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# **Common Fixed Point Theorem for Compatible Mapping**

# of Type (A)

Vishal Gupta Department pf Mathematics, Maharishi Markandeshwar University, Mullana, Ambala, Haryana, India. vishal.gmn@gmail.com, vkgupta09@rediffmail.com

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### Abstract

The purpose of this paper is to prove a common fixed point theorem involving two pairs of compatible mappings of type (A) using six maps using a contractive condition. This article represents a useful generalization of several results announced in the literature.

Key Words: Complete metric space, Compatible mapping of type (A), Commuting

mapping, Cauchy Sequence, Fixed points.

### 1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been studied by many mathematicians.Seesa(1982) introduce the concept of weakly commuting mapping and proved some theorem of commutativity by useing the condition to weakly commutativity, Jungck(1988) gave more generalized commuting and weakly commuting maps called compatible maps and use it for compatibility of two mappings. After that Jungck Muthy and Cho(1993) made another generalization of weak commuting mapping by defining the concept of compatible map of type (A).

We proposed to re-analysis the theorems of Aage C.T (2009) on common fixed point theorem compatibility of type (A)

### **2.Preliminaries**

*Definition 2.1.* Self maps S and T of metric space (X,d) are said to be weakly commuting pair

iff 
$$d(STx,TSx) \le d(Sx,Tx)$$
 for all x in X.

*Definition 2.2.* Self maps S and T of a metric space (X,d) are said to be compatible of type (A) if

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2011  $\lim_{n \to \infty} d(TSx_n, SSx_n) = 0 \text{ and } \lim_{n \to \infty} d(STx_n, TTx_n) = 0 \text{ as } n \to \infty \text{ whenever}$   $\{x_n\} \text{ is a}$ 

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sequence in X such that  $\lim Sx_n = \lim Tx_n = t$  as  $n \to \infty$  for some t in X.

Definition 2.3. A function  $\Phi: [0, \infty) \to [0, \infty)$  is said to be a contractive modulus if  $\Phi(0) = 0$  and

$$\Phi$$
 (t)  0.

#### 3.Main Result

*Theorem 3.1.* Let S, R, T, U, I and J are self mapping of a complete metric space (X,d) into itself satisfying the conditions

- (i)  $SR(X) \subset J(X), TU(X) \subset I(X)$
- (ii)  $d(SRx,TUy) \le \alpha \ d(Ix,Jy) + \beta \ [d(Ix,SRx) + d(Jy,TUy)] + \gamma \ [d(Ix,TUy) + d(Jy,SRx)]$

for all x,y  $\in$  X and  $\alpha,\beta$  and  $\gamma$  are non-negative reals such that  $\alpha{+}2\beta{+}2\gamma{<}1$ 

(iii) One of S,R,T,U,I and J is continuous.

(iv) (SR,I) and (TU.J) are compatible of type (A).Then SR,TU,I,J have a unique common

fixed point. Further if the pairs (S,R), (S,I), (R,I), T,U), (T,I), (U.J)

are commuting

pairs then S,R,T,U,I and J have a unique common fixed point.

*Proof:* Let  $x_0 \in X$  be arbitrary. Choose a point  $x_1$  in X such that  $SRx_0 = Jx_1$ .

This can be done since  $SR(X) ) \subset J(X)$ .

Let  $x_2$  be a point in X such that  $TUx_1 = Ix_2$ . This can be done since  $TU(X) \subset I(X)$ .

In general we can choose  $x_{2n}$ ,  $x_{2n+1}$ ,  $x_{2n+2}$  ..., such that  $SRx_{2n} = Jx_{2n+1}$  and  $TUx_{2n+1} = Ix_{2n+2}$ . So that we obtain a sequence  $SRx_0$ ,  $TUx_1$ ,  $SRx_2$ ,  $TUx_3$  .....

