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J2 optimality and Multi-level Minimum Aberration Criteria in fractional factorial design

Salawu, I.S., Adeleke B.L. and Oyeyemi, G.M.
Department of Statistics University of Ilorin
salahus74@yahoo.com

Abstract

The desirable properties of fractional factorial design: Balance and orthogonal; was examined for near balance and near orthogonal using the balance coefficient and J_2 optimality criteria respectively. Efficient orthogonal arrays with three factors having two, three and four levels were constructed with balance and orthogonal property for lowest common multiples of runs. The two forms of balance coefficient were used for classifying the designs into two and multi level minimum aberration criteria were used to determine designs with lesser aberration. It was observed that designs constructed using the maximum form of balance coefficient has the lesser aberration in both the generalized minimum aberration and minimum moment aberration criteria. The J_2 – optimality criterion reveals that the higher the run of a design, the lesser it's optimality value.

Keywords: Balance Coefficient, fractional factorial, Generalized Minimum Aberration (GMA), J_2 optimality and Minimum Moment Aberration (MMA).

1. Introduction

Factorial designs have broad applications in agricultural, engineering and scientific studies. In constructing and studying properties of factorial designs, traditional design theory treats all factors as nominal. However, this is not appropriate for experiments that involve quantitative factors. For designs with quantitative factors, level permutation of one or more factors in a design matrix could result in different geometric structures, and, thus, different design properties.

Basically two or three – levels factorial experiments are mostly used in the design of experimental research. In many situations, factors with more than two-three levels are desirable, when the factors are either qualitative or quantitative. As a result, designs with mixed-level factors have been used more often in designed experiments in modern industrial and agricultural trials, especially when only limited resources are allowed. Full factorial designs are test matrices that contain all possible combinations of the levels of the factors. For example, if a factor A has "a" levels, factor B has "b" levels and factor C has "c" levels, then the full factorial design will contain "abc" combinations.

The two basic properties of factorial experiments are balance and orthogonal. Balance requires a level of particular factors replicated the same number of times as any other levels of this factor in an experiment. Orthogonal designs are pair wise linearly independent, useful for assessing factor significance.

As the number of factors or factor levels increases, the number of runs increases and maintaining the balance property requires too many runs in some situations. For examples, consider a design with four factors having 3, 5, 5, and 2 levels. To generate a balanced design, at least 150 runs are needed. Suppose an experimenter only has resources for 50 tests and the main objective is screening. Then, a mechanism for creating mixed-level designs which is capable of meeting desirable resources is required.

Optimality properties and ease of interpretation have popularized the use of orthogonal designs in applied research. Traditionally, the construction of orthogonal S^n fractional factorial designs has been confined to using aliasing relations.

A two-level 2^{k-q} design is defined to be a fractional factorial design with k factors, each at two levels, consisting of 2^{k-q} runs. Therefore, it is a 2^{-q} fraction of the 2^k full factorial design in which the fraction is determined by q generators. The number of letters in a word is its word length and the word formed by the q defining words is called the defining relation.

For a 2^{k-q} design, let $A_K(d)$ be the number of words of length k in the defining contrast subgroup. The vector

$$W(d)=\left(A_1(d),A_2(d),\ldots,A_m(d)\right)$$

is called the word length pattern of the design d, (Fries and Hunter, 1980). The resolution of a 2^{m-q} design, R, is defined to be the smallest r such that $A_r(d) \geq 1$, that is, the length of the shortest word in the defining contrast subgroup for any two 2^{m-q} designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If no design exists with less aberration than d_1, then d_1 has minimum aberration (Fries and Hunter, 1980).

Minimum aberration has been widely recognized as a useful criterion for selecting regular fractional factorials. Recent work on minimum aberration designs includes Chen and Wu (2001), Tang and Wu (1996), Chen and Hedayat (1996), and Cheng et al. (1999). Minimum aberration mixed-level designs are also balanced, Cheng et al, (1999), Deng and Tang (1999), Mukerjee and Wu (2001), Xu and Wu (2001). For unbalanced mixed-level fractional factorial designs, the degree of balance was evaluated using a balance coefficient (Bashir (2003)).

