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A Six-Step Continuous Multistep Method For The Solution Of General Fourth Order Initial Value Problems Of Ordinary Differential Equations

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ABSTRACT

In this paper, continuous Linear Multistep Method (LMM) for the direct solution of fourth order initial value problems in ordinary differential equation is derived. The study provides the use of both collocation and interpolation techniques to obtain the schemes. Direct form of power series is used as basis function for approximation. An order six symmetric and zero-stable method is obtained. To implement our method, predictors of the same order of accuracy as the main method were developed using Taylor's series algorithm. This implementation strategy is found to be efficient and more accurate as the result has shown in the numerical experiments. The result obtained confirmed the superiority of our method over existing schemes

Keywords: Direct method; Fourth order; interpolation; collocation multistep methods,;power series; approximate solutions.

Introduction

In this article, the direct method of solving a fourth order initial value problems in ordinary differential equations of the form:

$$y^{iv} = f(x, y, y', y'', y'''), y(a) = y_0, y'(a) = \eta_1, y''(a) = \eta_2, y'''(a) = \eta_3, \quad \text{where } a, y, f \in R \quad (1)$$

This class of problem has a lot of applications in sciences and engineering especially in control theory, hence the study of the methods of solution is of great interest to researchers.

The conventional method of solving(1) is to first to reduce it to a system of first order differential equation .The approach of reducing to a system of first order has very serious drawback which includes wastage of human and computer time due to complicated computational work and lengthy execution time[1-5],

Direct method of solution of (1) using implicit linear multistep method has been found to be more more efficient in terms of speed and accuracy than the method of reduction to a system of first order ordinary differential equation[6-7]. Implicit linear multistep method is chosen because it has better stability properties than the explicit methods.

Direct method of solving higher order ordinary differential equations by continuous collocation multistep methods have been extensively discussed in [1,2, 7,9].

Several continuous LMM have been developed for the direct solution (1) (see [10-14]

The methods developed by some of these authors were implemented in predictor –corrector mode while those of the others were combined with additional methods obtained from continuous k-step LMMs to solve fourth orders ODES directly. Although the predictor- corrector methods yielded good results, the major setback of the method is that, apart from the inherent computational burden, the predictors which were developed have reducing order of accuracy.

In this paper therefore, we proposed a continuous LMM implemented in predictor-corrector mode using predictors of the same order of accuracy as the main method.

These authors in [10-14], have used a collocating function of the form:

$$Y(x)=\sum_{j=0}^{M}a_jx^j \tag{2}$$

In this work, we proposed a basis function of the form:

$$Y(x)=\sum_{j=0}^{M}a_j(x-x_k)^j \tag{3}$$

This is of the type in (2). The use of the above power series as basis function for approximate solution will enable us to derive continuous LMM of various orders and consequently the discrete formulae are obtained.
This paper is organized as follows: Section 2 considers the derivation of the methods and materials; Section3 considers the analysis of the basic properties of the method while Section 4 considers the implementation strategy and numerical experiments to test the efficiency of the new method.

2.0 Derivation of the Method

In this section, we shall approximate the exact solution $y(x)$ by a polynomial of degree M of the form:

$$Y(x)=\sum_{j=0}^{M}a_j\phi_j(x) \tag{4}$$

where $\phi_j(x)=(x-x_k)^j$, hence

$$Y(x)=\sum_{j=0}^{M}a_j(x-x_k)^j \tag{5}$$

We shall construct a k-step multistep method through collocation and interpolation techniques by imposing the following conditions

$$Y(x_{k+j})=y_{k+j},\ j=0(2)M-1 \tag{6}$$

$$D(x_{k+j})=f_{k+j},\ j=0(2)M \tag{7}$$

Substituting (4) into (7) yield

$$\sum_{j=4}^{M}j(j-1)(j-2)(j-3)a_j\phi_{j-4}(x_{n+i})=f(x,y,y',y'',y''') \tag{8}$$

By interpolating (4) at $x=x_{k+i}, i=2(1)M-1$ and collocating (7) at

$x=x_{k+2,}x_{k+4},x_{k+6}$ we obtained the system of equations as follows :

$$\sum_{j=4}^{M}j(j-1)(j-2)(j-3)a_j\phi_{j-4}(x_{k+i})=f_{k+i} \tag{9}$$

$$\sum_{j=0}^{M}a_j\phi_j(x_{k+i})=y_{k+i} \tag{10}$$

By solving the system above for the $a_j's$, $j=0(1)M$ and substituting into (5) for $M=6$,we obtain the polynomial

