n+1}(\omega)),$
$g(\omega, \zeta_n(\omega)) = g(\omega, \zeta_{n+1}(\omega)), \quad g(\omega, \rho_n(\omega)) = g(\omega, \rho_{n+1}(\omega)).$
Then, by (4), $(\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega))$ is a quadruple coincidence point of F and g. From now on, assume for any $n \in N$ that at least
$g(\omega, \xi_n(\omega)) \neq g(\omega, \xi_{n+1}(\omega)), \quad g(\omega, \eta_n(\omega)) \neq g(\omega, \eta_{n+1}(\omega)),$
$g(\omega, \zeta_n(\omega)) \neq g(\omega, \zeta_{n+1}(\omega)), \quad g(\omega, \rho_n(\omega)) \neq g(\omega, \rho_{n+1}(\omega)).$
Due to (1) and (4), we have

$$d\big(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))\big)$$
$$= d\Big(F\big(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega), \zeta_{n-1}(\omega), \rho_{n-1}(\omega))\big), F\big(\omega, (\xi_n(\omega), \eta_n(\omega), \zeta_n(\omega), \rho_n(\omega))\big)\Big)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))\big), d\big(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))\big), \\ d\big(g(\omega, \zeta_{n-1}(\omega)), g(\omega, \zeta_n(\omega))\big), d\big(g(\omega, \rho_{n-1}(\omega)), g(\omega, \rho_n(\omega))\big) \end{array}\right\}\right] \quad (6)$$

$$d\big(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))\big)$$
$$= d\Big(F\big(\omega, (\eta_{n-1}(\omega), \zeta_{n-1}(\omega), \rho_{n-1}(\omega), \xi_{n-1}(\omega))\big), F\big(\omega, (\eta_n(\omega), \zeta_n(\omega), \rho_n(\omega), \xi_n(\omega))\big)\Big)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))\big), d\big(g(\omega, \zeta_{n-1}(\omega)), g(\omega, \zeta_n(\omega))\big), \\ d\big(g(\omega, \rho_{n-1}(\omega)), g(\omega, \rho_n(\omega))\big), d\big(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))\big) \end{array}\right\}\right] \quad (7)$$

$$d\big(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))\big)$$
$$= d\Big(F\big(\omega, (\zeta_{n-1}(\omega), \rho_{n-1}(\omega), \xi_{n-1}(\omega), \eta_{n-1}(\omega))\big), F\big(\omega, (\zeta_n(\omega), \rho_n(\omega), \xi_n(\omega), \eta_n(\omega))\big)\Big)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega, \zeta_{n-1}(\omega)), g(\omega, \zeta_n(\omega))\big), d\big(g(\omega, \rho_{n-1}(\omega)), g(\omega, \rho_n(\omega))\big), \\ d\big(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))\big), d\big(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))\big) \end{array}\right\}\right] \quad (8)$$

$$d\big(g(\omega, \rho_n(\omega)), g(\omega, \rho_{n+1}(\omega))\big)$$
$$= d\Big(F\big(\omega, (\rho_{n-1}(\omega), \xi_{n-1}(\omega), \eta_{n-1}(\omega), \zeta_{n-1}(\omega))\big), F\big(\omega, (\rho_n(\omega), \xi_n(\omega), \eta_n(\omega), \zeta_n(\omega))\big)\Big)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega, \rho_{n-1}(\omega)), g(\omega, \rho_n(\omega))\big), d\big(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))\big), \\ d\big(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))\big), d\big(g(\omega, \zeta_{n-1}(\omega)), g(\omega, \zeta_n(\omega))\big) \end{array}\right\}\right] \quad (9)$$