As an extension of two level fractional factorial designs, Franklin (1984) and Suen, Chen and Wu (1997) discuss the construction of multi-level minimum aberration designs. Xu and Wu (2001) proposed a generalized minimum aberration for mixed –level fractional factorial designs. Wu and Zhang (1993) and Ankenman (1999) used minimum aberration designs in two-level and four – level designs. Murkerjee and Wu (2001) developed minimum aberration designs for mixed-level fractional factorial designs involving factors with two or three distinct levels.

Montgomery (2005) gives a slightly different formatted word length pattern from Wu and Zhang (1993), instead of using numbers of words of length k in the defining contrast subgroup, Montgomery (2005) directly shows the length of each word in the defining contrast group.

The objective of this paper is to screen designs using the measure of balance and orthogonal in an efficient fractional factorial design and to compare the designs using multi – level minimum aberration criteria at various run sizes.

2. Balance Coefficient

2.1 Form I

In form I of the balnced coefficient as defined by Guo (2009), the motivation behind the definition of the balance coefficient is a simple optimization problem. The balance coefficient of design matrices will be derived from the optimization problem stated below:

$$\text{Max} \qquad G = \prod_{K=1}^{m} X_K$$

$$\text{Subject to} \quad \sum_{k=1}^{m} X_K = C,$$

Where C is a constant

The balance coefficient for design matrix k, $F(k)$, is defined as the combination of the balanced coefficient of each column, F_j,

$$F(k) = \sum_{j=1}^{m} w_j F_J = \sum_{j=1}^{m} \left(\prod_{i=1}^{l_j} l_{ij} \right) w_j,$$

Where w_j are the weights for the corresponding column *j.* This balance coefficient depends on the runs. A standard number of levels will be used to standardize the balance coefficient. The notations f_{ij} is used instead of l_{ij}.

2.2 Form II

In form II as defined by Guo (2009), the definition of balance coefficient employs the concept of the distance function. Consider a distance function

$$H_J = \sum_{i=1}^{lj} (l_{ij} - T)^2,$$

where $T = n/l_j$, is a fixed value.

The balance coefficient under this definition becomes

$$H_j = \sum_{i=1}^{l_j} (l_{ij} - \frac{n}{l_j})^2$$

and

$$H=\sum_{j=1}^{m}H_j=\sum_{j=1}^{m}w_j\sum_{i=1}^{l_j}(l_{ij}-\frac{n}{l_j})^2$$

If f_{ij} are used instead of l_{ij}, then standardized H_j and H can be given by

$$\hat{H}_j=\sum_{i=1}^{l_j}(f_{ij}-\frac{1}{l_j})^2, \text{ and}$$

$$\hat{H}=\sum_{j=1}^{m}w_j\,H_j=\sum_{j=1}^{m}w_j\sum_{i=1}^{\hat{l}_j}(f_{ij}-\frac{1}{l_j})^2$$

3. J_2-Optimality

Consider a n x m, matrix D = [x_{ij}], with weights wk > 0, and S_k levels for each factor as defined by Guo (2009). The weights indicate the relative importance of each factor and if w_k = 1 are chosen , $\delta_{i\,j}(d)$ is the number of coincidences between the i[th] and j[th] rows. For $1\le i, j \le n$,

$$\delta_{i,j}(d)=\sum_{k=1}^{m}w_k\delta(x_{ik},x_{jk}),$$

Where δ(x,y) = 1 if x = y and 0 otherwise.

Then the J_2-optimality of this design matrix d is defined by Guo (2009) as

$$J_2(d)=\sum_{1\le i<j\le n}[\delta_{i,j}(d)]^2$$

3.1 Lower bound of J2-optimality for balanced designs

When the J_2 optimality of a balanced design matrix reaches the lower bound, then, the design is orthogonal. This conclusion was proved by Xu (2003). The lower bound, L(n), is given as follows,

$$L(n)=2^{-1}[\left(\sum_{k=1}^{m}ns_k^{-1}w_k\right)^2+\left(\sum_{k=1}^{m}(s_k-1)(ns_k^{-1}w_k)^2\right)-n\left(\sum_{k=1}^{m}w_k\right)^2]$$

4. Minimum Aberrations

Minimum aberration has been widely recognized as a useful criterion for selecting regular fractional factorials. Recent work on minimum aberration designs includes Chen and Wu (2001), Tang and Wu (1996), Chen and Hedayat (1996), and Cheng et al. (1999).

