Mathematical Model of Ethiopia's Population Growth

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Abstract
The purpose of this paper is to use mathematical models to predict the population growth of Ethiopia. Ethiopia is an overpopulated country in Africa next to Nigeria. It shares a border with Eritrea to the North and Northeast, Djibouti and Somalia to the East, Sudan and South Sudan to the West, and Kenya to the South. The Malthus's and the logistic growth models are applied to model the population growth of Ethiopia using data from 1980 to 2016. The data used was collected from International Data Base (IDB). We also use the least square method to compute the best population growth rate of Malthus's model. The Malthus's population model predicted a growth rate of 2.9% per year while the logistic growth model predicted the carrying capacity of 19767334710 and growth rate of 2.9% per annum. The growth rate for both models match well with the growth rate estimated by International Data Base for the past four years. The Mean Absolute Percentage Error is computed as 0.64% for Malthus's population model and 0.73% for logistic growth Model. This showed that the Malthus's population model seems to fit the original data the best among the models we tried.
Keywords: Malthus's population model, Logistic growth Model, Least square method, Population growth rate, Carrying Capacity, Mean Absolute Percentage Error.

1. Introduction

Ethiopia is an overpopulated country in Africa next to Nigeria, and the growth in resources has not been keeping pace with the growth in population. So, the increasing trend in population is a great threat to the nation. Population projections can be used for a number of purposes. Perhaps the most important use of population projections is in the role they can play as a rational basis for decision-making. Changes in population size and composition have many social, economic, environmental, and political implications. For this reason, population projections often serve as a basis for producing other projections (e.g., births, households, families, school enrollment, and labor force).

National population projections, for example, can be used to plan for future Social Security and Medicare obligations (Lee and Tuljapurkar, 1997; Miller, 2001). State projections can be used to determine future water demands (Texas Water Development Board, 1997). Local projections can be used to determine the need for new public schools (Swanson et al., 1998) and to select sites for fire stations (Tayman, Parrott, and Carnevale, 1994). Population projections can be used to forecast the demand for housing (Mason, 1996), the number of people with disabilities (Michaud, George, and Loh, 1996), and the number of sentenced criminals (Oregon Office of Economic Analysis, 2000).

Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumptions for their planning activities. To obtain this information, the behavior of the connected variables is analyzed based on the previous data by the statisticians and mathematicians at first, and using the conclusions drawn from the analysis, they make future projections of the aimed at variable. There are enormous concerns about the consequences of human population growth for social, environment and economic development. It is population growth that intensifies all these problems. Mathematical modeling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences. Thus, it is a process of mimicking reality by using the language of mathematics. Many people examine population growth through observation, experimentation or through mathematical modeling. Mathematical models can take many forms, including but not limited to dynamical systems, statistical models and differential equations.

In this paper we model the population growth of Ethiopia by using Malthus and logistic growth models. Furthermore, the paper gives an insight on how to determine the net growth rate of Malthus's population model by using the least square method.

2. Methodology

A research is best understood as a processing of arriving at dependent solutions to the problems through the systematic collection, analysis and interpretation of data. In this paper, secondary classified yearly population data of Ethiopia from $1980 - 2016$ (inclusive) were taken from International Data Base (IDB) [9]. The Malthus and logistic growth mathematical models and Least Square Method were applied to compute the projected population values by employing MATLAB. The goodness of the model was assessed using the Mean Absolute Percentage Error.

3.   The Malthus's Population Model

The simple differential equation

$$\frac{dP}{dt} = \beta P, \qquad \beta > 0 \qquad (1)$$

was proposed in 1798 by the English economist Thomas Malthus [10] as a basic model for population growth. Here the increase in the population is taken to be proportional to the total number of people, P, and β is the constant representing the rate of growth (the difference between the birth rate and death rate). This is the reasonable assumption for the population of bacteria or animal under ideal conditions (unlimited environment, adequate nutrition, absence of predators, and immunity from disease). Suppose we know the population P_0 at some given time $t = t_0$, and we are interested for projecting the population, P, at some future time t. The solution of Malthus population model in equation (1) is obtained as follows:

$$\int_{P_0}^{P(t)} \frac{1}{P} dP = \int_{t_0=0}^{t} \beta dt \qquad (2)$$

This gives that

$$P(t) = P_0 e^{\beta t} \qquad (3)$$

This model is often referred to as the exponential law and is widely regarded in the population ecology as the first principle of population of Dynamics. Rearranging equation (3), we can obtain the equation,

$$\beta = \frac{\ln P(t) - \ln P_0}{t} \qquad (4)$$

Thus, the Malthusian parameter β is estimated from equation (4).

