Computer Engineering and Intelligent Systems
ISSN 2222-1719 (Paper) ISSN 2222-2863 (Online)
Vol.5, No.1, 2014

www.iiste.org

Fixed Point Theorems Related to Fuzzy Metric Spaces

Rajesh Shrivastava, Jagrati Singhal*
Professor & Head, Department of Mathematics, Govt. Science & Commerce College Benazir Bhopal, M.P., India
Department of Mathematics, Sagar Institute of Research & Technology Bhopal, M.P., India

Abstract

In the present paper some fixed point and common fixed point theorems in complete Fuzzy 2-metric spaces and fuzzy 3- metric spaces are established which are motivated by Gahler [13-15], Sharma, Sharma and Isekey [30],Sharma , S.[31],

Key Words: Fuzzy metric spaces, fuzzy 2- metric spaces, fuzzy 3- metric spaces fixed point, Common fixed point .

2. Introduction:

In 1965, the concept of fuzzy sets was introduced by Zadeh [36]. After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [8], Erceg [10], Kaleva and Seikhala [23], Karamosil and Michalek [25], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [1,6,11,17,20,21,22,27,28,31,32] have also studied the fixed point theory in the fuzzy metric spaces and [2,3,4,5,19,26,33] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors.

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metic space. We [34, 35] have also worked on 2-Metric spaces and 2-Banach spaces for rational expressions.

We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-Metric space, which is suggested by the volume function.

SOME FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACES

Definition (3 A): A binary operation *: [0, 1] x [0,1] x [0,1] $\rightarrow$ [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever

$a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition (3 B): The 3-tuple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in X^3 x $[0,\infty)$ satisfying the followings

$$(FM^{'}-1): M(x,y,z,0) = 0$$

$$(FM^{'}-2): M(x,y,z,t) = 1, \forall t \succ 0, \Leftrightarrow x = y$$

$$(FM^{'}-3): M(x,y,t) = M(x,z,y,t) = M(y,z,x,t), symmetry\ about\ three\ var riable \quad .$$

$$(FM^{'}-4): M(x,y,z,t_{1,}\ t_{2,},t_{3,}) \geq M(x,y,u,t_{1,}) * M(x,u,z,t_{2,}) * M(u,y,z,t_{3,})$$

$$(FM^{'}-5): M(x,y,z):[0,1) \rightarrow [0,1]\ is\ left\ continuous, \forall x,y,z,u\,\varepsilon\, X, t_1,t_2,t_3 \succ 0$$

Definition (3C): Let (X, M, $*$) be a fuzzy 2-metric space. A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point x $\in$ X,

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1, \textit{ for all } a\,\varepsilon\, X \textit{ and } t \succ 0$$

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1, \textit{ for all } a\,\varepsilon\, X \textit{ and } t, p \succ 0$$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 3 D): A function M is continuous in fuzzy 2-metric space, iff whenever

For all a $\in$ X and t > 0.

$$x_n \to x, y_n \to y, \textit{then } \lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t), \forall\, a\,\varepsilon\, X \textit{ and } t \succ 0$$

Definition (3E): Two mappings A and S on fuzzy 2-metri space X are weakly commuting iff

$$M(ASu, SAu, a, t) \geq M(Au, Su, a, t), \forall\, u, a\,\varepsilon\, X \textit{and } t \succ 0$$

Theorem 3.1. Let (X,M,$*$) be a complete fuzzy 2-metric space. Let f and g be weakly compatible self maps of X satisfying

(3.1) M(gx,gy,a,kt) $\geq$ M(fx,fy,a,t) where 0<k<1, a>0

(3.2) g(X) $\subseteq$ f(X) .