Having in mind that $\varphi(t) < t$ for all > 0 , so from (6)-(9) we obtain that
$$0 < \max\left\{\begin{array}{l} d\big(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))\big), d\big(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))\big), \\ d\big(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))\big), d\big(g(\omega, \rho_n(\omega)), g(\omega, \rho_{n+1}(\omega))\big) \end{array}\right\}$$

$$\leq \varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\big), d\big(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\big), \\ d\big(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\big), d\big(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\big) \end{array}\right\}\right] \quad (10)$$
$$< \max\left\{\begin{array}{l} d\big(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\big), d\big(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\big), \\ d\big(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\big), d\big(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\big) \end{array}\right\}$$

It follows that

$$\max\left\{\begin{array}{l} d\big(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\big), \\ d\big(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\big), \\ d\big(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\big), \\ d\big(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\big) \end{array}\right\} < \max\left\{\begin{array}{l} d\big(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\big), \\ d\big(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\big), \\ d\big(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\big), \\ d\big(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\big) \end{array}\right\}$$

Thus, $\max\left\{\begin{array}{l} d\big(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\big), d\big(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\big), \\ d\big(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\big), d\big(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\big) \end{array}\right\}$ is a positive decreasing sequence. Hence there exist $r \geq 0$ such that

$$\lim_{n\to\infty}\max\left\{\begin{array}{l} d\big(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\big), d\big(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\big), \\ d\big(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\big), d\big(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\big) \end{array}\right\} = r$$

Suppose that $r > 0$. Letting $n \to \infty$ in (10),we obtain that

$$0 < r \leq \lim_{n\to\infty}\varphi\left[\max\left\{\begin{array}{l} d\big(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\big), d\big(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\big), \\ d\big(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\big), d\big(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\big) \end{array}\right\}\right]$$
$$\leq \lim_{t\to r^+}\varphi(t) < r$$

It is contraction. We deduce that

$$\lim_{n\to\infty}\max\left\{\begin{array}{l} d\big(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\big), d\big(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\big), \\ d\big(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\big), d\big(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\big) \end{array}\right\} = 0 \quad (11)$$

We shall show that there exists $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ and $\{g(\omega,\rho_n(\omega))\}$ are Cauchy sequences. Assume the contrary, that is one of the sequence $\{g(\omega,\xi_n(\omega))\}$,$\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ or $\{g(\omega,\rho_n(\omega))\}$ is not a cauchy sequence, that is,

$$\lim_{m,n\to\infty} d\big(g(\omega,\xi_m(\omega)),g(\omega,\xi_n(\omega))\big) \neq 0 \quad \text{or} \quad \lim_{m,n\to\infty} d\big(g(\omega,\eta_m(\omega)),g(\omega,\eta_n(\omega))\big) \neq 0$$

Or

$$\lim_{m,n\to\infty} d\big(g(\omega,\zeta_m(\omega)),g(\omega,\zeta_n(\omega))\big) \neq 0 \quad \text{or} \quad \lim_{m,n\to\infty} d\big(g(\omega,\rho_m(\omega)),g(\omega,\rho_n(\omega))\big) \neq 0$$

This means that there exist $\varepsilon > 0$, for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\max\left\{\begin{array}{l} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big), d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k}(\omega)\big)\Big), \\ d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k}(\omega)\big)\Big), d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k}(\omega)\big)\Big) \end{array}\right\} \geq \varepsilon \quad (12)$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (12). Then

$$\max\left\{\begin{array}{l} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big), d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k-1}(\omega)\big)\Big), \\ d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k-1}(\omega)\big)\Big), d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k-1}(\omega)\big)\Big) \end{array}\right\} < \varepsilon \quad (13)$$

By triangular inequality and (13), we have

$$d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)$$
$$\leq d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right)+d\left(g\left(\omega,\xi_{n_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)$$
$$<\varepsilon+d\left(g\left(\omega,\xi_{n_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)$$

Letting $k\to\infty$ and using (11), we get

$$\lim_{k\to\infty} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right)\leq\varepsilon$$

Similarly , we have

$$\lim_{k\to\infty} d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k-1}(\omega)\right)\right)\leq\varepsilon$$
$$\lim_{k\to\infty} d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k-1}(\omega)\right)\right)\leq\varepsilon$$
$$\lim_{k\to\infty} d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k-1}(\omega)\right)\right)\leq\varepsilon$$

