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On Optimal Power Allocation for Gaussian Broadcast Channel

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Abstract
We derive the optimal power allocation for Gaussian two users broadcast channel. To find the optimal power allocation between the two users, two optimization schemes are considered. In each optimization scheme, an analytical expression for the optimal power allocation between the two users is derived. The first optimization criterion finds the optimal power allocation between the two users such that they have equal rates. Then, the optimal power allocation that maximizes the sum rate capacity is studied. In addition, numerical examples are provided to verify the optimality of the derived schemes.
Keywords: Gaussian Broadcast Channel, Capacity Region, Optimization.

1. Introduction
Even thought the capacity region of the well known Gaussian broadcast channel (GBC) [Cover,1972] is established more than thirty years ago, the problem of optimal power allocation among the different users are still unknown. In its simpler form, GBC which contains a source that communicates with two different users is of particular interest. Specifically, the channels to these two users are degraded which means that the variance of the noise channels added over these two channels are not equal. Furthermore, the capacity region of this network is attained by superposition encoding of Gaussian inputs with different powers.

Transmit power allocation between these users is an effective for increasing the spectrum efficiency of a wireless communication system. In particular, the capacity region of the degraded GBC is achievable for arbitrary values of the power allocation between the two users. In practical systems, this is not the case, since one of the two users (the stronger user) has to decode both signals. This decoding can happen only in the case that one signal is stronger than the other such as in the case of very strong Gaussian interference channel [Carleial, 1975]. This means that much of the available power should be allocated to one of the two users sharing this channel whereas the rest of the power is allocated to the other user.

The problem of optimal power allocation for the broadcast channel (BC) was studied in many different scenarios. For instance, the authors in [Jindal & Goldsmith, 2003] characterized the optimal power allocation for slowly fading BC channel. Optimal power allocation over parallel GBC was studied in [Tse, 1997]. In addition, power allocation was investigated for broadcast relay channel in [Thakur etl, 2011]. Moreover, performance analysis for GBC was studied in [Bhat etl, 2010], where power allocation between the two users is randomly selected.

In this paper, we analytically characterize the optimal power allocation of the GBC using simple mathematical derivation. In particular, two optimization schemes are considered to optimally divide the total power between the two users. Specifically, the first scheme deals with deriving the optimal power allocation such that the two users have the same data rate. Then, the optimal power allocation that maximizes the sum rate capacity is studied. The associated numerical results show that i) the stronger user, with higher signal to noise ratio, has to be allocated less power than the weaker user, ii) the power allocated to the stronger user is inversely proportional with the total available power, and iii) the sum rate obtained from the sum rate maximization criterion is marginally better than the sum rate obtained from the equal rate optimization scheme.

The remainder of this paper is outlined as follows. Section 2 describes the communication model that we study in this work. Then, two optimization problems are studied, to allocate the available power between the two users sharing the GBC, in Section 3. In addition, some numerical examples that validate our theoretical results are also presented in Section 4. Finally, we conclude the paper in Section 5.

2. Gaussian Broadcast Channel
The GBC channel considered in this paper, with the structure of its transmitter, is depicted in Fig. 1. The sender transmits two different messages X_1 and X_2 to two different receivers at the same time. Thus, the transmitted signal is given as

$$\mathrm{X} = \sqrt{\alpha \mathrm{P}}\, \mathrm{X}_1 + \sqrt{(1-\alpha)\mathrm{P}}\, \mathrm{X}_2 \qquad (1)$$

where the parameter $\alpha \in [0,1]$ defines the power allocation between the two signals, and P is the power constraint at the transmitter. The received signal, Y_i, at a given destination $i \in \{0,1\}$ is given as

$$\begin{aligned} Y_i &= X + Z_i \\ &= \sqrt{\alpha P}\, X_1 + \sqrt{(1-\alpha) P}\, X_2 + Z \end{aligned} \tag{2}$$

where Z_i is a sequence of independent and identically distributed Gaussian random variables with zero mean and N_i variance. Further, $N_1 < N_2$ is assumed such that the first receiver can decode the common message X_2 better than the second destination. Then, after decoding the common signal X_2, the first destination can decode the desired signal, X_1. Practically, the authors in [Al-qudah & Rajan,2013] reported that the error of estimating the common signal at the first destination can dominate the error rate at this destination. In order to avoid this estimation error, the transmitter can employ dirty paper coding [Al-qudah & Rajan,2013, Mazzotti & Chiani, 2006] to completely remove the effect of the common message at the first destination. Finally, we note that the capacity region of the degraded GBC is characterized as

$$R_1 < C\left(\frac{\alpha P}{N_1}\right) \tag{3a}$$

$$R_2 < C\left(\frac{(1-\alpha) P}{N_2 + \alpha P}\right) \tag{3b}$$

where $C(x) = 0.5\log_2(1+x)$, R_1 and R_2 are the data rates for the first and second users, respectively.