4.   Logistic Growth Model

As population increases in size, the environment's ability to support the population decreases. As the population increases per capita, food availability decreases, waste products may accumulate and birth rates tend to decline while death rates tend to increase. Thus, it seems that reasonable to consider the mathematical model which explicitly incorporates the idea of carrying capacity (limiting value).

In 1838 Pierre-Francois Verhulst [11] proposed a model called the logistic model, for population growth. His model does not assume unlimited resources instead it incorporated the idea of the carrying capacity for the population. Thus, the population growth not only on how to depend on the population size but also on how this size is far from its upper limit (maximum supportable population). He modified Malthus's model to make the rate of change $\frac{dP}{dt}$ of the population P is proportional to the product of the current population P and $\left(1 - \frac{P}{K}\right)$. This gives us the logistic differential equation,

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \qquad r, K > 0 \qquad (5)$$

Where, r is the growth rate of population and K is the maximum sustainable population. When the population P is small compared to the parameter K, P^2 is very small, so equation (5) is approximated by $\frac{dP}{dt} = rP$. If the population P is above K, the population decrease, but if below, then it increases.

The Logistic equation has been solved by the method of separation of variables in [15] as follows: Separating the variables gives

$$\left(\frac{1}{P} + \frac{1}{K - P}\right) dP = rdt \qquad (6)$$

We can integrate both sides of equation (6) between the limits corresponding to times $t_0 = 0$ and t,

$$\int_{P_0}^{P(t)} \left(\frac{1}{P} + \frac{1}{K - P}\right) dP = \int_{t_0=0}^{t} rdt$$

To give

$$\frac{P(t)(K - P_0)}{P_0(K - P(t))} = e^{rt}$$

Finally, rearranging this gives

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}} \tag{7}$$

If we take the limit of equation (7) as $t \to \infty$, we get

$$P_{max} = \lim_{t \to \infty} P = K \tag{8}$$

He also explained that how the parameters r and K can be estimated from the population $P(t)$ in three different but equally spaced years. If P_0 is the population at time $t = 0$, P_T that at time $t = T$ and P_{2T} that at time $t = 2T$, then from equation (7), as in $[13]$, we can obtain r and K. Implies that

$$\frac{1}{K}(1 - e^{-rT}) = \frac{1}{P_T} - \frac{e^{-rT}}{P_0} \tag{9}$$

$$\frac{1}{K}(1 - e^{-2rT}) = \frac{1}{P_{2T}} - \frac{e^{-2rT}}{P_0} \tag{10}$$

Dividing equation (10) by the corresponding equation (9) to eliminate K we get

$$1 + e^{-rT} = \frac{\dfrac{1}{P_{2T}} - \dfrac{e^{-2rT}}{P_0}}{\dfrac{1}{P_T} - \dfrac{e^{-rT}}{P_0}}$$

So that

$$e^{-rT} = \frac{P_0(P_{2T} - P_T)}{P_{2T}(P_T - P_0)} \tag{11}$$

Equation (11) gives sense if $0 < \frac{P_0(P_{2T} - P_1)}{P_{2T}(P_T - P_0)} < 1$.

From equation (11), the growth rate computed as

$$r = \frac{1}{T}\ln\left(\frac{P_{2T}(P_T - P_0)}{P_0(P_{2T} - P_T)}\right) \tag{12}$$

Plugging this value of r into equation (9), we get the limiting value of P,

$$K = \frac{P_T(P_0 P_T - 2P_0 P_{2T} + P_T P_{2T})}{P_T{}^2 - P_0 P_{2T}} \tag{13}$$

In $[14]$, the inflection point of logistic function is a point $t = \tau$ at which the second derivative $P''(t)$ changes its sign only on either sides of zero or does not exist. In $[12]$, the inflection point of the logistic function has been solved as follows: For the logistic function in equation (7), $P'(t)$ and $P''(t)$ are given respectively by $P'(t) = rP'(t)\left(1 - \frac{P(t)}{K}\right)$ and $P''(t) = rP'(t)\left(1 - \frac{2P(t)}{K}\right)$. Here, we observe that $P''(t) = 0$ that is $\left(1 - \frac{2P(t)}{K}\right) = 0$ which gives $P(t) = \frac{K}{2}$ and this implies:

$$\tau = \frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right) \tag{14}$$

Equation (14) is the point of inflection, since $P''(t) = 0$ at the point and also it changes sign only on either sides of zero or does not exist. Hence, for logistic curve, the single point of inflection occurs when the growth reaches half of its final growth $P(\tau) = \frac{K}{2}$ at time given by (14).