If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof. Let $x_0 \epsilon$ X. Since g(X) $\subseteq$ f(X). Choose x_1 ϵ X such that g(x_0)= f(x_1). In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$ .Then by (3.1), we have

$$M(fx_n, fx_{n+1}, a, t) = M(gx_{n-1}, gx_n, a, t) \geq M(fx_{n-1}, fx_n, a, \tfrac{t}{k})$$

$$= M(gx_{n-2}, gx_{n-1}, a, \tfrac{t}{k}) \geq \dots\dots\dots \geq M(fx_0, fx_0, a, \tfrac{t}{k^n}).$$

Therefore, for any p,

$$M(fx_n, fx_{n+p}, a, t) \geq M(fx_n, fx_{n+1}, a, \tfrac{t}{p}) \geq \dots \geq M(fx_{n+p-1}, fx_{n+p}, a, \tfrac{t}{p})$$

$$\geq M(fx_0, fx_1, a, \tfrac{t}{pk^n}) \geq \dots \geq M(fx_0, fx_1, a, \tfrac{t}{pk^{n+p-1}}).$$

As n$\to\infty$. $\{fx_n\}$ = $\{y_n\}$ is a Cauchy sequence in fuzzy 2-metric space and so, by completeness of X, $\{y_n\}$ = $\{fx_n\}$ is convergent. We call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = z . As f(X) is complete, so there exist a point p in X such that fp=z. Now, from (3.1),

As n $\to\infty$, M(gp,gx_n,a,kt)$\geq$ M(fp,fx_n,a,t),

M(gp,z,a,kt)$\geq$ M(fp,z,a,t)

M(gp,z,a,kt)≥ M(z,z,a,t),

M(gp,z,a,kt)≥ 1,

M(gp,z,a,kt)= 1,

gp=z=fp.

As f and g are weakly compatible. Therefore fgp=gfp i.e. fz=gz. Now, we show that z is fixed point of f and g. From (3.1),

As n → ∞ M(gz,gx_n,a,kt)≥ M(fz,fx_n,a,t),

M(gz,z,a,kt)≥ M(fz,z,a,t),

M(gz,z,a,kt)≥ M(gz,z,a,t),

gz=z=fz.

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), M(gz,gw,a,kt) ≥ M(fz,fw,a,t), M(z,w,a,kt) ≥ M(z,w,a,t) and z=w.

Therefore z is unique common fixed point of f and g.

Theorem 3.2. Let (X,M, ∗) be a fuzzy 2-metric space. Let f and g weakly compatible self maps of X satisfying condition (3.1) and (3.2). If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof From the proof of above theorem. We conclude that $\{fx_n\}$ =$\{y_n\}$ is a Cauchy sequence in X. Now suppose that f(X) is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in f(X). Call it be u and f(v) = u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (3.1),

As n → ∞ M(gv,gx_n,a,kt)≥ M(fv,fx_n,a,t),

M(gv,u,a,kt)≥ M(fv,u,a,t),

M(gv,u,a,kt)≥ M(u,u,a,t),

M(gv,u,a,kt)≥ 1,

M(gv,u,a,kt)= 1,

gv=u=fv.

Which shows that pair (f,g) has a point of coincidence . Since, f and g are weakly compatible, fgv = gfv, i.e. fu = gu. Now, we show that u is a fixed point of f and g. From(3.1).

As n → ∞M(gu,gx_n,a,kt)≥ M(fu,fx_n,a,t),

M(gu,u,a,kt)≥ M(fu,u,a,t),

M(gu,u,a,kt)≥ M(gu,u,a,t),

gu=u=fu.

Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1),), M(gz,gw,a,kt) ≥ M(fz,fw,a,t), M(z,w,a,kt) ≥ M(z,w,a,t) and z=w. Therefore z is unique common fixed point of f and g.

SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACES

Definition (4.A): A binary operation *: $[0, 1]^4 \to [0, 1]$ is called a continuous t-norm if

([0, 1], *) is an abelian topological monoid with unit 1 such that

$a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$ Whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2 are in [0,1].