3. Optimal Transmit Power Allocation

In this section, two different optimization criteria are used to select the value of the power allocation parameter, α. The first optimization criterion is based on equal capacity. For instance, the sender can transmit to the both destinations at the same data rate. In the second optimization problem, the value of α is selected to maximize the sum rate.

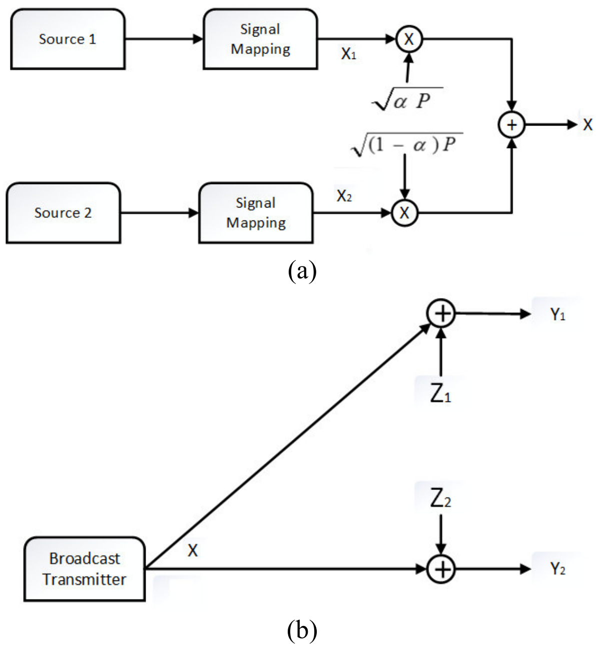

Figure 1. (a) Block diagram of the Broadcast Transmitter. (b) Gaussian Broadcast Channel.

3.1 Equal Capacity Criterion

The optimal transmit power allocation, α, is selected such that both users can have the same data rate, i.e., $R_1 = R_2$. Thus, it is sufficient to find the value of α such that

$$\frac{\alpha P}{N_1} = \frac{(1-\alpha) P}{N_2 + \alpha P} \tag{4}$$

which is equivalent to find the roots of $P^2\alpha^2 + P(N_1 + N_2)\alpha - P N_1 = 0$. Thus, the value of α that results in equal data rates for both users is given as

$$\alpha = \frac{-(N_1 + N_2) + \sqrt{(N_1 + N_2)^2 + 4 N_1 P}}{2P} \tag{5}$$

Further, we may note that this solution can not violate the degradedness condition since the following condition

$$C\left(\frac{(1-\alpha)P}{N_1 + \alpha P}\right) > C\left(\frac{(1-\alpha)P}{N_2 + \alpha P}\right) \tag{6}$$

is still valid, given that $N_1 < N_2$. Practically, it was indicated in [Al-qudah & Rajan,2013] that the error of estimating the common signal X_2 at the first destination can dominate the error rate at this destination. Thus, the probability of estimating the common signal at the first destination should be better than estimating the signal X_1 at its own receiver. Therefore, the following condition

$$C\left(\frac{(1-\alpha)P}{N_1 + \alpha P}\right) > C\left(\frac{\alpha P}{N_1}\right) \tag{7}$$

should also be satisfied. Actually, in this optimization problem, the previous condition is always valid since

$$C\left(\frac{(1-\alpha)P}{N_1 + \alpha P}\right) > C\left(\frac{(1-\alpha)P}{N_2 + \alpha P}\right) = C\left(\frac{\alpha P}{N_1}\right) \tag{8}$$

Numerically, Fig. 2 shows how the optimal value of α is varying versus the transmit power P. In particular, this figure shows that the power allocated to the first user (stronger user) is inversely proportional to i) the transmit power P, and ii) the noise variance N_2. It specifically shows that as either P or N_2 increases, the allocated power to the first user decreases. Remember that the first user's signal works as an additional noise (interference) at the second destination (weaker user). In addition, when N_2 increases, more power is allocated to the second user's signal for equal data rates.

