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Some Fixed Point and Common Fixed Point Theorems of Integral type on 2-Banach Spaces

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Abstract: In the present paper we prove some fixed point and common fixed point theorems in 2-Banach spaces for new rational expression. Which generalize the well-known results.

Keywords: Banach Space, 2-Banach Spaces, Fixed point, Common Fixed point.

2. INTRODUCTION

Fixed point theory plays basic role in application of various branches of mathematics from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many application in other disciplines.

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods especially Banach's contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [13] described about the application of Banach's contraction principle [4]. Ghalar [8] introduced the concept of 2-Banach spaces. Recently Badshah and Gupta [5], Yadava, Rajput and Bhardwaj [14], Yadava, Rajput, Choudhary and Bhardwaj [15] also worked for Banach and 2-Banch spaces for non-contraction mappings. In present paper we prove some fixed point and common fixed point theorems for non-contraction mappings, in 2-Banach spaces motivated by above, before starting the main result first we write some definitions .

Definition (2.A), 2-Banach Spaces: In a paper Gahler [8] define a linear 2-normed space to be pair $(L, \|.,\|)$ where L is a linear space and $\|.,\|$ is non- negative, real valued function defined on L such that a,b,c $\in L$

(i) $\|a, b\|$ = 0 if and only if a and b are Linearly dependent

(ii) $\|a, b\| = \|b, a\|$

(iii) $\|a, \beta b\| = |\beta| \|a, b\|$, β is real

(iv) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$

Hence $\|.,.\|$ is called a 2- norm.

Definition (2.B):

A sequence $\{x_n\}$ in a linear 2 – normed space L ,is called a convergent sequence if there is , x $\in L$, such that $\lim_{n\to\infty} \|x_n - x, y\|$ = 0 for all y$\in L$.

Definition (2.C):

A sequence $\{x_n\}$ in a linear 2 – normed space L,is called a Cauchy sequence if there exists y, z $\in L$, such that y and z are linearly independent and

$$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$$

Definition (2.D): A linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach spaces.

Theorem (2.E) (Banach's contraction principle) Let (X, d) be a complete metric space, c∈(0,1) and f: X→X be a mapping such that for each x, y ∈X,
d(fx, fy) ≤ cd (x, y) Then f has a unique fixed point a ∈X, such that for each
x∈ X, $\lim_{n\to\infty} f^n(x) = a.$

After the classical result, Kannan [11] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings Satisfying various type of contractive conditions.

In 2002, A. Branciari [3] analysed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem (2.F) (Branciari) Let (X,d) be a complete metric space ,c $\in (0,1)$ and let f : X→ X be a mapping such that for each x, y ∈X,
$\int_0^{d(fx,fy)} \emptyset(t)dt \le c \int_0^{d(x,y)} \emptyset(t)dt$ where $\emptyset$:[0,+ $\infty)$ →[0,+ $\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ $\infty)$, non-negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, then f has a unique fixed point a ∈X, such that for each x ∈ X,
$\lim_{n\to\infty} f^n(x) = a.$

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [5] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \emptyset(t)dt \le \int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \emptyset(t)dt.$$

Theorem (2.G):

Let T be a mapping of a 2 – Banach spaces into itself. If T satisfies the following conditions:

(1) $T^2 = I$, where I is identity mapping

(2) $\|Tx - Ty, a\| \ge \alpha \frac{\|x-Tx,a\|\,\|y-Ty,a\|}{\|x-y,a\|} + \beta \frac{\|y-Ty,a\|\,\|y-Tx,a\|\|x-Ty,a\| + \|x-y,a\|^3}{\|x-y,a\|^2} + \gamma \left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right] + \delta \left[\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}\right] + \eta \|x-y,a\|$

Where x$\neq y$, a > 0 is real with 8 $\alpha + 10\beta + 4\gamma + 2\delta + 3\eta > 4.$ Then T has unique fixed point.

Our main result is modified the above result in integral type mapping.

