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On the Probability Density Function of the Rank Statistics

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Abstract

The purpose of this paper is to investigate properties of the probability density function of the rank statistics for a Brownian motion. Fujita and Miura (2006) defined the rank statistics for a Brownian motion and derived closed form for its density function. However, since their density function includes multiple integrals, it is hard to imagine the shape of the density function. Thus we will evaluate the multiple integrals numerically to investigate the properties of the density function of the rank statistics. We also prove some properties of the distribution of the rank statistics mathematically.

Keywords

Rank statistics, Brownian motion, arcsine law

1 Introduction

This short paper reports the properties of the probability density function of the rank statistics for a Brownian motion. Fujita and Miura [1] defined the rank statistics for a Brownian motion and derived closed form for its density function. Further Miura [2] studied an exotic derivative called stochastic corridor, which is related to the rank statistics. However, since the density function derived in Fujita and Miura [1] includes multiple integrals, it is hard to imagine the shape of the density function. Thus we will evaluate the multiple integrals numerically and investigate the properties of the density function of the rank statistics. We also prove some properties of the distribution of the rank statistics mathematically.

For a stochastic process $\{X_s\}_{s\in[0,t]}$, the rank of X_t is defined by

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$$\frac{1}{T}\int_0^T I(X_s < X_t)ds, \tag{1}$$

where $I(\cdot)$ is an indicator function. In this paper we only consider the case where X_s is a Brownian motion, that is $X_s = \mu s + W_s$, where W_s is a one-dimensional standard Brownian motion and $\mu \in \mathbb{R}$. We use the notation

$$R^{\mu}_{t,T} = \frac{1}{T}\int_0^T I(\mu s + W_s < \mu t + W_t)ds.$$

The density function $f_{R^{\mu}_{t,T}}$ for $R^{\mu}_{t,T}$ is given by

$$f_{R^{\mu}_{t,T}}(x) = \int_{-\infty}^{\infty} e^{\mu y - \frac{1}{2}\mu^2 T} f_{(W_T, R^0_{t,T})}(y, x)dy, \tag{2}$$

where $f_{(W_T, R^0_{t,T})}$ is the joint probability density function of $(W_T, R^0_{t,T})$. The joint density function $f_{(W_T, R^0_{t,T})}$ is given as follows. If $0 < t < T/2$,

$$f_{(W_T, R^0_{t,T})}(x, y) = T\iint_A f_t^+(x_1, y_1) f^-_{T-t}(x - x_1, Ty - y_1)dx_1 dy_1,$$

where

$$A = \begin{cases} \{(x_1, y_1): -\infty < x_1 < \infty,\ 0 < y_1 < Ty\}, \text{ if } 0 < y < \dfrac{t}{T} \\ \{(x_1, y_1): -\infty < x_1 < \infty,\ 0 < y_1 < t\}, \text{ if } \dfrac{t}{T} < y < \dfrac{T-t}{T} \\ \{(x_1, y_1): -\infty < x_1 < \infty,\ Ty - (T-t) < y_1 < t\}, \text{ if } \dfrac{T-t}{T} < y < 1. \end{cases}$$

If $T/2 < t < 1$,

$$f_{(W_T, R^0_{t,T})}(x, y) = T\iint_B f_t^+(x - x_2, Ty - y_2) f^-_{T-t}(x_2, y_2)dx_2 dy_2,$$

where

$$B = \begin{cases} \{(x_2, y_2): -\infty < x_2 < \infty,\ 0 < y_2 < Ty\}, \text{ if } 0 < y < \dfrac{T-t}{T} \\ \{(x_2, y_2): -\infty < x_2 < \infty,\ 0 < y_2 < T-t\}, \text{ if } \dfrac{T-t}{T} < y < \dfrac{t}{T} \\ \{(x_2, y_2): -\infty < x_2 < \infty,\ Ty - (T-t) < y_2 < T-t\}, \text{ if } \dfrac{t}{T} < y < 1. \end{cases}$$

Here

$$f_t^-(a, u) = \begin{cases} \displaystyle\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2(t-s)}\right\}ds,\ a > 0 \\ \displaystyle\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2s}\right\}ds,\ a < 0 \end{cases}$$

$$f_t^+(a, u) = \begin{cases} \displaystyle\int_0^u \frac{a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2s}\right\}ds,\ a > 0 \\ \displaystyle\int_u^t \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2(t-s)}\right\}ds,\ a < 0 \end{cases}$$

See Fujita and Miura [1] for derivation of $f_{R^{\mu}_{t,T}}$ and $f_{(W_T, R^0_{t,T})}$. In the next section, we investigate the properties of $f_{R^{\mu}_{t,T}}$.