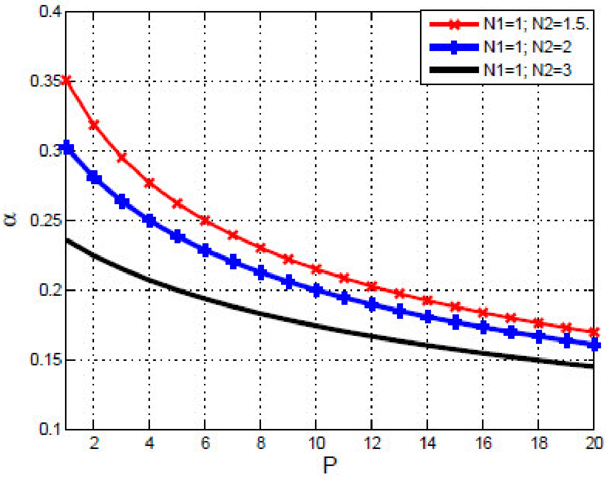

Figure 2. Optimal value of α as function of the power constraint at the transmitter for different values of N_1 and N_2.

Further, Fig. 3 shows how the achievable rate to the first user, R_1, can vary, with the power constraint at the transmitter, for different values of the noise variance over the channels to the two destinations. In particular, the noise variance over the channel to the first user is normalized to 1 whereas the noise variance over the channel to the second user is varied. In addition, we remind that in this numerical example, the optimal power allocation is derived to make $R_1 = R_2$. This figure shows that as N2 increases, the data rate R_1 decreases since more power is allocated to transmit X2, to keep $R_1 = R_2$.

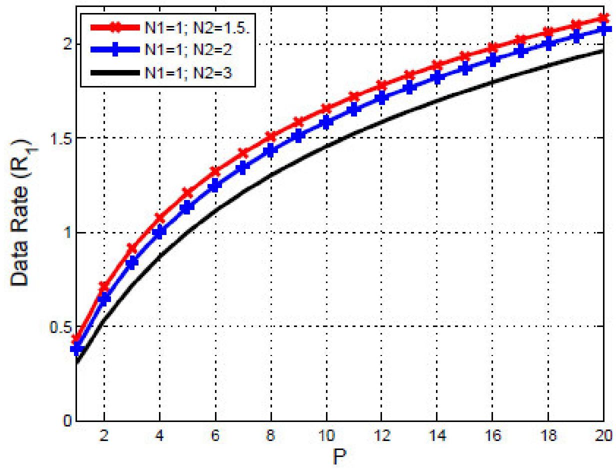

Figure 3. The first users data rate as a function of the power constraint at the transmitter for different vales of N_1 and N_2, and optimal values of α. In this case, the two users have the same rate i.e., $R_1 = R_2$.

3.2 Sum Rate Maximization Criterion

In this subsection, the optimal power allocation factor α is selected to maximize the sum rate capacity. Analytically, this optimization problem is given as,

$$\begin{aligned} &\text{maximize} \quad R = R_1 + R_2 \\ &\text{subject to} \quad 0 \le \alpha < 0.5 \\ &C\left(\frac{(1-\alpha)P}{N_1+\alpha P}\right) > C\left(\frac{(1-\alpha)P}{N_2+\alpha P}\right) \\ &C\left(\frac{(1-\alpha)P}{N_1+\alpha P}\right) > C\left(\frac{\alpha P}{N_1}\right) \end{aligned} \tag{9}$$

In this optimization problem, the second constraint is always valid since $N_1 < N_2$. Now, the third constrain implies that the optimal value of α is given as

$$\alpha \le \frac{-2N_1 + \sqrt{4N_1^{\,2} + 4N_1 P}}{2P} \tag{10}$$

Further, the objective function $R = R_1 + R_2$ is given as

$$\begin{aligned} R = R_1 + R_2 &\le \log_2\left(1+\frac{\alpha P}{N_1}\right) + \log_2\left(1+\frac{(1-\alpha)P}{N_2+\alpha P}\right) \\ &\le \log_2\left(1+\frac{\alpha P}{N_1}\right)\left(1+\frac{(1-\alpha)P}{N_2+\alpha P}\right) \end{aligned} \tag{11}$$

since the logarithm is monotonically increasing function, it is sufficient to incorporate the first objective function as

$$J(\alpha) = \left(1+\frac{\alpha P}{N_1}\right)\left(1+\frac{(1-\alpha)P}{N_2+\alpha P}\right) \tag{12}$$