5. Method of Least Squares

In many branches of Applied Mathematics, Engineering and Sciences we come across experiments and problems, which involve two or more variables. Here, our focus is to fit the exponential function,

$$P(t) = be^{dt}. \tag{15}$$

Here, b and d are constants to be determined. For this purpose, we take several sets of observations of time and the corresponding population. The problem is to find the best values for b and d using the observed values of t and P. Thus, the principle of least square provides unique values to the constants b and d and hence, suggests a

curve of the best fit to the given data. The method of least square is probably the most systematic procedure to fit a unique curve through the observed data points.

Suppose that we are given the points $(t_0, P_0), (t_1, P_1), \ldots, (t_{n-1}, P_{n-1})$ and we want to fit an exponential function of the form equation (15).

Taking the logarithm of both sides:

$$\ln P = \ln b + dt$$

Then, introduce change of variables:

$$Y = \ln P, \quad c = \ln b$$

This results in a linear relation between the variables Y and t,

$$Y = c + dt \tag{16}$$

Thus, the residual e_i is given as

$$e_i = (Y_i - c - dt_i), \quad i = 0,1,2,\ldots,n-1.$$

Now, we have the sum squared deviation,

$$S = \sum_{i=0}^{n-1} e_i^2 = \sum_{i=0}^{n-1} (Y_i - c - dt_i)^2 \tag{17}$$

By the least squares criterion, the best-fitting curve is that for which the sum of the squared deviations between the estimated and actual values of the dependent variable for the sample data is minimized. Thus,

$$\frac{\partial S}{\partial c} = -2 \sum_{i=0}^{n-1} (Y_i - c - dt_i) = 0, \quad \frac{\partial S}{\partial d} = -2 \sum_{i=0}^{n-1} (Y_i - c - dt_i)(t_i) = 0 \tag{18}$$

On solving equation (18), we have the system equations,

$$nc + d \sum_{i=0}^{n-1} t_i = \sum_{i=0}^{n-1} Y_i$$

$$c \sum_{i=0}^{n-1} t_i + d \sum_{i=0}^{n-1} t_i^2$$

$$= \sum_{i=0}^{n-1} t_i Y_i \tag{19}$$

Equation (19) is called a normal equation which is used to find the parameter values of b and d that will minimize the least square error $[16]$. After c and d have been found, the parameter b is computed as $b = e^c$. Putting the value of b and d in equation (15), we get the equation of the line that best fits.

6. Estimating Optimal Solution to Malthus's Model

In this section, we use least square method to predict the value of β so that the Malthus's population model to have the optimal solution. Thus, method of least square serves as benchmark to compute the best value of parameter β to the Malthus's model.

For example, the exponential function in (3) has an option to have different values of β. In other words, for $(n+1)$ number of population, we have (n) values of β computed from equation (4). However, all these values of β may not permit the exponential function in (3) nicely approximate the actual population. As a result, it is compulsory to look for a unique and the best value of β so that the exponential function in (3) approximates the actual population excellently. So, here we propose that $P_0 e^{\beta t} \approx b e^{dt}$ which gives that $e^{(\beta - d)t} \approx \frac{b}{P_0}$. After simplification we have $(\beta - d) \approx \frac{1}{t} \ln\left(\frac{b}{P_0}\right)$. Thus, $P_0 e^{\beta t} \approx b e^{dt}$ whenever $\beta \approx d$. Now, let we define the absolute deviation (ϵ) as

$$\epsilon = |\beta - d|$$

Previously we have stated that method of least square gives unique and best values of b and d to equation (15). Once we have determined the value of d and all values of β from equations (19) and (4), respectively, we

need to compute the absolute deviation (ϵ). Thus, the value at which the absolute deviation is a minimum is the best value of β.

Mathematically, first, compute β's values from equation (4) modified as:

$$\beta_i = \frac{\ln P(t_i) - \ln P_0}{t_i}, \quad i = 1, 2, \ldots, n.$$

Then, compute the absolute deviation (ϵ), i.e.

$$\epsilon_i = |\beta_i - d|, \quad i = 1, 2, \ldots, n. \tag{20}$$

Once we have obtained the absolute deviations from equation (20), we choose the most minimum absolute deviation (i.e. $\min\{\epsilon\}$). Finally, we can determine the time and corresponding population at which the absolute deviation is a minimum. Substituting these values of β and P_0 into equation (3), and it approximates the actual population nicely than others.

Earlier, we have stated that the logistic growth model is approximated by the Malthus's population model when $P \ll K$. Under this assumption, it is possible to obtain that the solution of the logistic growth model, and for the solution of logistic growth model, we need to use the time t and the corresponding population P at which the absolute deviation in equation (20) is diminishing, so as to estimate the values of the parameters r and K to equation (7). However, as P tends towards K, the above assumption may be rejected.