2 Density functions of the rank statistics

In this section, we will examine the properties of the density function of the rank statistics. First we prove several properties mathematically.

Proposition 1. *The rank statistics have the following properties.*

(i) $f_{R^{\mu}_{t,T}}(x) = f_{R^{-\mu}_{t,T}}(1-x)$.

(ii) $R^{\mu}_{t,T} \overset{\text{(law)}}{=} R^{-\mu}_{T-t,T}$.

(iii) $R^0_{0,T}$ *and* $R^0_{T,T}$ *follow an arcsine law.*

Proof. We first show (i). We have

$$
\begin{aligned}
R^{\mu}_{t,T} &= \frac{1}{T}\int_0^T I(\mu s + W_s < \mu t + W_t)ds \\
&= \frac{1}{T}\int_0^t I(W_t - W_s + \mu(t-s) > 0)ds \\
&\quad + \frac{1}{T}\int_t^T I(W_s - W_t + \mu(s-t) < 0)ds \\
&= \frac{1}{T}\int_0^t I(\hat{W}_s - \mu s > 0)ds + \frac{1}{T}\int_0^{T-t} I(\widetilde{W}_s - \mu s < 0)ds, \qquad (3)
\end{aligned}
$$

where $\hat{W}_s = W_t - W_{t-s}$ and $\widetilde{W}_s = W_{t+s} - W_t$, which are independent Brownian motions. It then follows that

$$
\begin{aligned}
R^{\mu}_{t,T} < x &\Leftrightarrow 1 - \frac{1}{T}\int_0^t I(\hat{W}_s + \mu s > 0)ds - \frac{1}{T}\int_0^{T-t} I(\widetilde{W}_s + \mu s < 0)ds > 1 - x \\
&\Leftrightarrow \frac{1}{T}\left(t - \int_0^t I(\hat{W}_s + \mu s > 0)ds\right) \\
&\qquad + \frac{1}{T}\left(T - t - \int_0^{T-t} I(\widetilde{W}_s + \mu s < 0)ds\right) > 1 - x \\
&\Leftrightarrow \frac{1}{T}\int_0^t I(\hat{W}_s + \mu s < 0)ds + \frac{1}{T}\int_0^{T-t} I(\widetilde{W}_s + \mu s > 0)ds > 1 - x \\
&\Leftrightarrow \frac{1}{T}\int_0^t I(-\hat{W}_s - \mu s > 0)ds + \frac{1}{T}\int_0^{T-t} I(-\widetilde{W}_s - \mu s < 0)ds > 1 - x
\end{aligned}
$$

for $\mu \in \mathbb{R}$. Since $-\hat{W}_s$ and $-\widetilde{W}_s$ are standard Brownian motions,

$$
\frac{1}{T}\int_0^T I(-\hat{W}_s - \mu s > 0)ds + \frac{1}{T}\int_0^{T-t} I(-\widetilde{W}_s - \mu s < 0)ds \overset{\text{(law)}}{=} R^{-\mu}_{t,T}.
$$

We then obtain

$$
P(R^{\mu}_{t,T} < x) = P(R^{-\mu}_{t,T} > 1 - x) = 1 - P(R^{-\mu}_{t,T} < 1 - x),
$$

which implies (i).