We need to note that $J(\alpha)$ is not guaranteed to be a concave function of α . Thus, all possible boundary points and the extreme points should be searched to find the optimal power allocation. Therefore, the values of $J(\alpha)$ at the boundary points are given as

$$J(\alpha = 0) = \left(1+\frac{P}{N_2}\right) \tag{13a}$$

$$J(\alpha = 1) = \left(1+\frac{P}{N_1}\right) \tag{13b}$$

Now, to find the extreme points, let us define $J(\alpha)=\frac{C}{D}$, then $J^{'}=\frac{\partial J}{\partial \alpha}=\frac{DC^{'}-CD^{'}}{D^{2}}$. Hence, since D^{2} is always positive, then $J^{'}=\frac{\partial J}{\partial \alpha}$ is equivalent to solve $DC^{'}-CD^{'}=0$. In particular, solving $DC^{'}-CD^{'}=0$ is equivalent to find the roots of

$$N_{1}P(\alpha P+N_{2})(P+N_{2})-N_{1}P(P+N_{2})(\alpha P+N_{2})=0 \quad (14)$$

which reduces to $N_{1}-N_{2}=0$. This means that this solution does not add any constraints on the value of α. Therefore, by using the first and third constraints, the value of α that can maximize the objective function in (9) should be searched in $\left[0,\frac{-2N_{1}+\sqrt{4N_{1}^{2}+4N_{1}P}}{2p}\right]$.

Fig. 4 shows how the optimal power allocation is varying with the power constraint at the broadcast transmitter. This result is similar to that shown in Fig. 2 except that in this case, α does not change with N_{2}. In particular, this figure shows that increasing P results in increasing the allocated power to the second user's signal X_{2} since this user requires more power to correctly decode its signal due to additional noise signal from the first user's signal.
Fig. 5 compares between the achieved sum rate in the two optimization schemes discussed before. It clearly shows that the sum rate obtained from the optimization problem considered in section 3.2 marginally outperforms that attained in section 3.1.

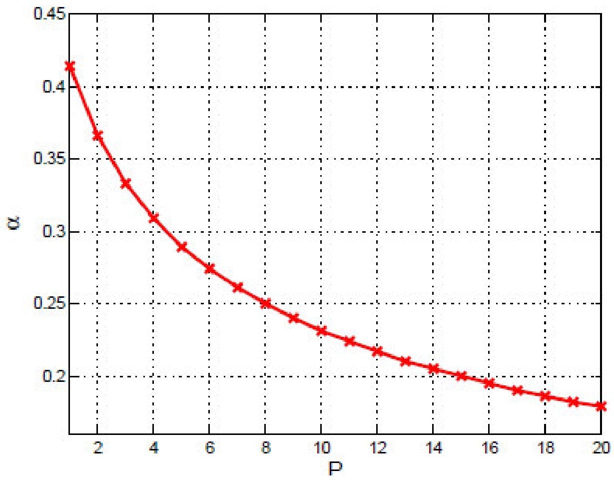

Figure 4. Optimal value of α as function of the power constraint at the transmitter.

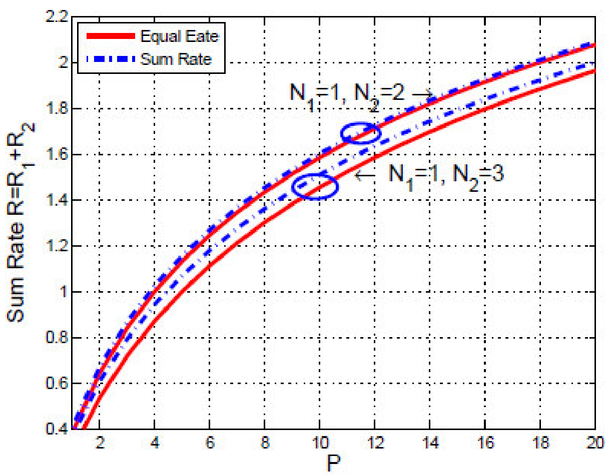

Figure 5. Sum rate as a function of the power constraint at the transmitter for different values of N_{1} and N_{2}, and also for different optimization schemes.

4. Conclusion

In this letter, the optimal power allocation between two users sharing the GBC has been considered. Two optimization problems have been considered to optimize the available power between these two users. In particular, an analytical expression that shows how the power should be allocated between the two users has been derived. Specifically, the first optimization scheme has assumed that equal rates should offered to the two users whereas maximizing the sum rate has been considered in the second optimization criterion. Further, numerical results have also been shown to verify the validity of our mathematical analysis.

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