4.1 Generalized Minimum Aberration Criterion

Xu and Wu (2001) proposed a generalized minimum aberration (*GMA*) criterion for multi-level and mixed-level designs. For a design *d*, the ANOVA model has the following form

$$Y=X_0\alpha_0+X_1\alpha_1+\ldots+X_m\alpha_m+\varepsilon,$$

where Y is the response, α_k is the vector of all *k*-factor interactions and $X_k=\left[x_{ij}^{(k)}\right]$ is the matrix of contrast coefficients for α_k. Let

$$A_k(d) = n^{-2}\sum_j \left|\sum_{i=1}^{n} x_{ij}^{(k)}\right|^2 .$$

The $A_k(d)$ are invariant with respect to the choice of orthogonal contrasts. The vector $(A_1(d), A_2(d), \ldots A_m(d))$ is called the generalized word length pattern. Then the generalized minimum aberration criterion is to sequentially minimize $A_k(d)$ for $k=1, \ldots, m$.

4.2 Minimum Moment Aberration Criterion

The Minimum Generalized Aberration (*MGA),* Minimum G_2 Abberation *(MG_2A),* and Generalized Minimum Aberration (*GMA)* criteria all require contrast coefficients of factors. Xu (2003) developed a Minimum Moment Aberration criterion (MMA), which does not need contrast coefficients. For a design matrix *d*, with d_{ij} as the elements of i^{th} row and j^{th} column. The coincidence between two elements d_{ij} and d_{lj} is defined by $\delta(d_{ij}, d_{lj})$, where $\delta(d_{ij}, d_{lj})$ = 1 if d_{ij} = d_{lj} and 0 otherwise. The value of $\sum_{j=1}^{m}\delta(d_{ij}, d_{lj})$ measures the coincidence between i^{th} and j^{th} rows of *d.* The k^{th} power moment is defined by Xu (2003) as

$$K_k(d) = [n(n-1)/2]^{-1}\sum_{1\le i\le l\le n}\left[\sum_{j=1}^{m}(d_{ij}, d_{lj})\right]^k .$$

For two designs d_1 and d_2, d_1 is said to have less moment aberration than d_2 if there exists an r such that $K_r(d_1) < K_r(d_2)$ and $K_t(d_1) = K_t(d_2)$ for all $t=1, \ldots, r-1$. Therefore, d_1 is said to have minimum moment aberration if there is no other design with less moment aberration than d_1.

Table 4.1: Design selected using balance coefficient and their J_2 – optimality in $OA(n, 2^1 3^1 4^1)$

Runs	Balance coefficient methods	Column 1	Column 2	Column 3	J_2 optimality
6	Distance Function	[1 5]	[1 4 1]	[1 1 1 3]	39
	Optimization Procedure	[3 3]	[2, 2, 2]	[1, 1, 1, 3]	24
7	Distance Function	[1, 6]	[1, 5, 1]	[1, 4, 1, 1]	77
	Optimization Procedure	[3, 4]	[1, 3, 3]	[1, 2, 2, 2]	36
8	Distance Function	[1, 7]	[1, 6, 1]	[1, 5, 1, 1]	116
	Optimization Procedure	[4, 4]	[3, 3, 2]	[2, 2, 2, 2]	47
9	Distance Function	[1, 8]	[1, 7, 1]	[1, 1, 6, 1]	166
	Optimization Procedure	[4, 5]	[3, 3, 3]	[1, 2, 3, 3]	70
10	Distance Function	[1, 9]	[1, 8, 1]	[7, 1, 1, 1]	201
	Optimization Procedure	[5, 5]	[3, 3, 4]	[2, 2, 3, 3]	82
11	Distance Function	[2, 9]	[1, 9, 1]	[8, 1, 1, 1]	231
	Optimization Procedure	[5, 6]	[2, 4, 5]	[3, 2, 3, 3]	112
12	Distance Function	[3, 9]	[1, 9, 2]	[1, 9, 1, 1]	276
	Optimization Procedure	[6, 6]	[4, 4, 4]	[3, 3, 3, 3]	128
13	Distance Function	[9, 4]	[3, 9, 1]	[9, 1, 2, 1]	272
	Optimization Procedure	[7, 6]	[3, 5, 5]	[3, 3, 3, 4]	162
14	Distance Function	[9, 5]	[1, 9, 4]	[1, 3, 1, 9]	271
	Optimization Procedure	[7,7]	[5, 5, 4]	[2, 3, 4, 5]	220
15	Distance Function	[9, 6]	[1, 9, 5]	[1, 4, 9, 1]	312
	Optimization Procedure	[8, 7]	[4, 5, 6]	[2, 3, 4, 6]	249
16	Distance Function	[7, 9]	[2, 9, 5]	[1, 9, 1, 5]	336
	Optimization Procedure	[8, 8]	[3, 6, 7]	[1, 8, 3, 4]	313
17	Distance Function	[9, 8]	[1, 9, 7]	[1, 7, 1, 8]	839
	Optimization Procedure	[9, 8]	[4, 6, 7]	[1, 7, 1, 8]	670
18	Distance Function	[9, 9]	[1, 9, 8]	[1, 7, 1, 9]	515
	Optimization Procedure	[9, 9]	[6, 6, 6]	[3, 4, 5, 6]	341