Next we will show (ii). We have

$$
\begin{aligned}
R^{-\mu}_{T-t,T} &= \frac{1}{T}\int_0^T I(-\mu s + W_s < -\mu(T-t) + W_{T-t})ds \\
&= \frac{1}{T}\int_0^{T-t} I(W_{T-t} - W_s - \mu(T-t-s) > 0)ds \\
&\quad + \frac{1}{T}\int_{T-t}^{T} I(W_s - W_{T-t} - \mu(s-(T-t)) < 0)ds \\
&= \frac{1}{T}\int_0^{T-t} I(\hat{W}_s - \mu s > 0)ds + \frac{1}{T}\int_0^t I(\widetilde{W}_s - \mu s < 0)ds \\
&= \frac{1}{T}\int_0^{T-1} I((-\widetilde{W}_s) + \mu s < 0) + \frac{1}{T}\int_0^t I((-\widetilde{W}_s) + \mu s > 0)ds, \qquad (4)
\end{aligned}
$$

where $\hat{W}_s = W_{T-t} - W_{T-t-s}$ and $\widetilde{W}_s = W_{T-t+s} - W_{T-t}$, which are independent standard Brownian motions. Since $-\hat{W}_s$ and $-\widetilde{W}_s$ are also independent standard Brownian motions, it follows from (3) and (4) that $R^{\mu}_{t,T} \overset{\text{(law)}}{=} R^{-\mu}_{T-t,T}$.

Finally we prove (iii). It follows from (3) that

$$
R^0_{0,T} = \frac{1}{T}\int_0^T I(\widetilde{W}_s < 0)ds \quad \text{and} \quad R^0_{T,T} = \frac{1}{T}\int_0^T I(\hat{W}_s > 0)ds,
$$

where $\widetilde{W}_s$ and $\hat{W}_s$ are Brownian motions. Thus $R^0_{0,T}$ and $R^0_{T,T}$ follow an arcsine law. □

This proposition implies that, to know the density functions of the rank statistics, it is enough to consider $f_{R^{\mu}_{t,T}}(x)$ for $\mu > 0$ and $t \in [0, T/2]$. Indeed, if $\mu < 0$ and $t \in [0, T/2]$, then the density function is derived from the property (i). If $\mu > 0$ and $t \in [T/2, T]$, it follows from (ii) that $f_{R^{\mu}_{t,T}}(x) = f_{R^{-\mu}_{T-t,T}}(x)$. Together with (i), we obtain the density function by

$$
f_{R^{\mu}_{t,T}}(x) = f_{R^{-\mu}_{T-t,T}}(x) = f_{R^{\mu}_{T-t,T}}(1-x).
$$

If $\mu < 0$ and $t \in [T/2, T]$, then the density function is derived from the property (ii). Further the property (i) implies that $f_{R^0_{t,T}}(x)$ is symmetric with respect to $x = T/2$.

Next we will investigate the shape of the density function of $R^{\mu}_{t,T}$ numerically. For simplicity, we put $T = 1$. We adopt Monte Carlo integration to calculate the value $f_{R^{\mu}_{t,1}}(x)$. We use Sobol' sequence as random numbers. We calculate $f_{R^{\mu}_{t,1}}(x)$ at $x = 0.01, 0.05, 0.1, 0.15, \ldots, 0.99$ and draw the graph by interpolation. Note that (2) includes integral over infinite interval. We then transform (2) into integral over [0, 1], that is,

$$
\begin{aligned}
\int_{-\infty}^{\infty} h(y)dy &= \int_{-\infty}^{-1} h(y)dy + \int_{-1}^{0} h(y)dy + \int_0^1 h(y)dy + \int_1^{\infty} h(y)dy \\
&= \int_0^1 \frac{h(-1/z)}{z^2}dz + \int_0^1 h(z+1)dz + \int_0^1 h(z)dz + \int_0^1 \frac{h(1/z)}{z^2}dz,
\end{aligned}
$$

where $h(y)$ is the integrand in (2). So we adopt Monte Carlo integration to these four integrals. Figure 1–5 show the graphs of the density functions of $R^{\mu}_{0,1}$. From these graphs, in addition to the properties in Proposition 1, we can find the followings.

- When $t \neq 0, 0.5, 1$, $f_{R^{\mu}_{t,1}}(x)$ is bimodal.
- $f_{R^{\mu}_{0.5,1}}(x)$ is unimodal.

These properties also will be proved mathematically, but we leave this issue to future work.