7. Mean Absolute Percentage Error (MAPE)

The mean absolute percentage error as given in $[17]$, is defined as

$$MAPE = \frac{1}{N} \sum \left| \frac{X_t - \breve{X}}{X_t} \right| * 100 \tag{21}$$

Where X_t, $\breve{X}$ and N are the actual, fitted and number of observation of the population respectively.

The MAPE is a relative measure which expresses errors *as a* percentage of the actual data. This is its biggest advantage as it provides an easy and intuitive way of judging the extent, or importance of errors. Lower MAPE values are better because they indicate that smaller percentages' errors are produced by the forecasting model.

8. Result

To estimate the future population of Ethiopia, first we need to determine the constants b and d using the exponential function in (15). Using the actual population of Ethiopia in Table 1 given below, and MATLAB programs, [appendix [A]], we find the values of c and d from the following system of equations:

$$37c + 666d = 663$$
$$666c + 16206d = 12058 \tag{22}$$

So, $c = 17.398$, $d = 0.02904093 \approx 0.029$ and $b = e^c = 35963083$.

Substituting these values of b and d into equation (15), we obtain:

$$P(t) = 35963083 e^{0.029t} \tag{23}$$

This equation can be used to estimate the value of the dependent variable P given the value of the independent variable t, and it is plotted below, and compared with the actual population of Ethiopia.

Table 2 given below is computed from the equation (20). From this table one can read the minimum absolute deviation $\epsilon = 1.42E - 05$, and then we can easily understand that time $t = 18 \ (1998)$ and $\beta = 0.0290267 \approx 0.029$.

Substituting these values of $\beta = 0.029$ and $P_0 = 36036457$ into equation (3), the Malthus's population model has the optimal solution,

$$P(t) = 36036457 e^{0.029t} \tag{24}$$

However, it needs to be recognized that this solution is optimal only with respect to the data being used in Table 1. Thus, the value of the population growth rate of Ethiopia is approximately 2.9% per annum with the Malthus's population model. Equation (24) is sketched below, and compared with the data for the growth of

the population of Ethiopia.

For the logistic growth model, we use $t = T = 18$ and $t = 2T = 36$ with the corresponding values of $P_{18} = 60764879$ and $P_{36} = 10237404$, respectively to predict the parameters r and K.

Substituting the values of P_0, P_{18} and P_{36} into equation (13), we get $K = 19767334710$. This value of K is the predicted carrying capacity or the limiting value of the population of Ethiopia. From equation (12), we obtain the value of $r = 0.029096383 \approx 0.029$. This implies that the predicted rate of Ethiopia's growth is approximately 2.9% per year with the logistic growth model.

Finally, substituting the values of P_0, r and K into equation (7) gives the general solution of the logistic growth model,

$$P(t) = \frac{19767334710}{1 + 547.54e^{-0.029t}} \qquad (25)$$

Based on the data given in Table 1, from equation (14), we obtain the time (year) $t = 2197$ at which the growth reaches half of its final growth $P(2197) = 9883667355$. In other words Ethiopian population is growing most rapidly when there are 9883667355 populations.

From equation (21), the mean absolute percentage errors of equations $(23), (24)$ and (25) are computed as 0.66%, 0.64% and 0.73% respectively.

Table 1: Shows the actual population of Ethiopia from $1980 - 2016$.

Year	Actual Pop.	Year	Actual Pop.	Year	Actual Pop.	Year	Actual Pop.
1980	36036457	1990	47554116	2000	64365225	2010	86042927
1981	37720093	1991	49447575	2001	66230452	2011	88588758
1982	39478541	1992	51205192	2002	68167888	2012	91195675
1983	40673915	1993	52525039	2003	70174826	2013	93877025
1984	40072730	1994	54030146	2004	72238990	2014	96633458
1985	40683742	1995	55725462	2005	74354300	2015	99465819
1986	41642553	1996	57424029	2006	76535275	2016	102374044
1987	43058596	1997	59076269	2007	78799447		
1988	44596107	1998	60764879	2008	81135648		
1989	46093687	1999	62538426	2009	83548430		

Table 2: Shows the absolute deviation computed using equation (20) with the corresponding values of t, P, β and d where time (t) is $(1981 - 2016)$ and P is the corresponding actual population.