Again by (13) ,we have

$$\begin{aligned}d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)&\leq d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{m_k-1}(\omega)\right)\right)\\&\quad+\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right)+d\left(g\left(\omega,\xi_{n_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\\&\leq d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{m_k-1}(\omega)\right)\right)+\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{m_k}(\omega)\right)\right)\\&\quad+\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right)+d\left(g\left(\omega,\xi_{n_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\\&< d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{m_k-1}(\omega)\right)\right)+\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{m_k}(\omega)\right)\right)\\&\quad+\varepsilon+d\left(g\left(\omega,\xi_{n_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\end{aligned}$$

Letting $k\to\infty$ and using (11), we get

$$\lim_{k\to\infty} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right)\leq\varepsilon \qquad (14)$$

Similarly , we have

$$\lim_{k\to\infty} d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k-1}(\omega)\right)\right)\leq\varepsilon \qquad (15)$$
$$\lim_{k\to\infty} d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k-1}(\omega)\right)\right)\leq\varepsilon \qquad (16)$$
$$\lim_{k\to\infty} d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\leq \lim_{k\to\infty} d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k-1}(\omega)\right)\right)\leq\varepsilon \qquad (17)$$

Using (12) and (14)-(17), we have

$$\begin{aligned}&\lim_{k\to\infty}\max\left\{\begin{matrix}d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{matrix}\right\}\\&=\lim_{k\to\infty}\max\left\{\begin{matrix}d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k-1}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k-1}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k-1}(\omega)\right)\right)\end{matrix}\right\}\\&=\varepsilon\end{aligned} \qquad (18)$$

Now using inequality (1) we obtain

$$\begin{aligned}&d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\\&=d\left(F\left(\omega,\left(\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\xi_{n_k}(\omega),\eta_{n_k}(\omega),\zeta_{n_k}(\omega),\rho_{n_k}(\omega)\right)\right)\right)\\&\leq\varphi\left[\max\left\{\begin{matrix}d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{matrix}\right\}\right]\end{aligned} \qquad (19)$$

$$d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)$$
$$=d\left(F\left(\omega,\left(\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\eta_{n_k}(\omega),\zeta_{n_k}(\omega),\rho_{n_k}(\omega),\xi_{n_k}(\omega)\right)\right)\right)$$
$$\leq\varphi\left[\max\left\{\begin{array}{l}d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\end{array}\right\}\right] \quad (20)$$

$$d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)$$
$$=d\left(F\left(\omega,\left(\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\zeta_{n_k}(\omega),\rho_{n_k}(\omega),\xi_{n_k}(\omega),\eta_{n_k}(\omega)\right)\right)\right)$$
$$\leq\varphi\left[\max\left\{\begin{array}{l}d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)\end{array}\right\}\right] \quad (21)$$

$$d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)$$
$$=d\left(F\left(\omega,\left(\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\rho_{n_k}(\omega),\xi_{n_k}(\omega),\eta_{n_k}(\omega),\zeta_{n_k}(\omega)\right)\right)\right)$$
$$\leq\varphi\left[\max\left\{\begin{array}{l}d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)\end{array}\right\}\right] \quad (22)$$

From (19) – (22) we deduce that

$$\max\left\{\begin{array}{l}d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{array}\right\}$$
$$\leq\varphi\left[\max\left\{\begin{array}{l}d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{array}\right\}\right] \quad (23)$$

Letting $k\to\infty$ in (23) and having in mind (18) we get that

$$0<\varepsilon\leq\lim_{t\to\varepsilon^+}\varphi(t)<\varepsilon,$$

It is contraction. Thus $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ and $\{g(\omega,\rho_n(\omega))\}$ are Cauchy sequences in (X,d).