3 Concluding remarks

In this paper, we investigate the properties of the density function of the rank statistics. In particular, we use Monte Carlo integration to draw the graphs of the density functions. Alternatively, we can imagine the shape of the graph by discretization of (1) and drawing the histogram. Let $0=t_0<t_1<\cdots<t=t_i<\cdots<t_N=T$ be a sequence of the interval $[0, T]$. We simulate the path of X_s and obtain the sequence $X_{t_0}, X_{t_1}, \ldots, X_{t_i}, \ldots, X_{t_N}$. We then approximate $1/T\int_0^T I(X_s<X_t)ds$ by the rank of X_{t_i} in $X_{t_0}, X_{t_1}, \ldots, X_{t_N}$. Repeating this simulation, we obtain the histogram of approximated rank statistics. Of course, we will obtain similar histograms to Figure 1-5.

References

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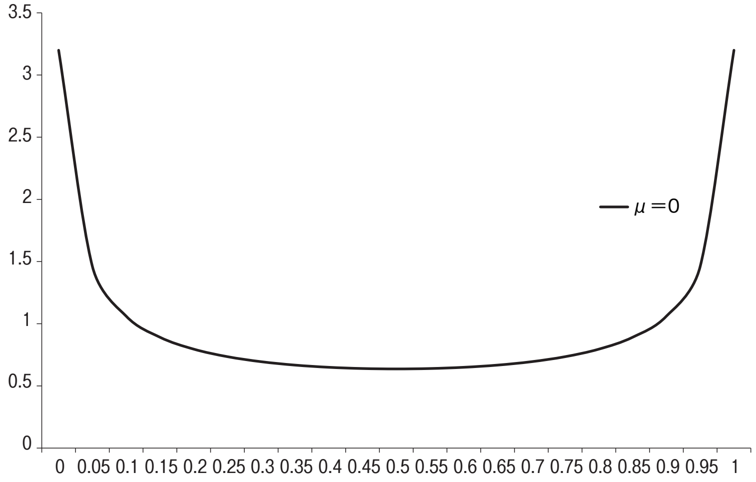