$$Y(x)=\frac{1}{2880h^2}\left[2880h^2-22560h(x-x_k)+5760(x-x_k)^2-\frac{480}{h}(x-x_k)^3\right]y_{k+2}$$
$$\frac{1}{2880h^2}\left[-57600h^2+54720h(x-x_k)-15840(x-x_k)^2+(\frac{1440}{h})(x-x_k)^3\right]y_{k+3}$$
$$\frac{1}{2880h^2}\left[43200h^2-44640h(x-x_k)+14400(x-x_k)^2-(\frac{1440}{h})(x-x_k)^3\right]y_{k+4}$$
$$\frac{1}{2880h^2}\left[-11520h^2+12480h(x-x_k)-4320(x-x_k)^2+(\frac{480}{h})(x-x_k)\right]y_{k+5}$$
$$\frac{1}{2880h^2}\left[\begin{array}{l}-840h^6+598h^5(x-x_k)+239h^4(x-x_k)^2-340h^3(x-x_k)^3\\+120h^2(x-x_k)^4-18h(x-x_k)^5+(x-x_k)^6\end{array}\right]f_{k+6} \quad (11)$$
$$\frac{1}{2880h^2}\left[\begin{array}{l}13200h^6-13100h^5(x-x_k)+2642h^4(x-x_k)^2+1040h^3(x-x_k)^3\\-480h^2(x-x_k)^4+60h(x-x_k)^5-2(x-x_k)^6\end{array}\right]f_{k+4}$$
$$\frac{1}{2880h^2}\left[\begin{array}{l}2040h^6-5978h^5(x-x_k)+5639h^4(x-x_k)^2-2380h^3(x-x_k)^3\\+480h^2(x-x_k)^4-42h(x-x_k)^5+(x-x_k)^6\end{array}\right]f_{k+2}$$

Evaluating (11) at $x=x_{k+6}$, we obtained the discrete scheme:

$$y_{k+6}-4y_{k+5}+6y_{k+4}-4y_{k+3}+y_{k+2}=\frac{h^4}{24}\left(f_{k+6}+22f_{k+4}+f_{k+2}\right) \quad (12)$$

3.0 Analysis and Implementation of the Method

The main method is a specific member of the conventional LMM which can be written as

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h^4\sum_{j=0}^{k}\beta_j f_{n+j}. \quad (13)$$

And can be written symbolically as

$$\rho(E)y_n-h^n\sigma(E)f_n=0\ ,\ f_n=f(x_n,y_n)\ . \quad (14)$$

where E is the shift operator defined $E^i y_n=y_{n+j}$ and $\rho(E)$ and $\sigma(E)$ are respectively the first and second characteristics polynomial of the LMM defined as

$$\rho(E)=\sum_{j=0}^{k}\alpha_j E^j,\quad \sigma(E)=\sum_{j=0}^{k}\beta_j E^j\ ,\ \alpha_k\neq 0.$$

Following [15,16] we define the linear operator associated with the method (12) as

$$L\left[y(x),h\right]=\sum_{j=0}^{k}\left[\alpha_j y(x+jh)-h^4\beta_j y''''(x+jh)\right]. \tag{15}$$

where the function $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Therefore expanding (15) in Taylor series about the point x to obtain the expression

$$L\left[y(x),h\right]=C_0 y(x)+C_1 hy'(x)+C_2 h^2 y''(x)+...+C_{p+2}h^{p+2}. \tag{16}$$

and $$C_0=\sum_{j=0}^{k}\alpha_j \ ,$$

$$C_1=\sum_{j=0}^{k} j\alpha_j \ ,$$

$$C_2=\frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j ,$$

$$\vdots$$

$$C_q=\frac{1}{q!}\left[\sum j^q q(q-1)(q-2)(q-3)\alpha_j\sum_{j=1}^{k}\beta_j j^{q-4}\right],q=0,1,2,...,p+2 \tag{17}$$

In the sense of [15] ,we say that the method (12) is of order p and error constant C_{p+2} if

$$C_0=C_1=C_2=...=C_p=C_{p+1}=0,C_{p+2}\neq 0$$

This concept is used to calculate the order and error constant of the method (12).

The order of the six-step method (12) is p =6 and error constant $C_{p+2}=\frac{-31}{720}=-0.043055555.$

3.1 Zero stability of the six –step method

Given that the first characteristics polynomial of (12) is:

$$\rho(r)=r^6-4r^5+6r^4-4r^3+r^2=0 \ ,.$$

On solving $\rho(r)$, we obtained $r^2(r-1)^4=0.$.

Therefore $r=0,1,1,1,1$. Thus $\rho(r)=0$ satisfies $\left|R_j\right|\leq 1,\ j=1,...,k$. That is the roots lie in the unit circle and the multiplicity of $|r|=1$ did not exceed four. Hence the method is zero stable.