t	P	β	d	$\|\beta - d\|$	t	P	β	d	$\beta - d$
1	37720093	0.045662	0.029	0.0166209	19	62538426	0.029013	0.029	2.78E-05
2	39478541	0.045613	0.029	0.0165721	20	64365225	0.029002	0.029	3.88E-05
3	40673915	0.040352	0.029	0.011311	21	66230452	0.028981	0.029	5.95E-05
4	40072730	0.026541	0.029	0.0024997	22	68167888	0.028975	0.029	6.63E-05
5	40683742	0.024259	0.029	0.0047814	23	70174826	0.028976	0.029	6.45E-05
6	41642553	0.024099	0.029	0.0049424	24	72238990	0.028977	0.029	6.39E-05
7	43058596	0.025433	0.029	0.003608	25	74354300	0.028972	0.029	6.85E-05
8	44596107	0.026639	0.029	0.0024015	26	76535275	0.02897	0.029	7.09E-05
9	46093687	0.027349	0.029	0.0016915	27	78799447	0.028977	0.029	6.41E-05
10	47554116	0.027734	0.029	0.0013072	28	81135648	0.028985	0.029	5.55E-05
11	49447575	0.028762	0.029	0.0002789	29	83548430	0.028996	0.029	4.45E-05
12	51205192	0.029276	0.029	0.0002349	30	86042927	0.029011	0.029	3.04E-05
13	52525039	0.028981	0.029	5.95E-05	31	88588758	0.029015	0.029	2.56E-05
14	54030146	0.028929	0.029	0.0001116	32	91195675	0.029015	0.029	2.60E-05
15	55725462	0.02906	0.029	1.95E-05	33	93877025	0.029014	0.029	2.72E-05
16	57424029	0.029121	0.029	7.98E-05	34	96633458	0.029012	0.029	2.93E-05
17	59076269	0.029076	0.029	3.54E-05	35	99465819	0.029008	0.029	3.28E-05
18	**60764879**	**0.029027**	**0.029**	**1.42E-05**	36	102374044	0.029003	0.029	3.81E-05

Table 3: Shows the projected population of Ethiopia using Malthus's model, Logistic Model and Least Square Method.

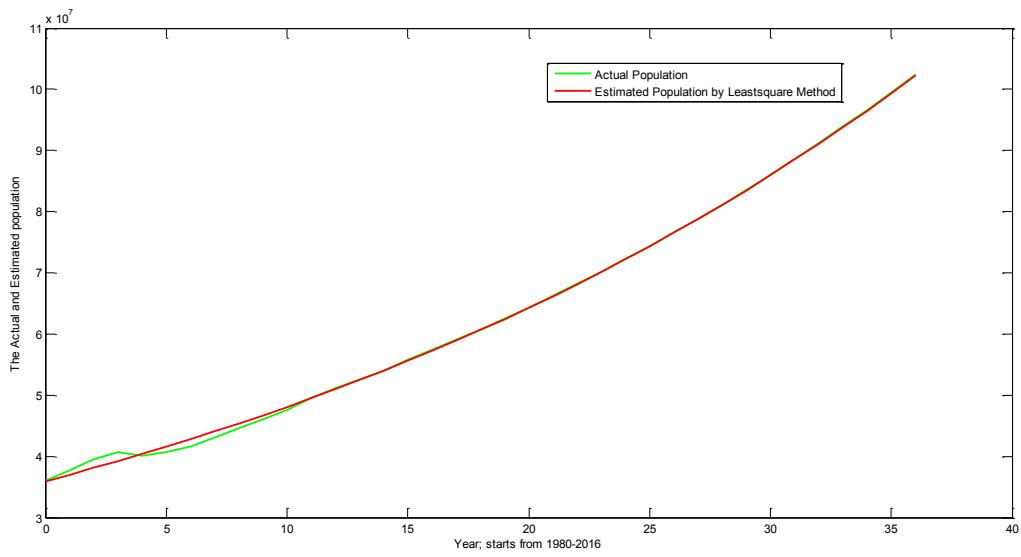| Year	Actual Population	Projected Population		Logistic Model
		Malthus's Model	Fitted by Least Square	
1980	36036457	36036457	35963083	36036457
1981	37720093	37096815	37022797	37094825
1982	39478541	38188374	38113738	38184217
1983	40673915	39312051	39236825	39305538
1984	40072730	40468792	40393006	40459720
1985	40683742	41659570	41583256	41647723
1986	41642553	42885386	42808579	42870533
1987	43058596	44147271	44070008	44129165
1988	44596107	45446287	45368607	45424663
1989	46093687	46783525	46705471	46758104
1990	47554116	48160112	48081729	48130592
1991	49447575	49577203	49498540	49543266
1992	51205192	51035993	50957100	50997296
1993	52525039	52537706	52458639	52493885
1994	54030146	54083607	54004424	54034275
1995	55725462	55674995	55595758	55619738
1996	57424029	57313209	57233983	57251586
1997	59076269	58999627	58920481	58931168
1998	60764879	60735667	60656675	60659873
1999	62538426	62522789	62444029	62439127
2000	64365225	64362497	64284051	64270399
2001	66230452	66256337	66178292	66155201
2002	68167888	68205903	68128349	68095085
2003	70174826	70212834	70135869	70091651
2004	72238990	72278819	72202544	72146542
2005	74354300	74405594	74330117	74261449
2006	76535275	76594948	76520382	76438113
2007	78799447	78848723	78775188	78678321
2008	81135648	81168815	81096435	80983914
2009	83548430	83557174	83486082	83356784
2010	86042927	86015810	85946144	85798878
2011	88588758	88546791	88478696	88312196
2012	91195675	91152244	91085874	90898798
2013	93877025	93834362	93769876	93560799
2014	96633458	96595400	96532968	96300375
2015	99465819	99437680	99377479	99119766
2016	102374044	102363594	102305808	102021273