Definition (4.B): The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm monoid and M is a fuzzy set in $X^4 \times [0, \infty]$ satisfying the following conductions:

$$(FM''-1): M(x,y,z,w,0) = 0$$
$$(FM''-2): M(x,y,z,w,t) = 1, \forall t \succ 0,$$
$$\textit{Only when the three simplex } \langle x,y,z,w \rangle \textit{ degenerate}$$
$$(FM''-3): M(x,y,z,w,t) = M(x,w,z,y,t) = M(z,w,x,y,t) = ----$$
$$(FM''-4): M(x,y,z,w,t+t_2+t_3) \geq M(x,y,z,u,t_{1,})*$$
$$M(x,y,u,w,t_{2,}) * M(x,u,,z,w,t_{3,}) * M(u,,y,z,w,t_4)$$
$$(FM''-5): M(x,y,z,w) : [0,1) \to [0,1] \textit{ is left continuous},$$
$$\forall x,y,z,u,w \,\varepsilon\, X, t_1,t_2,t_3,t_4 \succ 0$$

Definition (4.C): Let (X, M, *) be a fuzzy 3-metric space:

(1)A sequence {Xn} in fuzzy 3-metric space X is said to be convergent to a point x ∈ X, if

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, \textit{ for all } a, b \,\varepsilon\, X \textit{ and } t \succ 0$$

(2)A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1, \textit{ for all } a, b \,\varepsilon\, X \textit{ and } t, p \succ 0$$

(3)A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (4.D) A function M is continuous in fuzzy 3-metric space if

$$x_n \to x, y_n \to y, \textit{then } \lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, t), \forall a, b \,\varepsilon\, X \textit{ and } t \succ 0$$

Definition (4.E): Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff,

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t) \;\forall u, a, b \,\varepsilon\, X \textit{and } t \succ 0$$

Theorem 4.1. Let (X,M,∗) be a complete fuzzy 3-metric space. Let f and g be weakly compatible self maps of X satisfying

(4.1) M(gx,gy,a,b,kt) ≥ M(fx,fy,a,t) where 0<k<1, a,b >0

(4,2) g(X) ⊆ f(X) .

If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof. Let $x_0 \epsilon$ X. Since g(X) ⊆ f(X). Choose x_1 ϵ X such that g(x_0)= f(x_1). In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$ .Then by (4.1), we have

$$M(fx_n, fx_{n+1}, a, b, t) = M(gx_{n-1}, gx_n, a, b, t) \geq M(fx_{n-1}, fx_n, a, b, \tfrac{t}{k})$$

$$= M(gx_{n-2}, gx_{n-1}, a, b, \tfrac{t}{k}) \geq \dots\dots\dots \geq M(fx_0, fx_0, a, b, \tfrac{t}{k^n}).$$

Therefore, for any p,

$$M(fx_n, fx_{n+p}, a, b, t) \geq M(fx_n, fx_{n+1}, a, b, \tfrac{t}{p}) \geq \dots \geq M(fx_{n+p-1}, fx_{n+p}, a, b, \tfrac{t}{p})$$

$$\geq M(fx_0, fx_1, a, b, \tfrac{t}{pk^n}) \geq \dots \geq M(fx_0, fx_1, a, b, \tfrac{t}{pk^{n+p-1}}).$$

As n→ ∞ . $\{fx_n\}$ = $\{y_n\}$ is a Cauchy sequence in fuzzy 3-metric space and so, by completeness of X, $\{y_n\}$ = $\{fx_n\}$ is convergent. We call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = z . As f(X) is complete, so there exist a point p in X such that fp=z. Now, from (3.1),

As n → ∞, M(gp,gx_n,a,b,kt)≥ M(fp,fx_n,a,b,t),

M(gp,z,a,b,kt)≥ M(fp,z,a,b,t)

M(gp,z,a,b,kt)≥ M(z,z,a,b,t),

M(gp,z,a,b,kt)≥ 1,

M(gp,z,a,b,kt)= 1,

gp=z=fp.