3.2 Interval of absolute stability of the six-step method

The first and second characteristics polynomials of the method (3.6) are given as

$$\rho(r)=(r^6-4r^5+6r^4-4r^3+r^2) \ , \ \sigma(r)=\frac{1}{24}(r^6+22r^4+r^2).$$

$$h(r)=\frac{\rho(r)}{\sigma(r)}=\frac{24(r^6-4r^5+6r^4-4r^3+r^2)}{(r^6+22r^4+r^2)}.$$

By letting $r = e^{i\theta}$,where $e^{i\theta} = \cos\theta + i\sin\theta$,then

$$h(\theta) = \frac{24\begin{bmatrix}\cos 6\theta - 4\cos 5\theta + 6\cos 4\theta - 4\cos 3\theta + \cos 2\theta \\ +i(\sin 6\theta - 4\sin 5\theta + 6\sin 4\theta - 4\sin 3\theta + \sin 2\theta\end{bmatrix}}{(\cos 6\theta + 22\cos 4\theta + \cos 2\theta) + i(\sin 6\theta + 22\sin 4\theta + \sin 2\theta)}$$

By setting $y(\theta) = 0$, we have

$$x(\theta) = \frac{24(134 - 184\cos\theta + 56\cos 2\theta - 8\cos 3\theta + 2\cos 4\theta)}{486 + 88\cos 2\theta + 2\cos 4\theta}.$$

Thereof re evaluating $x(\theta)$ for $0^\circ \le \theta \le 180^\circ$ in the interval of 30°; we have

The interval of absolute stability of the six –step method is (0, 16)

3.3 Consistency of the method

An LMM is said to be consistent if it has order p≥1. Hence our method is consistent.

3.4 Convergence

The basic property which is demanded of an acceptable LMM is that it solution $\{y_n\}$ generated by the method converges to the theoretical solution $y(x)$ as the step-length tends to zero.

An LMM is convergent if and only if it is consistent and zero stable. The method (12) is consistent, zero stable and hence convergent.

4.0 Implementation

Consider the initial value problem in (1). For our method of order p =6, we shall develop the predictors of the same order of accuracy by using Taylor series expansion of the form

$$Y(x) = \sum_{i=0}^{3}\frac{(jh)^i}{i!}y_n^i + h^4\sum_{\lambda=0}^{p}\frac{\partial^\lambda}{\partial x^\lambda}f(x,y,y',y'',y''')_{(x_0,y_0,y_0',y_0'',y_0''')} \tag{18}$$

where $\frac{\partial^\lambda}{\partial x^\lambda}f(x,y,y',y'',y'') = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + f\frac{\partial}{\partial y'''}\right)f_j = Df_j$

$$f(x_j, y_j, y_j', y_j'', y_j''') = f_j.$$

$$D = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + f\frac{\partial}{\partial y'''}\right) \quad , \quad D^2 = D(D)$$

and p is the order of the method.

4.2 Numerical Experiments

Our methods of order p = 6 was used to solve some initial value problems of both general and special nature using Taylor's series. Our results were compared with the results of other researchers in this area as seen in table1. In tables 2 and 3,the accuracy of our method is seen in the small error values

The following initial value problems were used as our test problems:

Problem 1

$y''' = x \quad , y(0) = 0 \quad , y'(0) = 1 \, , y''(0) = 0 \, , y'''(0) = 0, \; h = 0.1.$

Theoretical solution: $y(x) = \dfrac{x^2}{120} + x..$

Problem 2:

$$y'''' = \frac{-(8+25x+30x^2+12x^3+x^4)}{(1+x^2)} \quad , y(0) = 0 \, , \; y'(0) = 1 \, , \; y''(0) = 0 \, , \; y'''(0) = -3 \, ,$$

$h = 0.103125$

Theoretical solution: $y(x) = x(1-x)e^x..$

Problem 3:

. $y''' = -y'', y(0) = 0, y'(0) = \dfrac{1.1}{72-50\pi}, y'' = \dfrac{1.2}{144-100\pi}, h = \dfrac{0.1}{32}.$

Theoretical solution: $y(x) = \dfrac{1 - x - \cos x - 1.2\sin x}{144-100\pi}.$

Problem 4:

$y^{(iv)} - 4y'' = 0 \quad y(0) = 1, y'(0) = 3, y''(0) = 0, y'''(0) = 16 \;\; , 0 \le x \le 1 \quad h = 0.1.$

Theoretical solution: $y(x) = 1 - x + e^{2x} - e^{-2x}.$

Table 1: Result for test problem 1: (h=0.1)