3. MAIN RESULTS

Theorem 3.1

Let T be a mappings of a 2- Banach space X into itself. T satisfy the following conditions :

(1) $T^2 = I$, where I is identity mapping,

(2) $$\int_0^{\|Tx-Ty,a\|} \emptyset(t)dt \geq \alpha \int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|}} \emptyset(t)dt + \beta \int_0^{\frac{\|y-Ty,a\|\|y-Tx,a\|\|x-Ty,a\|+\|x-y,a\|^3}{\|x-y,a\|^2}} \emptyset(t)dt + \gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt$$
$$+ \delta \int_0^{\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|x-y,a\|} \emptyset(t)dt$$
$$+\Psi \int_0^{\frac{\|x-Ty,a\|\|y-Tx,a\|}{\|x-y,a\|}} \emptyset(t)dt$$

For every x, y ∈X, $\alpha, \beta, \gamma, \delta, \eta \in [0,1]$ with x≠and $8\alpha + 10\beta + 4\gamma + 3\delta + 2\eta + 4\Psi > 4$. Also $\emptyset$:[0,+ ∞) →[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, Then T has unique fixed point .

Proof : Suppose x is any point in 2- Banach space X.

Taking $y = \frac{1}{2}(T+I)x$, $z = T(y)$

$$\int_0^{\|z-x,a\|} \emptyset(t)dt = \int_0^{\|Ty-T^2x,a\|} \emptyset(t)dt = \int_0^{\|Ty-T(Tx),a\|} \emptyset(t)dt$$
$$\geq \alpha \int_0^{\frac{\|y-Ty,a\|\|Tx-T(Tx),a\|+}{\|y-Tx,a\|}} \emptyset(t)dt$$
$$+ \beta \int_0^{\frac{\|Tx-T(Tx),a\|\|Tx-Ty,a\|\|y-T(Tx),a\|+\|y-Tx,a\|^3}{\|y-Tx,a\|^2}} \emptyset(t)dt$$
$$+ \gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-T(Tx),a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|y-T(Tx),a\|+\|Tx-Ty,a\|}{2}} \emptyset(t)dt$$
$$+ \eta \int_0^{\|y-Tx,a\|} \emptyset(t)dt + \Psi \int_0^{\frac{\|y-T(Tx),a\|\|Tx-Ty,a\|}{\|y-Tx,a\|}} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|y-Ty,a\|\|Tx-x,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt$$
$$+ \beta \int_0^{\frac{\|Tx-x,a\|[\|Tx-y,a\|+\|y-Ty,a\|]\|y-x,a\|+\|y-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$
$$+ \gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|y-x,a\|+\|Tx-y,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\|y-Tx,a\|} \emptyset(t)dt + \Psi \int_0^{\frac{\|y-x,a\|\|Tx-Ty,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt$$

$$\geq 2\,\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt$$
$$+\beta \int_0^{\frac{\|Tx-x,a\|\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|\right]\frac{1}{2}\|x-Tx,a\|+\frac{1}{8}\|x-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$
$$+\gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt$$
$$+\eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt + \Psi \int_0^{\|Tx-y,a\|+\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq 2\,\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt + \frac{\beta}{2} \int_0^{4\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|\right]+\frac{\|x-Tx,a\|^3}{\|x-Tx,a\|^2}} \emptyset(t)dt + \gamma$$
$$\int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \frac{\eta}{2} \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$
$$+\Psi \int_0^{\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq 2\,\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt + \frac{\beta}{2} \int_0^{[3\|x-Tx,a\|+4\|y-Ty,a\|]} \emptyset(t)dt + \gamma$$
$$\int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \frac{\eta}{2} \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$
$$+\Psi \int_0^{\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq \left(\frac{3\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}+\frac{\Psi}{2}\right) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \left(2\alpha+2\beta+\frac{\gamma}{2}+\frac{\delta}{2}+\Psi\right)$$
$$\int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq \frac{1}{2}\,(3\beta+\gamma+\delta+\eta+\Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha+4\beta+\gamma+\delta+2\Psi)$$
$$\int_0^{\|y-Ty,a\|} \emptyset(t)dt \quad \text{----------------------------------} \ (3.1.1)$$