Since (X,d) is complete and $g(\omega\times X)=X$ then there exist $\theta_0,\theta_0,\mu_0,\upsilon_0\in\Theta$ such that

$$\left.\begin{array}{l}\lim_{n\to\infty} g(\omega,\xi_n(\omega)) = g(\omega,\theta_0(\omega)),\ \lim_{n\to\infty} g(\omega,\eta_n(\omega)) = g(\omega,\vartheta_0(\omega)),\\ \lim_{n\to\infty} g(\omega,\zeta_n(\omega)) = g(\omega,\mu_0(\omega)),\ \lim_{n\to\infty} g(\omega,\rho_n(\omega)) = g(\omega,\upsilon_0(\omega)).\end{array}\right\} \quad (24)$$

Since $g(\omega,\theta_0(\omega)), g(\omega,\vartheta_0(\omega)), g(\omega,\mu_0(\omega))$ and $g(\omega,\upsilon_0(\omega))$ are measurable, then the function $\xi(\omega),\eta(\omega),\zeta(\omega)$ and $\rho(\omega)$, defined by

$$\left.\begin{array}{l}\xi(\omega) = g(\omega,\theta_0(\omega)),\ \eta(\omega) = g(\omega,\vartheta_0(\omega)),\\ \zeta(\omega) = g(\omega,\mu_0(\omega)),\ \rho(\omega) = g(\omega,\upsilon_0(\omega))\end{array}\right\} \quad (25)$$

Are measurable too. Thus

$$\left.\begin{array}{l}\lim_{n\to\infty} g(\omega,\xi_n(\omega)) = \xi(\omega),\ \lim_{n\to\infty} g(\omega,\eta_n(\omega)) = \eta(\omega),\\ \lim_{n\to\infty} g(\omega,\zeta_n(\omega)) = \zeta(\omega),\ \lim_{n\to\infty} g(\omega,\rho_n(\omega)) = \rho(\omega)\end{array}\right\} \quad (26)$$

Since g is continuous, (26) implies that

$$\left.\begin{array}{l}\lim_{n\to\infty} g\left(\omega, g(\omega,\xi_n(\omega))\right) = g(\omega,\xi(\omega)),\ \lim_{n\to\infty} g\left(\omega, g(\omega,\eta_n(\omega))\right) = g(\omega,\eta(\omega)),\\ \lim_{n\to\infty} g\left(\omega, g(\omega,\zeta_n(\omega))\right) = g(\omega,\zeta(\omega)),\ \lim_{n\to\infty} g\left(\omega, g(\omega,\rho_n(\omega))\right) = g(\omega,\rho(\omega)).\end{array}\right\} \quad (27)$$

by using the fact that F and g are commutative, From (4)

$$\begin{aligned} F\Big(\omega,\big(g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega))\big)\Big) &= g\Big(\omega, F\big(\omega,(\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega))\big)\Big)\\ &= g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big) \end{aligned} \quad (28)$$

$$\begin{aligned} F\Big(\omega,\big(g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega))\big)\Big) &= g\Big(\omega, F\big(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega))\big)\Big)\\ &= g\big(\omega, g(\omega,\eta_{n+1}(\omega))\big) \end{aligned} \quad (29)$$

$$\begin{aligned} F\Big(\omega,\big(g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega))\big)\Big) &= g\Big(\omega, F\big(\omega,(\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega))\big)\Big)\\ &= g\big(\omega, g(\omega,\zeta_{n+1}(\omega))\big) \end{aligned} \quad (30)$$

$$\begin{aligned} F\Big(\omega,\big(g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega))\big)\Big) &= g\Big(\omega, F\big(\omega,(\rho_n(\omega),\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega))\big)\Big)\\ &= g\big(\omega, g(\omega,\rho_{n+1}(\omega))\big) \end{aligned} \quad (31)$$