Figure 1: Shows the graph of actual population and the projected population using Least square method.
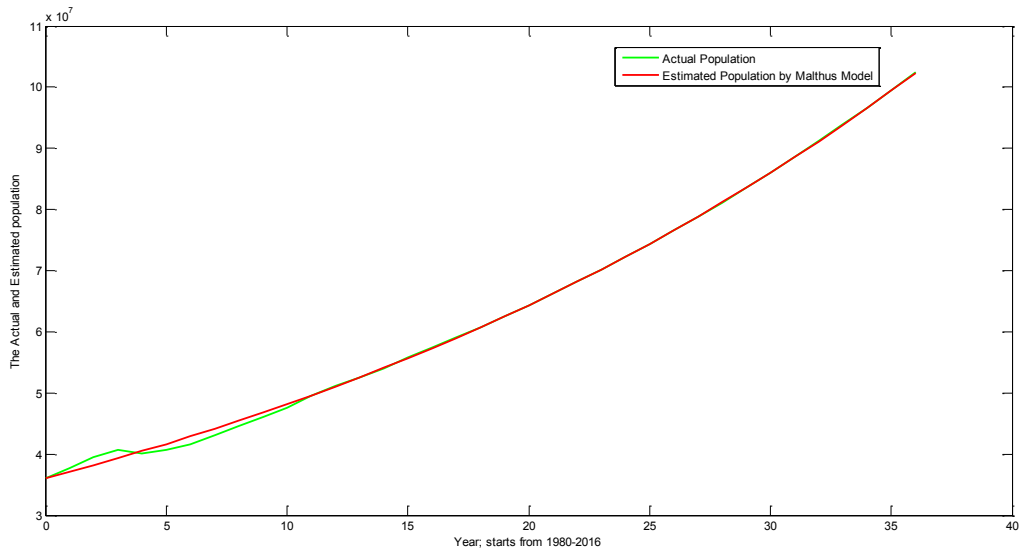

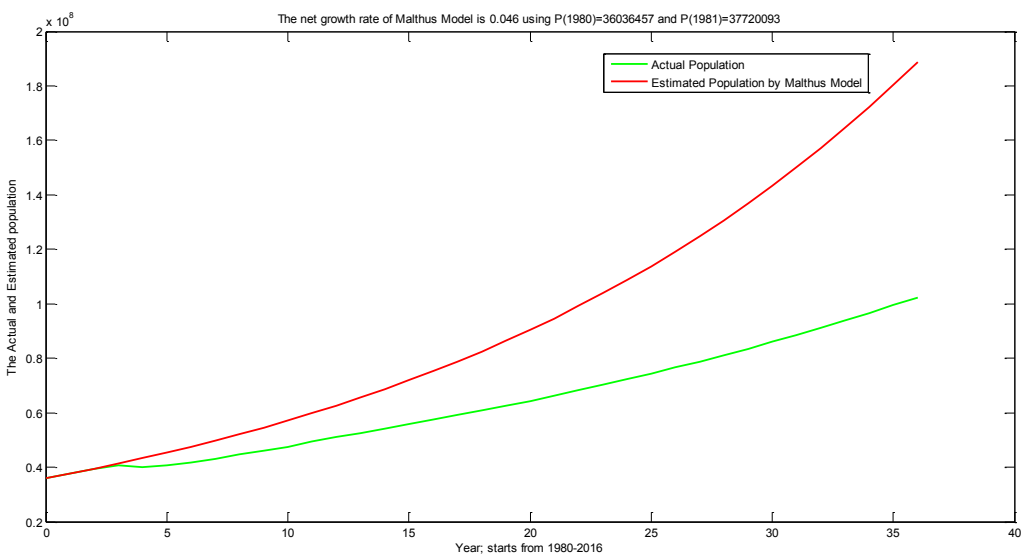Figure 2: Shows the graph of actual population and the projected population using Malthus's model.