Using condition (ii) we have

 $\begin{array}{l} d(SRx_{2n},TUx_{2n+1}) \leq \alpha \ d(I_{2n},Jx_{2n+1}) \ + \beta \ [d(Ix_{2n},SRx_{2n}) \ + \ d(\ Jx_{2n+1},\ TUx_{2n+1})] \ + \gamma \ [d(Ix_{2n},TUx_{2n+1}) \ + \ d(Ix_{2n+1},TUx_{2n+1})] \end{array} \right) \\ + \gamma \ [d(Ix_{2n},TUx_{2n+1}) \ + \ d(Ix_{2n},TUx_{2n+1}) \ + \ d$ 

 $\begin{array}{l} & d(Jx_{2n+1},SRx_{2n})] \\ = \alpha \ d \ (TUx_{2n-1},SRx_{2n}) + \beta \ [d(TUx_{2n-1},SRx_{2n}) + d(SRx_{2n},TUx_{2n+1})] + \\ & \gamma[d(TUx_{2n-1},TUx_{2n+1}) + \ \ d(SRx_{2n},SRx_{2n})] \\ \leq \alpha \ d(TUx_{2n-1},SRx_{2n}) + \beta \ [d(TUx_{2n-1},SRx_{2n}) + d(SRx_{2n},TUx_{2n+1})] + \\ & \Gamma(TUx_{2n-1},SRx_{2n}) + \beta \ [d(TUx_{2n-1},SRx_{2n}) + d(SRx_{2n},TUx_{2n+1})] + \\ \end{array}$ 

 $\gamma [d(TUx_{2n-1}, SRx_{2n}) + d(SRx_{2n}, TUx_{2n+1})]$ 

 $= (\alpha + \beta + \gamma) d (TUx_{2n-1}, SRx_{2n}) + (\beta + \gamma) (SRx_{2n}, TUx_{2n+1})$ 

Hence  $d(SRx_{2n}, TUx_{2n+1}) \le kd(SRx_{2n}, TUx_{2n-1})$  where  $k=(\alpha+\beta+\gamma)/(1-(\beta+\gamma)) < 1$ , Similarly we can show  $d(SRx_{2n}, TUx_{2n-1}) \le k d(SRx_{2n-2}, TUx_{2n-1})$ Therefore  $d(SRx_{2n}, TUx_{2n+1}) \le k^2 d(SRx_{2n-2}, TUx_{2n-1})$ 

$$\leq k^{2n} d(SRx_0, TUx_1)$$

Which implies that the sequence is a Cauchy sequence and since (X,d) is complete so the sequence has a limit point z in X. Hence the subsequences  $\{SRx_{2n}\} = \{Jx_{2n-1}\}$  and  $\{TUx_{2n-1}\} = \{Ix_{2n}\}$  also converges to the point z in X.