As f and g are weakly compatible. Therefore fgp=gfp i.e. fz=gz. Now, we show that z is fixed point of f and g. From (4.1),

As n → ∞ M(gz,gx_n,a,b,kt)≥ M(fz,fx_n,a,b,t),

M(gz,z,a,b,kt)≥ M(fz,z,a,b,t),

M(gz,z,a,b,kt)≥ M(gz,z,a,b,t),

gz=z=fz.

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (4.1), M(gz,gw,a,b,kt) ≥ M(fz,fw,a,b,t), M(z,w,a,b,kt) ≥ M(z,w,a,b,t) and z=w.

Therefore z is unique common fixed point of f and g.

Theorem 4.2. Let (X,M, $*$) be a fuzzy 3-metric space. Let f and g weakly compatible self maps of X satisfying condition (4.1) and (4.2). If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof. From the proof of above theorem. We conclude that $\{fx_n\}$ =$\{y_n\}$ is a Cauchy sequence in X. Now suppose that f(X) is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in f(X). Call it be u and f(v) = u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (4.1),

As n → ∞ M(gv,gx_n,a,b,kt)≥ M(fv,fx_n,a,b,t),

M(gv,u,a,b,kt)≥ M(fv,u,a,b,t),

M(gv,u,a,b,kt)≥ M(u,u,a,b,t),

M(gv,u,a,b,kt)≥ 1,

M(gv,u,a,b,kt)= 1,

gv=u=fv.

Which shows that pair (f,g) has a point of coincidence . Since, f and g are weakly compatible, fgv = gfv, i.e. fu = gu. Now, we show that u is a fixed point of f and g. From(4.1).

As n → ∞M(gu,gx_n,a,b,kt)≥ M(fu,fx_n,a,b,t),

M(gu,u,a,b,kt)≥ M(fu,u,a,b,t),

M(gu,u,a,b,kt)≥ M(gu,u,a,b,t),

gu=u=fu.

Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (4.1),), M(gz,gw,a,b,kt) ≥ M(fz,fw,a,b,t), M(z,w,a,b,kt) ≥ M(z,w,a,b,t) and z=w. Therefore z is unique common fixed point of f and g.

References

1. Badard, R. "Fixed point theorems for fuzzy numbers" Fuzzy sets and systems 13 (1984) 291-302.
2. Bose, B.K. and Sahani, D. "Fuzzy mappings and fixed point theorems" Fuzzy sets and Systems 21 (1984) 53-58.
3. Butnariu, D. "Fixed point for fuzzy mappings" Fuzzy sets and Systems 7 (1982) 191-207.
4. Chang, S.S. "Fixed point theorems for fuzzy mappings" Fuzzy Sets and Systems 17(1985) 181-187.
5. Change, S.S., Cho, Y.J., Lee, B.S. and Lee, G.M. "Fixed degree and fixed point theorems for fuzzy mappings" Fuzzy Sets and Systems 87 (1997) 325-334.
6. Change, S.S., Cho, Y.J. , Lee B.S., June, J.S. and Kang, S.M. "Coincidence point and minimization theorems in fuzzy metric spaces" Fuzzy Sets and Systems, 88 (1) (1997) 119-128.
7. Cho, Y.J. "Fixed points and fuzzy metric spaces" J. Fuzzy Math.5 (1997) no. 4, 949-962.
8. Deng, Z. "Fuzzy pseudo-metric space" J. Math. Anal. Appl.86 (1982) 74-95.
9. Ekland, I. and Gahler, S. "Basic notions for fuzzy topology" Fuzzy Sets and System 26 (1988) 336-356.
10. Erceg, M.A. "Metric space in fuzzy set theory" J. Math. Anal. Appl.69 (1979) 205-230.
11. Fang, J.X. "On fixed point theorems in fuzzy metric spaces" Fuzzy Sets and Systems 46 (1979) 107-113.
12. Fisher, B. "Mappings with a common fixed point" Math. Seminar notes Kobe university 7 (1979) 81-84.
13. Gahler, S. Linear 2-normierte Raume" Math. Nachr. 28 (1964) 1-43.