Figure 3: Shows the graph of actual population and the projected population using Malthus's model with
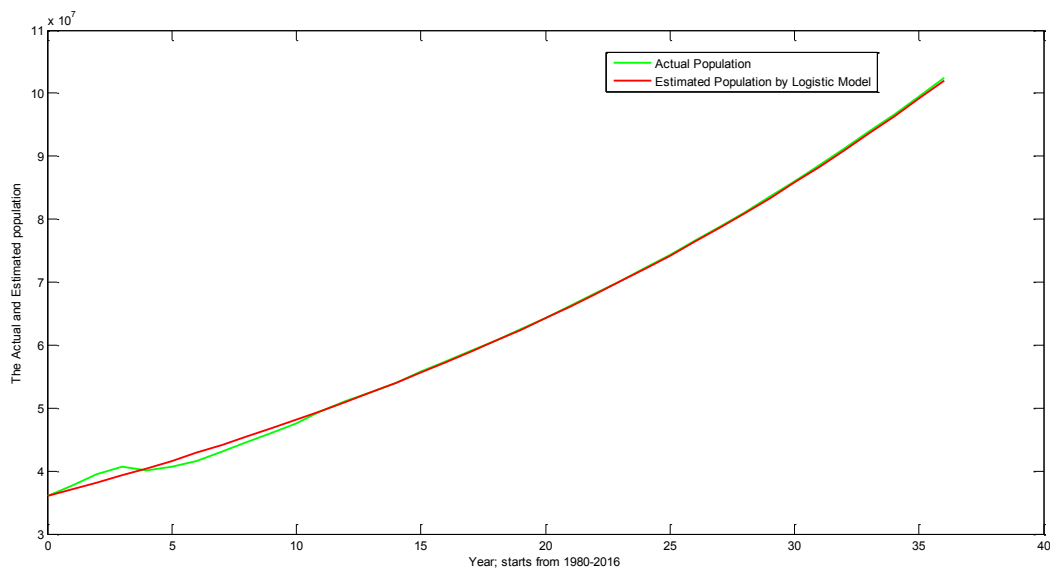
$\beta = 0.046$.

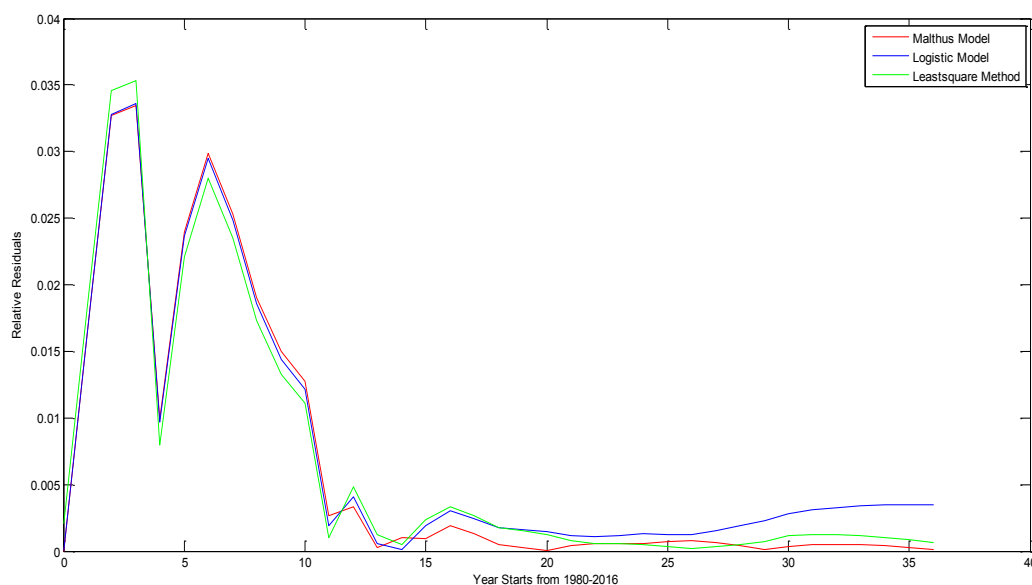Figure 4: Shows the graph of actual population and the projected population using Logistic model.

Figure 5: Shows the graph of relative residuals of the projected population using three models.