Suppose that the mapping I is continuous. Then  $I^2x_{2n} \rightarrow Iz$  and  $ISRx_{2n} \rightarrow Iz$  as  $n \rightarrow \infty$ . Since the pair (SR,I) is compatible of type (A). we get  $SRIx_{2n} \rightarrow Iz$  as  $n \rightarrow \infty$ . **Computer Engineering and Intelligent Systems** www.iiste.org ISSN 2222-1719 (Paper) ISSN 2222-2863 (Online) IISTE Vol 2, No.8, 2011 Now by (ii)  $d(SRIx_{2n}, TUx_{2n+1}) \le \alpha d(I^2x_{2n}, Jx_{2n+1}) + \beta [d(I^2x_{2n}, SRIx_{2n}) + d(Jx_{2n+1}, TUx_{2n+1})] + \beta [d(I^2x_{2n}) + d(Jx_{2n}) +$  $\gamma [d(I^2 x_{2n}, TU x_{2n+1}) + d(J x_{2n+1}, SRI x_{2n})]$ letting  $n \rightarrow \infty$ , we get  $d(Iz,z) \leq \alpha d(Iz,z) + \beta [d(Iz,z) + d(z,z)] + \gamma [d(Iz,z) + d(z,Iz)]$  $=(\alpha+2\gamma) d(Iz,z)$ This gives d(Iz,z)=0 since  $0 \le \alpha + 2\gamma < 1$ , Hence Iz=z. Further  $d(SRz, TUx_{2n+1}) \le \alpha d(Iz, Jx_{2n+1}) + \beta [d(Iz, SRz) + d(Jx_{2n+1}, TUx_{2n+1})] + \beta [d(Iz, SRz) + d(Jx_{2n+1}, TUx_{2n+1})] + \beta [d(Iz, SRz) + d(Jx_{2n+1}, TUx_{2n+1})]$  $\gamma [d(Iz,TUx_{2n+1}) + d(Jx_{2n+1}, SRz)]$ Letting  $Jx_{2n+1}$ ,  $TUx_{2n+1} \rightarrow z$  as  $n \rightarrow \infty$  and Iz=z we get  $d(SRz,z) \le \alpha d(z,z) + \beta [d(z,SRz) + d(z,z)] + \gamma [d(z,z) + d(z,SRz)]$  $= (\beta + \gamma) d(SRz,z)$ Hence d(SRz,z) = 0 i.e SRz=z, since  $0 \le \beta + \gamma < 1$ . Thus SRz=Iz=zSince  $SR(X) \subset J(X)$ , there is a point  $z_1$  in X such that  $z=SRz=-Jz_1$ Now by (ii)  $d(z,TUz_1) = d(SRz,TUz_1)$  $\leq \alpha d(Iz,Jz_1) + \beta [d(Iz,SRz) + d(Jz_1,TUz_1)] + \gamma [d(Iz,TUz_1) + d(Jz_1,SRz)]$  $= \alpha d(z,z) + \beta [d(z,z) + d(z,TUz_1)] + \gamma [d(z,TUz_1) + d(z,z)]$  $=(\beta+\gamma) d(z,TU z_1)$ Hence  $d(z,TUz_1) = 0$  i.e TU  $z_1 = z = Jz_1$ , since  $0 \le \beta + \gamma < 1$ , Take  $y_n = z_1$  for  $n \ge 1$ Then  $TUy_n \rightarrow Tz_1 = z$  and  $Jy_n \rightarrow J z_1 = z$  as  $n \rightarrow \infty$ Since the pair (TU,J) is compatible of type (A), we get Lim d(TUJy<sub>n</sub>, JJy<sub>n</sub>) =0 as  $n \rightarrow \infty$  implies d(TUz,Jz)=0 since Jy<sub>n</sub> =z for all  $n \ge 1$ . Hence TUz=Jz. Now d(z,TUz) = d(SRz,TUz) $\leq \alpha d(Iz,Jz) + \beta [d(Iz,SRz) + d(Jz,TUz)] + \gamma [d(Iz,TUz) +$ d(Jz, SRz)  $= \alpha d(z,TUz) + \beta [d(z,z) + d(TUz,TUz)] + \gamma [d(z,TUz) + d(TUz,z)]$  $=(\alpha+2\gamma) d(z, TUz)$ Since  $\alpha + 2\gamma < 1$ , we get TUz=z, hence z=TUz=Jz therefore z is common fixed point of SR,TU,I,J when the continuity of I is assumed. Now suppose that SR is continuous then  $S^2R x_{2n} \rightarrow SRz$ ,  $SRIx_{2n} \rightarrow SRz$  as  $n \rightarrow \infty$ . By condition (ii), we have  $d(S^{2}Rx_{2n}, TUx_{2n+1}) \leq \alpha d(ISRx_{2n} Jx_{2n+1}) + \beta [d(ISRx_{2n}, S^{2}Rx_{2n}) + d(Jx_{2n+1}, TUx_{2n+1}) + \beta [d(ISRx_{2n}, S^{2}Rx_{2n}) + d(Jx_{2n+1}, S^{2}Rx_{2n+1}) + \beta [d(ISRx_{2n}, S^{2}Rx_{2n}) + \beta [d(ISRx_{2n}, S^{2}$  $\gamma[d(ISRx_{2n}, TUx_{2n+1}) + d(Jx_{2n+1}, S^2Rx_{2n})]$ letting  $n \rightarrow \infty$  and using the compatibility of type (A) of the pair (SR,I), we get  $d(SRz,z) \le \alpha d(SRz,z) + \beta \left[ d(SRz,SRz) + d(z,z) \right] + \gamma \left[ d(SRz,z) + d(z,SRz) \right]$  $=(\alpha+2\gamma) d(SRz,z)$ Since  $\alpha + 2\gamma < 1$  we get SRz=z. But SR(X)  $\subset$  J(X) there is a point p in X such that ,Now by (ii) z=SRz=Jp  $d(S^2Rx_{2n}, TUp) \leq \alpha d(ISRx_{2n}, Jp) + \beta [d(ISRx_{2n}, S^2Rx_{2n}) + d(Jp, TUp)] +$  $\gamma$ [d(ISRx<sub>2n</sub>,TUp)+d(Jp,S<sup>2</sup>Rx<sub>2n</sub>) letting  $n \rightarrow \infty$  we have d(z,TUp) = d(SRz,TUp) $\leq \alpha d(z,z) + \beta [d(z,z) + d(z,TUp)] + \gamma [d(z,TUp) + d(z,z)]$  $=(\beta+\gamma) d(z,TUp)$ Since  $\beta + \gamma < 1$ , we get TUp=z. Thus z=Jp=TUp. Let  $y_n = p$  then  $TUy_n \rightarrow TUp = z$  and  $Jy_n \rightarrow TUp = z$ Since (TU,J) is compatible of type (A), we have Lim d(TUJy<sub>n</sub> ,JJy<sub>n</sub>) =0 as  $n \rightarrow \infty$ This gives TUJp=JTUp or TUz=Jz Further