Now for

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$$\int_0^{\|u-x,a\|} \emptyset(t)dt = \int_0^{\|2y-z-x,a\|} \emptyset(t)dt = \int_0^{\|Tx-Ty,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|}} \emptyset(t)dt + \beta \int_0^{\frac{\|y-Ty,a\|\|y-Tx,a\|\|x-Ty,a\|+\|x-y,a\|^3}{\|x-y,a\|^2}} \emptyset(t)dt + \gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|x-y,a\|} \emptyset(t)dt$$
$$+ \Psi \int_0^{\frac{\|x-Ty,a\|\|y-Tx,a\|}{\|x-y,a\|}} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt + \beta \int_0^{\frac{\|y-Ty,a\|\frac{1}{2}\|x-Tx,a\|\frac{1}{2}\|x-Tx,a\|+\frac{1}{8}\|x-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$
$$+ \gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|}{2}} \emptyset(t)dt$$
$$+ \eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt + \Psi \int_0^{\frac{\frac{1}{2}\|x-Tx,a\|\frac{1}{2}\|x-Tx,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt$$

$$\geq 2\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt + \beta \int_0^{\|y-Ty,a\|+\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt + \gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt$$
$$+ \Psi \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt$$

$$\geq \left(\frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} + \frac{\Psi}{2}\right) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \left(2\alpha + \beta + \frac{\gamma}{2}\right) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq \frac{1}{2} (\beta + \gamma + \delta + \eta + \Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha + 2\beta + \gamma) \int_0^{\|y-Ty,a\|} \emptyset(t)dt \quad \text{-----} (3.1.2)$$

Now

$$\int_0^{\|z-u,a\|} \emptyset(t)dt = \int_0^{\|z-x,a\|} \emptyset(t)dt + \int_0^{\|x-u,a\|} \emptyset(t)dt$$

$$\geq \frac{1}{2} (3\beta + \gamma + \delta + \eta + \Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha + 4\beta + \gamma + \delta + 2\Psi) \int_0^{\|y-Ty,a\|} \emptyset(t)dt + \frac{1}{2} (\beta + \gamma + \delta + \eta + \Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$+\frac{1}{2}(4\alpha + 2\beta + \gamma\,)\ \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$\geq \frac{1}{2}\ (3\beta + \gamma + \delta + \eta + \Psi + \beta + \gamma + \delta + \eta + \Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$
$+\frac{1}{2}(4\alpha + 4\beta + \gamma + \delta + 2\Psi + 4\alpha + 2\beta + \gamma\,)\ \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$\geq \frac{1}{2}\ (4\beta + 2\gamma + 2\delta + 2\eta + 2\Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$
$+\frac{1}{2}(8\alpha + 6\beta + 2\gamma + \delta + 2\Psi\,)\ \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

On the other hand

$\int_0^{\|z-u,a\|} \emptyset(t)dt = \int_0^{\|T(y)-(2y-z),a\|} \emptyset(t)dt$

$= \int_0^{\|T(y)-2y+T(y),a\|} \emptyset(t)dt$

$= 2\int_0^{\|Ty-y,a\|} \emptyset(t)dt$

So

$2\int_0^{\|Ty-y,a\|} \emptyset(t)dt\ \ \geq \frac{1}{2}\ (4\beta + 2\gamma + 2\delta + 2\eta + 2\Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$
$+\frac{1}{2}(8\alpha + 6\beta + 2\gamma + \delta + 2\Psi\,)\ \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$[4 - (8\alpha + 6\beta + 2\gamma + \delta + 2\Psi\,)] \int_0^{\|Ty-y,a\|} \emptyset(t)dt \geq$
$(4\beta + 2\gamma + 2\delta + 2\eta + 2\Psi) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$

$\int_0^{\|x-Tx,a\|} \emptyset(t)dt \leq \frac{4-(8\alpha+6\beta+2\gamma+\delta+2\Psi\,)}{4\beta+2\gamma+2\delta+2\eta+2\Psi}\ \int_0^{\|Ty-y,a\|} \emptyset(t)dt$

$\int_0^{\|x-Tx,a\|} \emptyset(t)dt \leq k\int_0^{\|Ty-y,a\|} \emptyset(t)dt$

as ($8\alpha + 10\beta + 4\gamma + 3\delta + 2\eta + 4\Psi > 4$)

Where $k = \frac{4-(8\alpha+6\beta+2\gamma+\delta+2\Psi\,)}{4\beta+2\gamma+2\delta+2\eta+2\Psi} < 1$