In the above Table 4.1, the balance coefficient criteria are used to select the designs of columns 1, 2, and 3 and their J_2 optimality obtained. It was observed that the designs selected using the optimization procedure method of balance coefficient has a lower J_2 optimality in all the designs selected.

The two basic criteria of fractional factorial design: balance and orthogonality are measured using the balance coefficient and J_2-optimality was used to select designs between Optimization Procedure and Distance Function pattern of balance coefficient. Minimum aberration criterion is used to select the best among efficient designs. The selected designs were subjected to comparison using multi-level aberration criteria such as minimum moment aberration criteria (MMAC) and generalized minimum aberration criteria (GMAC). These criteria were selected because; the constructed design involves more than two levels i.e. mixed level design that does not make use of design generator.

Table 4.2: Design Comparison using Generalized Minimum Aberration Criterion (GMAC) in $(n,2^1 3^1 4^1)$

		Generalized Minimum Aberration Criteria		
Runs	Designs	Sum of Squares	$A_i(d_i)$	
6	Distance Function	16.99	0.472	
	Optimization Procedure	2.99	0.083	d_2
7	Distance Function	29.89	0.61	
	Optimization Procedure	3.92	0.08	d_2
8	Distance Function	46.04	0.72	
	Optimization Procedure	0.67	0.01	d_2
9	Distance Function	67.21	0.82	
	Optimization Procedure	3.24	0.04	d_2
10	Distance Function	91.62	0.92	
	Optimization Procedure	1.67	0.02	d_2
11	Distance Function	103.86	0.86	
	Optimization Procedure	5.91	0.05	d_2
12	Distance Function	103.95	0.72	
	Optimization Procedure	0	0	d_2
13	Distance Function	91.87	0.54	
	Optimization Procedure	3.92	0.02	d_2
14	Distance Function	83.63	0.43	
	Optimization Procedure	5.67	0.03	d_2
15	Distance Function	79.22	0.35	
	Optimization Procedure	11.25	0.05	d_2
16	Distance Function	70.64	0.28	
	Optimization Procedure	34.66	0.14	d_2
17	Distance Function	77.89	0.27	
	Optimization Procedure	5.91	0.02	d_2
18	Distance Function	88.97	0.27	
	Optimization Procedure	4.99	0.02	d_2

The generalized minimum aberration criteria in the comparison of the designs, $OA(n,2^1 3^1 4^1)$, using both Optimization Procedure and Distance Function methods of balance coefficient for $6 \le n \le 18$, it was shown that at $6 \le n \le 18$, $A_1(\mathrm{d}_2) < A_1(\mathrm{d}_1)$, i.e. design d_2 has less aberration than d_1. Therefore, d_2 is better than d_1 by the GMAC.