Figure 1: Density function of $R^{\mu}_{0,1}$

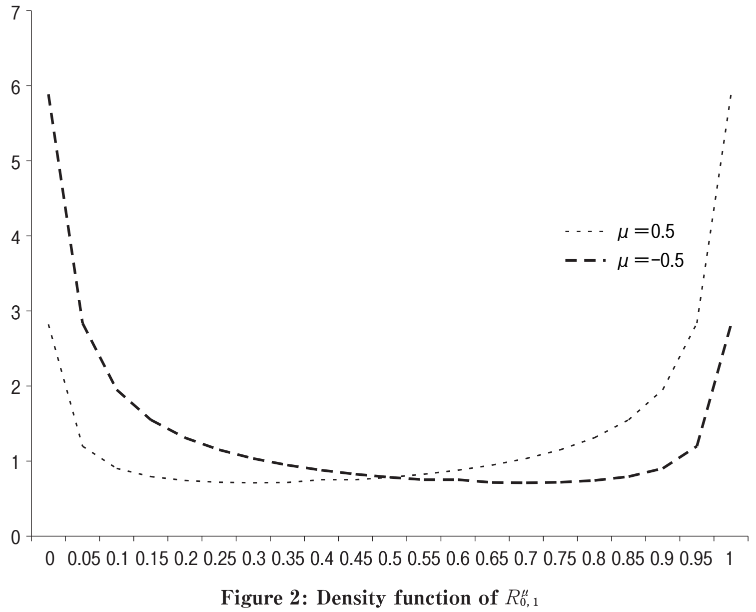

Figure 2: Density function of $R^{\mu}_{0,1}$

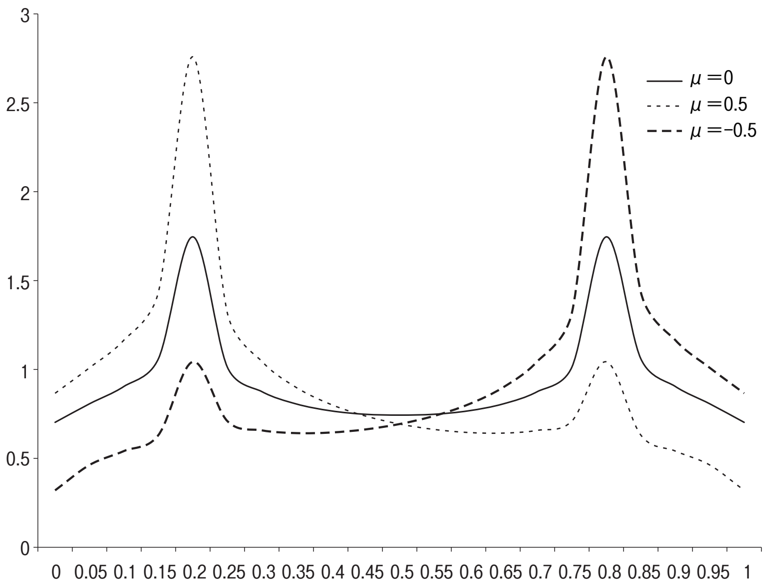

Figure 3: Density function of $R^{\mu}_{0.2,1}$

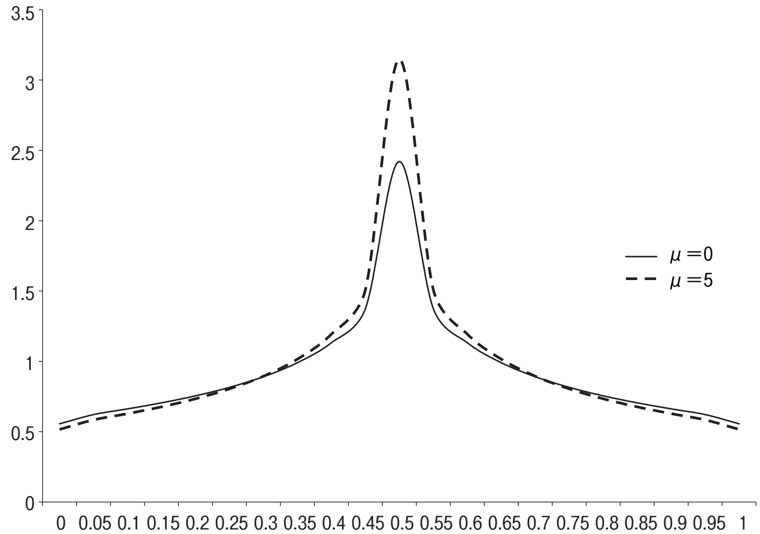

Figure 4: Density function of $R^{\mu}_{0.5,1}$

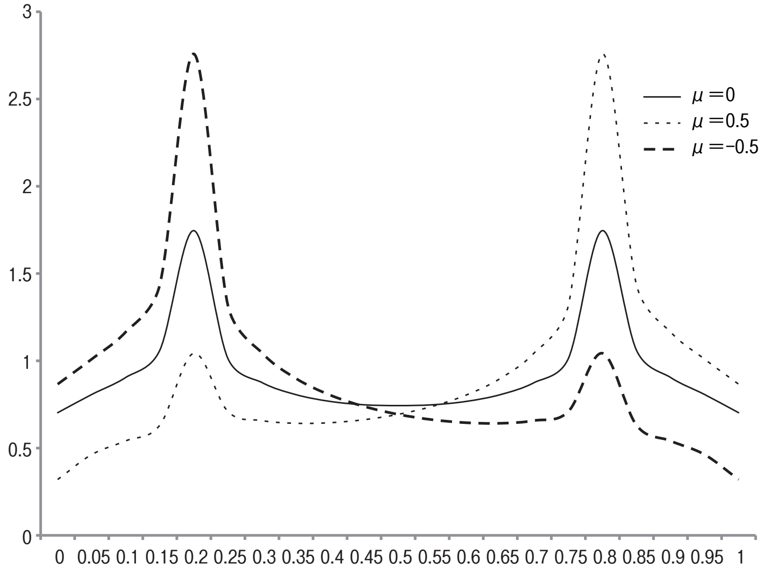

Figure 5: Density function of $R^{\mu}_{0.8,1}$