Now we will show that if the assumption (a) and (b) hold, then

$$\left.\begin{array}{l} F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big) = g(\omega,\xi(\omega)),\\ F\big(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))\big) = g(\omega,\eta(\omega)),\\ F\big(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))\big) = g(\omega,\zeta(\omega)),\\ F\big(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))\big) = g(\omega,\rho(\omega)),\end{array}\right\} \quad \text{For all } \omega\in\Omega .$$

Suppose (a) hold from (26), (27), (28) and the continuity of F, we obtain

$$\begin{aligned} g(\omega,\xi(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big)\\ &= \lim_{n\to\infty} F\Big(\omega,\big(g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega))\big)\Big)\\ &= F\Big(\omega,\big(\lim_{n\to\infty} g(\omega,\xi_n(\omega)), \lim_{n\to\infty} g(\omega,\eta_n(\omega)), \lim_{n\to\infty} g(\omega,\zeta_n(\omega)), \lim_{n\to\infty} g(\omega,\rho_n(\omega))\big)\Big) \end{aligned}$$

$$= F\big(\omega, (\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega))\big)$$

and similarly

$$\begin{aligned}
g(\omega, \eta(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega, \eta_{n+1}(\omega))\big) \\
&= \lim_{n\to\infty} F\Big(\omega, \big(g(\omega, \eta_n(\omega)), g(\omega, \zeta_n(\omega)), g(\omega, \rho_n(\omega)), g(\omega, \xi_n(\omega))\big)\Big) \\
&= F\Big(\omega, \big(\lim_{n\to\infty} g(\omega, \eta_n(\omega)), \lim_{n\to\infty} g(\omega, \zeta_n(\omega)), \lim_{n\to\infty} g(\omega, \rho_n(\omega)), \lim_{n\to\infty} g(\omega, \xi_n(\omega))\big)\Big) \\
&= F\big(\omega, (\eta(\omega), \zeta(\omega), \rho(\omega), \xi(\omega))\big) \\
g(\omega, \zeta(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega, \zeta_{n+1}(\omega))\big) \\
&= \lim_{n\to\infty} F\Big(\omega, \big(g(\omega, \zeta_n(\omega)), g(\omega, \rho_n(\omega)), g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega))\big)\Big) \\
&= F\Big(\omega, \big(\lim_{n\to\infty} g(\omega, \zeta_n(\omega)), \lim_{n\to\infty} g(\omega, \rho_n(\omega)), \lim_{n\to\infty} g(\omega, \xi_n(\omega)), \lim_{n\to\infty} g(\omega, \eta_n(\omega))\big)\Big) \\
&= F\big(\omega, (\zeta(\omega), \rho(\omega), \xi(\omega), \eta(\omega))\big) \\
g(\omega, \rho(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega, \rho_{n+1}(\omega))\big) \\
&= \lim_{n\to\infty} F\Big(\omega, \big(g(\omega, \rho_n(\omega)), g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega)), g(\omega, \zeta_n(\omega))\big)\Big) \\
&= F\Big(\omega, \big(\lim_{n\to\infty} g(\omega, \rho_n(\omega)), \lim_{n\to\infty} g(\omega, \xi_n(\omega)), \lim_{n\to\infty} g(\omega, \eta_n(\omega)), \lim_{n\to\infty} g(\omega, \zeta_n(\omega))\big)\Big) \\
&= F\big(\omega, (\rho(\omega), \xi(\omega), \eta(\omega), \zeta(\omega))\big)
\end{aligned}$$

Thus , we proved that $(\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega)) \in X^4$ is a quadruple random coincidence of F and g.