X	Exact solution	New result (k=6,p=6)	Errors in[13] (P=6)	Errors in our new result (k=6,p=6)
0.1	0.1000000848E+00	0.1000000848E+00	7.000000024E-10	0.00000E+00
0.2	0.2000026696E+00	0.2000026696E+00	8.999999912E-10	0.00000E+00
0.3	0.3000262545E+00	0.3000262545E+00	2.599999993E-09	0.00000E+00
0.4	0.4000853393E+00	0.4000853393E+00	5.100000033E-09	0.00000E+00
0.5	0.5002604241E+00	0.5002604241E+00	7.799999979E-09	0.00000E+00
0.6	0.6006480090E+00	0.6006480090E+00	1.180000009E-08	1.11022E-16
0.7	0,7014005939E+00	0,7014005939E+00	1.240000003E-08	3.33067E-16
0.8	0.8027306788E+00	0.8027306788E+00	1.410000006E-08	5.55112E-16
0.9	0.9049207638E+00	0.9049207638E+00	1.880000000E-08	9.99201E-16
1.0	0.1008333349E+00	0.1008333349E+00	2.600000015E-08	1.55431E-15

Table 2: Result for test problem 2: $(h = \frac{1}{320})$.

X-value	Exact solution	New result(k=6,p=6)	Error in[14] (p=7)	Error in our New result (k=6,p=6)
0.003125	0.3124984756E-02	0. 3124984756E-02	1.990205E-14	2.4874E-14
0.006250	0.6249877513E-02	0.6249877512E-02	6.379298E-13	7.9720E-13
0.009375	0.9374585568E-02	0.9374585562E-02	4.852393E-12	6.3116E-14
0.001250	0.1249901545E-01	0.1249901542E-01	2.048206E-11	4.4102E-12
0.015625	0.1562307290E-01	0.1562307282E-01	6.261025E-11	5.7680E-12
0.018750	0.1874666289E-01	0.1874666270E-01	1.560543E-10	1.4918E-11
0.021875	0.2186968961E-01	0.2186968919E-01	3.378600E-10	9.1931E-11
0.025000	0.2499120564E-01	0.2499120556E-01	6.598189E-10	2.7786E-10
0.028125	0.2811366598E-01	0.2811366450E-01	1.191010E-09	6.4684E-10
0.031250	0.3123442003E-01	0.3123441752E-01	2.020367E-09	1.2977E-09

Table 3 : Results of test problem 3 : $(h = 0.103125)$

X-value	Exact solution	New result (p=6)	Errors in (p=6)	Errors in [11] (P=6)
0.103125	0.11192647E+01	0.11192647E+01	2.11164E-13	4.68429E-12
0.206250	0.12715995E+01	0.12715995E+01	5.69866E-12	2.06871E-10
0.306250	0.14582861E+01	0.14582861E+01	6.80311E-10	9.04219E-10
0.406250	0.16807458E+01	0.16807456E+01	2.20723E-09	2.91379E-09
0.506250	0.19405540E+01	0.19405533E+01	1.27407E-08	7.51140E-09
0.606250	0.22394574E+01	0.22394562E+01	3.45612E-06	1.62313E-08
0.703125	0.25793924E+01	0.25793915E+01	6.55238E-06	3.22371E-08
0.803125	0.29625057E+01	0.29625076E+01	9.58653E-06	5.88918E-08
0.903125	0.33911774E+01	0.33911869E+01	1.04933E-06	1.00799E-07
1.031250	0.38680458E+01	0.38680722E+01	5.69624E-06	1.63736E-08

Table 4 : Results of test problem 4 : $(h = \frac{1}{320})$.

X-value	Exact solution	New results of our method(k=6,p=6)	Errors in the new results (k=6,p=6)
0.003125	0.100937508152E+01	0.1009375082E+01	0.00000E+00
0.006250	0.101875065133E+01	0.1018750651E+01	0.00000E+00
0.009375	0.102812719772E+01	0.1028127198E+01	2.22045E-16
0.001250	0.103750520906E+01	0.1037505209E+01	2.44249E-15
0.015625	0.104688517372E+01	0.1046885174E+01	1.15463E-14
0.018750	0.105626758020E+01	0.1056267580E+01	3.30846E-14
0.021875	0.106565291706E+01	0.1065652917E+01	7.28306E-14
0.025000	0.107504167299E+01	0.1075041673E+01	1.37002E-13
0.028125	0.108443433682E+01	0.1084434337E+01	2.30926E-13
0.031250	0.109383139751E+01	0.1093831398E+01	3.60822E-13

Conclusion

We have developed a k-step linear multistep method (LMM) and implemented same using predictors of the same order of accuracy. A new scheme of order p=6 is obtained which was applied to solve some special and general fourth order initial value problems in ordinary differential equations. Evidence of the better accuracy of our method over existing methods is as given in Tables 1,
2, 3 and 4 respectively.

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