9. Discussion

From Table 1, it can be seen that in 1980 population of Ethiopia was 36036457 and continued to increase till 1983 when it decreased considerably from 40673915 to 40072730 in 1984. This reduction was probably due to famine, starvation, war and some people's migration to other countries. The population began to rise once again from 40072730 in 1984 to 102374044 in 2016. There has been an improvement in the education, agricultural productivity, water and sanitation and health services in the country. There were also early marriages and long time belief that more children one had, one would have a higher social and economic status, have higher work force in their farms and receive better care in old age. All these with other factors had an overall effect on the increase in population.

From figure 1 and 2, we see that the actual population and predicted values computed from Least Square method and Malthus's model respectively, are very close to each other. As we see in Figure 3 for the

parameter value $\beta = 0.046$ of Malthus's model, the actual and the predicted population are not matching very well as time (year) increases. This shows that the values of β computed from equation (4) could not be the best value so that the Malthus's model to have the best solution. Thus, equation (20) is better assumption as β has optimal value. Figure 4 shows that the predicted population has a better agreement with actual population. As time (year) increases, logistic model has greater mean absolute percentage error than others. This is may be due to the assumption that $P \ll K$. This is presented in Figure 5.

The population growth rate of Ethiopia according to information in International Data Base (IDB) was approximately 2.9% in 2012 and 2013, 2.89% in 2014 and 2015 which corresponds well with the findings in this research work of a growth rate of approximately 2.9% per annum.

10. Conclusions and Recommendations

In this study, a mathematical analysis of the future population of Ethiopia is carried out based on ordinary differential equation models which are Malthus's population model and Logistic model. The study applies least square method in order to determine the optimal value of the growth rate β of the Malthus's model. The Malthus's population model predicts a growth rate of approximately 2.9% per year with a MAPE of 0.64%. The Logistic Model on the other hand, predicts a carrying capacity for the population of Ethiopia to be $K = 19767334710$ while the population growth rate of Ethiopia is approximately 2.9% per annum with a MAPE of 0.73%. Based on this, we can conclude that the Malthus's population model seems to fit the original data the best among the models we tried. From Logistic model we also found out that the population of Ethiopia is expected to grow most rapidly when there are 9883667355 (half of its final growth) populations in the year $t = 2197$. MATLAB program based on an algorithm in appendix is used for calculation of future population. Furthermore, the growth rate for both models is approximately 2.9% which matches well with the growth rate estimated by International Data Base (IDB) for the past four years. As a recommendation, we forward that it is better to find the most appropriate value of the growth rate r of Logistic model since it has many values and use the proposed method to find the best value of the growth rate of the Malthus's population model.

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Appendix

Figure 6: Map of Ethiopia

MATLAB Programs

A. MATLAB program used to find the constants b and c by least Square method.

```
function [A,b,a,f,r]=expleast(xdata,ydata,m)
n=length (t);
A=zeros (m); b=zeros (m, 1); f=zeros (n, 1);
r=zeros (n, 1);
for i=1:n
y(i)=log(ydata(i));
end
for k=1:m
for i=1:n
b(k)=b(k)+xdata(i)^(k-1)*y(i);
end
end
for k=1:m
for j=1:m
for i=1:n
A(k,j)=A(k,j)+xdata(i)^(j+k-2);
end
end
end
a=A\b;
a(1)=exp(a(1));
for i=1:n
f(i)=a(1)*exp(a(2)*xdata(i));
end
for i=1:n
```

r(i)=abs((ydata(i)-f(i))./ydata(i));

end

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xdata=[0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36]';

ydata=[36036457 37720093 39478541 40673915 40072730 40683742 41642553 43058596 44596107 46093687 47554116 49447575 51205192 52525039 54030146 55725462 57424029 59076269 60764879 62538426 64365225 66230452 68167888 70174826 72238990 74354300 76535275 78799447 81135648 83548430 86042927 88588758 91195675 93877025 96633458 99465819 102374044]';

m=2; % number of parameters or coefficients

[A,b,a,f,r]=expleast(xdata,ydata,m)

Where xdata and ydata represent year (t) and actual population (P).

B. MATLAB used to plot the actual and predicted population

Plot(xdata,ydata,'g',xdata,f,'r','linewidth',2)

Where f=35963083*exp(0.029*xdata) % equation (23)

Plot(xdata,ydata,'g',xdata,W,'r','linewidth',2)

Where W=36036457*exp(0.029*xdata) % equation (24)

Plot(xdata,ydata,'g',xdata,Q,'r','linewidth',2)

Where Q=19767334710./(1+547.54*exp(-0.029*xdata)) % equation (25)