Suppose, now the assumption (b) holds. Since

$$g(\omega, \xi_n(\omega)) \le g(\omega, \theta_0(\omega)) = \xi(\omega),$$

$$g(\omega, \eta_n(\omega)) \ge g(\omega, \theta_0(\omega)) = \eta(\omega),$$

$$g(\omega, \zeta_n(\omega)) \le g(\omega, \mu_0(\omega)) = \zeta(\omega),$$

$$g(\omega, \rho_n(\omega)) \ge g(\omega, \upsilon_0(\omega)) = \rho(\omega).$$

Therefore, by the triangle inequality

$$\begin{aligned}
&d\Big(g(\omega, \xi(\omega)), F\big(\omega, (\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega))\big)\Big) \le d\Big(g(\omega, \xi(\omega)), g\big(\omega, g(\omega, \xi_{n+1}(\omega))\big)\Big) \\
&\qquad + d\Big(g\big(\omega, g(\omega, \xi_{n+1}(\omega))\big), F\big(\omega, (\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega))\big)\Big) \\
&\le d\Big(g(\omega, \xi(\omega)), g\big(\omega, g(\omega, \xi_{n+1}(\omega))\big)\Big) \\
&+ d\Big(F\Big(\omega, \big(g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega)), g(\omega, \zeta_n(\omega)), g(\omega, \rho_n(\omega))\big)\Big), F\big(\omega, (\xi(\omega), \eta(\omega), \zeta(\omega), \rho(\omega))\big)\Big) \\
&\le d\Big(g(\omega, \xi(\omega)), g\big(\omega, g(\omega, \xi_{n+1}(\omega))\big)\Big) \\
&+ \varphi\left[\max\left\{\begin{array}{l} d\Big(g\big(\omega, g(\omega, \xi_n(\omega))\big), g(\omega, \xi(\omega))\Big), d\Big(g\big(\omega, g(\omega, \eta_n(\omega))\big), g(\omega, \eta(\omega))\Big), \\ d\Big(g\big(\omega, g(\omega, \zeta_n(\omega))\big), g(\omega, \zeta(\omega))\Big), d\Big(g\big(\omega, g(\omega, \rho_n(\omega))\big), g(\omega, \rho(\omega))\Big) \end{array}\right\}\right]
\end{aligned}$$

And since $\varphi(t) < t$, we have

$$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) < d\Big(g(\omega,\xi(\omega)),g\big(\omega,g(\omega,\xi_{n+1}(\omega))\big)\Big)$$

$$+\max\left\{\begin{matrix} d\Big(g\big(\omega,g(\omega,\xi_n(\omega))\big),g(\omega,\xi(\omega))\Big),d\Big(g\big(\omega,g(\omega,\eta_n(\omega))\big),g(\omega,\eta(\omega))\Big), \\ d\Big(g\big(\omega,g(\omega,\zeta_n(\omega))\big),g(\omega,\zeta(\omega))\Big),d\Big(g\big(\omega,g(\omega,\rho_n(\omega))\big),g(\omega,\rho(\omega))\Big) \end{matrix}\right\}$$

Letting $n \to \infty$ and by (27), we get

$$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) \le 0$$

But $$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) \ge 0$$

Hence $$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) = 0$$

Hence $$F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big) = g(\omega,\xi(\omega))$$

Similarly, we can show that

$$F\big(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))\big) = g(\omega,\eta(\omega)),$$

$$F\big(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))\big) = g(\omega,\zeta(\omega)),$$

$$F\big(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))\big) = g(\omega,\rho(\omega)).$$

For all $\omega \in \Omega$.

Thus we showed that $(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega)) \in X^4$ is a quadruple random coincidence of F and g.