23 | P a g e www.iiste.org **Computer Engineering and Intelligent Systems** www.iiste.org ISSN 2222-1719 (Paper) ISSN 2222-2863 (Online) IISTE Vol 2, No.8, 2011  $d(SRx_{2n}, TUz) \leq \alpha d(Ix_{2n}, Jz) + \beta [d(Ix_{2n}, SRz) + d(Jz, TUz)] + \gamma [d(Ix_{2n}, TUz) + d(Jz, SRx_{2n})]$ Letting  $n \rightarrow \infty$ , we get  $d(z, TUz) \leq \alpha d(z, TUz) + \beta [d(z, z) + d(TUz, TUz)] + \gamma [d(z, TUz) + d(TUz, z)]$  $=(\alpha+2\gamma) d(z,TUz)$ Since  $0 \le \alpha + 2\gamma < 1$  we get z = TUzAgain we have  $TU(X) \subset I(X)$  there is a point q in X such that z=TUz=Iq Now d(SRq,z) =d(SRq,TUz)d(Iq,Jz)[d(Iq,SRq)] $\leq$ α  $+\beta$ +d(Jz,TUz)]+ $\gamma[d(Iq,TUz)+d(Jz,SRq)]$  $=\alpha d(z,z) + \beta \left[ d(z,SRq) + d(z,z) \right] + \gamma \left[ d(z,TUz) + d(z,SRq) \right]$  $=(\beta+\gamma) d(z,SRq)$ Since  $0 \le \beta + \gamma < 1$  we get SRq=z, take  $y_n = q$  then SRy<sub>n</sub>  $\rightarrow$  SRq =z, Iy<sub>n</sub>  $\rightarrow$  Iq=z Since (SR,I) is compatible of type (A), we get Lim d( ISRy<sub>n</sub>, IIy<sub>n</sub>) = 0 as  $n \rightarrow \infty$ This implies that SRIq=ISRq or SRz=Iz. Thus we have z=SRz=Iz=Jz=TUz Hence z is a common fixed point of SR,TU,I and J, when S is continuous The proof is similar that z is common fixed point of SR,TU,I and J when I is continuous, R and U is continuous. For uniqueness let z and w be two common fixed point os SR,TU,I and J, then by condition (ii)  $d(z,w) = d(SRz,TUw) \leq \alpha d(Iz,Jw) + \beta [d(Iz,SRz) + d(Jw,TUw)] + \gamma [d(Iz,TUw)]$ +d(Jw,SRz)]  $=\alpha d(z,w) + \beta [d(z,z) + d(w,w)] + \gamma [d(z,w) + d(w,z)]$  $= (\alpha + 2\gamma) d(z,w)$ Since  $\alpha + 2\gamma < 1$  we have z=w. Again let z be the unique common fixed point of both the pairs (SR,I), (TU,J) then Sz=S(SRz) = S(RSz) = SR(Sz)Sz=S(Iz)=I(Sz)Rz=R(SRz)=(RS)(RS)=(SR)(Rz)Rz=R(Iz)=I(Rz)Which shows that Sz and Rz is the common fixed point of (SR,I) yielding thereby Sz=z=Rz=Iz=SRz In view of uniqueness of the common fixed point of the pair (SR,I). Similarly using the commutativity of (T,U), (T,J), (U,J) it can be shown that Tz=z=Uz=Jz=TUz. Thus z is the unique common fixed point of S,R,T,U,I and J. Hence the proof.

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