Table 4.3: Designs using Minimum Moment Aberration Criteria (MMA) in $(n, 2^1 3^1 4^1)$

N	Designs	Minimum Moment Aberration Criteria $(K_1(d), K_2(d), K_3(d), K_4(d))$	
6	Distance function	(1.267, 2.6, 5.667, 13)	
	Optimization Procedure	(0.8, 1.6, 3.6, 8.8)	d_2
7	Distance function	(1.429, 3.667, 9.476, 26.43)	
	Optimization Procedure	(0.857, 1.714, 4.809, 10.286)	d_2
8	Distance function	(1.643, 4.143, 11.286, 32)	
	Optimization Procedure	(0.821, 1.679, 3.964, 10.607)	d_2
9	Distance function	(1.75, 4.611, 12.778, 36.611)	
	Optimization Procedure	(0.88, 1.94, 4.88, 13.28)	d_2
10	Distance function	(1.889, 4.467, 11.622, 31.933)	
	Optimization Procedure	(0.88, 1.82, 3.67, 10.62)	d_2
11	Distance function	(1.727, 4.2, 10.545, 28.2)	
	Optimization Procedure	(0.872, 2.036, 4.873, 12.509)	d_2
12	Distance function	(1.697, 4.181, 11.060, 30.879)	
	Optimization Procedure	(0.909, 1.939, 4.727, 12.485)	d_2
13	Distance function	(1.513, 3.487, 8.897, 24.103)	
	Optimization Procedure	(1.025, 2.077, 5.103, 13.462)	d_2
14	Distance function	(2.494, 2.978, 7.132, 17.703)	
	Optimization Procedure	(1, 2.418, 10.099, 16.923)	d_2
15	Distance function	(1.352, 2.971, 7.038, 18.371)	
	Optimization Procedure	(0.96, 2.37, 6.314, 17.457)	d_2
16	Distance function	(1.267, 2.75, 7.05, 19)	
	Optimization Procedure	(1.1, 2.608, 6.55, 17.483)	d_2
17	Distance function	(1.235, 6.169, 9.279, 26.779)	
	Optimization Procedure	(0.98, 4.93, 5.78, 15.93)	d_2
18	Distance function	(1.261, 3.366, 10.189, 26.634)	
	Optimization Procedure	(0.987, 2.229, 5.693, 15.562)	d_2

The minimum aberration criteria for two selected designs using Optimization Procedure and Distance Function method of balance coefficient, for $6 \le n \le 18$.

The observation shows that at $6 \le n \le 18$

$$K_1(d_2) < K_1(d_1);$$

This indicated that in all the runs mentioned, $K_1(d_2)$ has a lesser aberration than $K_1(d_1)$, that is, the design d_2 is a better fractional factorial of all possible designs in the runs considers.

Table 4.4: Summary of designs evaluated in Table 4.2 and 4.3

		$(n, 2^1 3^1 4^1)$					
	n	**MMAC**	**GMAC**		**n**	**MMAC**	**GMAC**
Optimization procedure	**6**	d_2	d_2		**13**	d_2	d_2
Optimization Procedure	**7**	d_2	d_2		**14**	d_2	d_2
Optimization Procedure	**8**	d_2	d_2		**15**	d_2	d_2
Optimization Procedure	**9**	d_2	d_2		**16**	d_2	d_2
Optimization Procedure	**10**	d_2	d_2		**17**	d_2	d_2
Optimization Procedure	**11**	d_2	d_2		**18**	d_2	d_2
Optimization Procedure	**12**	d_2	d_2				

5.0 Conclusion

In this paper, constructed efficient fractional factorial designs with balance coefficient and J_2 optimality criteria were used to compare the two forms of balanced coefficient methods using the generalized minimum aberration and minimum moment aberration criteria. It was observed that designs constructed using the maximum form of balance coefficient has the less aberration in both the generalized minimum aberration and minimum moment aberration criteria.

The major contributions of this paper include:

The screening of efficient mixed level designs associated with optimal near-balance and near orthogonal properties. The efficient mixed-level fractional designs are the solution to the excessive run requirements associated with many balanced mixed-level designs.

Balance coefficient of form I using maximum methods produces an efficient mixed fractional factorial designs as shown by the minimum aberrations

J_2 optimality values are lower in efficient fractional factorial designs

The relationship between the balance coefficient and J_2 optimality is inverse, that is, as one increase the other decreases

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