References

[1] I. Beg and N. Shahzad, "Random fixed point theorems for non expansive and contractive-type random operators on Banach spaces," Journal of Applied Mathematics and Stochastic Analysis,
vol. 7, no. 4, pp. 569–580,1994.
[2] A. T. Bharucha-Reid, "Fixed point theorems in probabilistic analysis," Bulletin of the American Mathematical Society, vol. 82, no. 5, pp. 641–657, 1976.
[3] O. Hans, "Reduzierende zulliallige transformaten," Czechoslovak Mathematical Journal, vol. 7, no. 82, pp. 154–158, 1957.
[4] O. Hans, "Random operator equations," in Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, vol. 2, pp. 185–202, University California Press, Berkeley, Calif, USA, 1961.
[5] C. J. Himmelberg, "Measurable relations," Fundamenta Mathematical ,vol. 87, pp. 53–72, 1975.
[6] S. Itoh, "A random fixed point theorem for a multi valued contraction mapping," Pacific Journal of Mathematics, vol. 68, no. 1, pp. 85–90, 1977.
[7] S. Itoh, "Random fixed-point theorems with an application to random differential equations in Banach spaces," Journal of Mathematical Analysis and Applications, vol. 67, no. 2, pp. 261–273, 1979.
[8] N. S. Papageorgiou, "Random fixed point theorems for measurable multifunctions in Banach spaces," Proceedings of the American Mathematical Society, vol. 97, no. 3, pp. 507–514, 1986.
[9] R. T. Rockafellar, "Measurable dependence of convex sets and functions on parameters," Journal of Mathematical Analysis and Applications, vol. 28, pp. 4–25, 1969.
[10] A. ˇSpaˇcek, "Zuf¨allige gleichungen," Czechoslovak Mathematical Journal, vol. 5, no. 80, pp. 462–466, 1955.
[11] D. H.Wagner, "Survey of measurable selection theorems," SIAM Journal on Control and Optimization, vol. 15, no. 5, pp. 859–903,1977.
[12] H. K. Xu, "Some random fixed point theorems for condensing and non-expansive operators," Proceedings of the American Mathematical Society, vol. 110, no. 2, pp. 395–400, 1990.

[13] M. Turinici, “Abstract comparison principles and multivariable Gronwall-Bellman inequalities,” Journal of Mathematical Analysis and Applications, vol. 117, no. 1, pp. 100–127, 1986.
[14] A. C. M. Ran and M. C. B. Reurings, “A fixed point theorem in partially ordered sets and some applications to matrix equations,” Proceedings of the American Mathematical Society,vol. 132, no. 5, pp. 1435–1443, 2004.
[15] T. G. Bhaskar and V. Lakshmikantham, “Fixed point theorems in partially ordered metric spaces and applications,” Nonlinear Analysis: Theory, Methods &Applications, vol. 65,no. 7, pp. 1379–
1393, 2006.
[16] V. Lakshmikantham and L. B. ´ Ciric, “Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces,”NonlinearAnalysis:Theory,Methods&Applications, vol. 70, no. 12, pp. 4341–4349, 2009.
[17] L. B. ´ Ciri´c and V. Lakshmikantham, “Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces,” Stochastic Analysis and Applications, vol. 27, no.
6, pp. 1246–1259, 2009.
[18] V. Berinde and M. Borcut, “Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces,” Nonlinear Analysis: Theory,Methods & Applications, vol. 74, no.
15, pp. 4889–4897, 2011.
[19] E. Karapinar, “Quartet fixed point for nonlinear contraction,”submitted to General Topology, 2011, http://arxiv.org/pdf/1106 .5472.pdf.
[20] E. Karapinar and V. Berinde, “Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces,” Banach Journal of Mathematical Analysis, vol. 6, no. 1, pp. 74–89, 2012.
[21] E. Karapınar andN. V. Luong, “Quadruple fixed point theorems for nonlinear contractions,” Computers & Mathematics with Applications, vol. 64, no. 6, pp. 1839–1848, 2012.
[22] Z. Mustafa, H. Aydi, and E. Karapınar, “Mixed g-monotone property and quadruple fixed point theorems in partially orderedmetric spaces,” Fixed Point Theory andApplications, vol.2012, article 71, 2012.
[23] E. Karapınar, W. Shatanawi, and Z. Mustafa, “Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces,” Journal of Applied Mathematics,vol. 2012, Article ID951912, 17 pages, 2012.
[24] E. J. McShane and R. B. Warfield, Jr., “On Filippov’s implicit functions lemma,” Proceedings of the American Mathematical Society, vol. 18, pp. 41–47, 1967.