14. Gahler, S. Uber 2-Banach Raume, Math. Nachr. 42 (1969) 335-347
15. Gahler, S. 2- metrische Raume and ihre topologische structure, Math. Nachr.26 (1983) 115-148.
16. George, A. and Veramani, P. "On some results in fuzzy metric spaces" Fuzzy Sets and Systems 64 (1994) 395-399.
17. Grabiec, M. "Fixed points in fuzzy metric space" Fuzzy Sets and Systems 27 (1988) 385-389.
18. Hu, C. "Fuzzy topological space" J. Math. Anal. Appl. 110(1985)141-178.
19. Heipern, S. "Fuzzy mappings and fixed point theorems" J. Math. Anal. Appl. 83 (1981) 566-569.
20. Hadzic, O. "Fixed point theorems, for multi-valued mappings in some classes of fuzzy metric space" Fuzzy Sets and Systems 29(1989) 115-125.
21. Jung, J.S., Cho. Y.J. and Kim J.K.: Minimization theorems for fixed point theorems in fuzzy metric spaces and applications, Fuzzy Sets and Systems 61 (1994) 199-207.
22. Jung, J.S., Cho, Y.J., Chang, S.S. and Kang, S.M. "Coincidence theorems for set valued mappings and Ekland's variational principle in fuzzy metric spaces" Fuzzy Sets and Systems 79 (1996) 239-250.
23. Kaleva, O. and Seikkala, S. "On Fuzzy metric spaces" Fuzzy Sets and Systems 12 (1984) 215-229.
24. Kaleva, O. "The Completion of fuzzy metric spaces" J. Math. Anal. Appl. 109(1985) 194-198.
25. Kramosil, I. and Michalek, J. "Fuzzy metric and statistical metric spaces" Kybernetica 11 (1975) 326-334.
26. Lee, B.S., Cho, Y.J. and Jung, J.S. "Fixed point theorems for fuzzy mappings and applications" Comm. Korean Math. Sci. 11 (1966) 89-108.
27. Kutukcu, S., Sharma, S. and Tokgoz, H. "A Fixed point theorem in fuzzy metric spaces" Int. Journal of Math. Analysis 1 (2007) no.18, 861-872.
28. Mishra, S.N., Sharma, N. and Singh, S.L. "Common fixed points of maps on fuzzy metric spaces" Internet. J. Math. & Math Sci.17 (1966) 89-108.
29. Schweitzer, B. and Sklar, A. "Statistical metric spaces" Pacific Journal Math.10 (1960) 313-334.
30. Sharma. P.L., Sharma. B.K. and Iseki, K. "Contractive type mapping on 2-metric spaces" Math. Japonica 21 (1976) 67-70.
31. Sharma, S. "On fuzzy Metric space" Southeast Asian Bulletin of Mathematics 26 (2002) 133-145.
32. Som, T. "Some results on common fixed point in fuzzy Metric spaces" Soochow Journal of Mathematics 4 (2007) 553-561.
33. Turkoglu, D and Rhodes, B.E. "A fixed fuzzy point for fuzzy mapping in complete metric spaces" Mathematical Communications10 (2005)115-121.
34. Yadava, R.N., Rajput, S.S. Choudhary, S. and Bhardwaj, R.K "Some fixed point theorems for rational inequality in 2- metric spaces" Acta Ciencia Indica 33 (2007) 709-714.
35. Yadava, R.N., Rajput, S.S. Choudhary, S. and Bhardwaj, R.K. "Some fixed point theorems for non contraction type mapping on 2- Banach spaces" Acta Ciencia Indica 33 (2007) 737-744.
36. Zadeh, L.A. "Fuzzy Sets" Information and control 8 (1965) 338-353.