Let $R = \frac{1}{2}(T+I)$, then

$\int_0^{\|R^2(x)-R(x),a\|} \emptyset(t)dt = \int_0^{\|R(R(x))-R(x),a\|} \emptyset(t)dt$

$$= \int_0^{\|R(y)-y,a\|} \emptyset(t)dt = \frac{1}{2}\int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$< \frac{k}{2}\int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

By the definition of R we claim that $\{R^n(x)\}$ is a Cauchy sequence in X , $\{R^n(x)\}$ is converges to so element x_0 in X . So $\lim_{n\to\infty}\{R^n(x)\} = x_0$. Hence $T(x_0) = x_0$

So x_0 is a fixed point of T.

Uniqueness:

If possible let $y_0 \neq x_0$ is another fixed point of T . Then

$$\int_0^{\|x_0-y_0,a\|} \emptyset(t)dt = \int_0^{\|Tx_0-Ty_0,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x_0-Tx_0,a\|\|y_0-Ty_0,a\|}{\|x_0-y_0,a\|}} \emptyset(t)dt + \beta \int_0^{\frac{\|y_0-Ty_0,a\|\|y_0-Tx_0,a\|\|x_0-Ty_0,a\|+\|x_0-y_0,a\|^3}{\|x_0-y_0,a\|^2}} \emptyset(t)dt + \gamma \int_0^{\frac{\|x_0-Tx_0,a\|+\|y_0-Ty_0,a\|}{2}} \emptyset(t)dt +$$
$$\delta \int_0^{\frac{\|x_0-Ty_0,a\|+\|y_0-Tx_0,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

$$+ \Psi \int_0^{\frac{\|x_0-Ty_0,a\|\|y_0-Tx_0,a\|}{\|x_0-y_0,a\|}} \emptyset(t)dt$$

$$\geq \beta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt + \delta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt + \eta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt + \Psi \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

$$\geq (\beta+\delta+\eta+\Psi) \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

Which is contradiction as $8\alpha + 10\beta + 4\gamma + 3\delta + 2\eta + 4\Psi > 4$

so $x_0 = y_0$. Hence fixed point in unique.

Theorem 3.2
Let T and G be two expansion mappings of a 2- Banach space X into itself. T and G satisfy the following conditions:

(1) T and G commute
(2) $T^2 = I$ and $G^2 = I$, where I is identity mapping,

(3) $\int_0^{\|Tx-Ty,a\|} \emptyset(t)dt \geq \alpha \int_0^{\frac{\|Gx-Tx,a\|\|Gy-Ty,a\|}{\|Gx-Gy,a\|}} \emptyset(t)dt + \beta \int_0^{\frac{\|Gy-Ty,a\|\|Gy-Tx,a\|\|Gx-Ty,a\|+\|Gx-Gy,a\|^3}{\|Gx-Gy,a\|^2}} \emptyset(t)dt + \gamma \int_0^{\frac{\|Gx-Tx,a\|+\|Gy-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|Gx-Ty,a\|+\|Gy-Tx,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|Gx-Gy,a\|} \emptyset(t)dt + \mu \int_0^{max\{\|Gx-Tx,a\|\,\|Gx-Ty,a\|,\}} \emptyset(t)dt$

For every x, y ∈X, $\alpha,\beta,\gamma,\delta,\eta$ ∈ [0,1] with x$\neq y$ and $\|Gx-Gy\| \neq 0$ and $\beta+\delta+\eta+\mu > 1$. Also $\emptyset$:[0,+ ∞) →[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$

Then there exists a unique common fixed point of T and G such that T$(x_0) = x_0$ and G$(x_0) = x_0$.

Proof:-

Suppose x is point in 2- Banach space X ,it is clear that $(TG)^2 = I$

$$\int_0^{\|TG.G(x)-TG.G(y),a\|} \emptyset(t)dt \geq \alpha \int_0^{\frac{\|G\,(G^2x)-T\,(G^2x),a\|\|G\,(G^2y)-T\,(G^2y),a\|}{\|G\,(G^2x)-G\,(G^2y),a\|}} \emptyset(t)dt + \beta$$

$$\int_0^{\frac{\|G\,(G^2y)-T\,(G^2y),a\|\|G\,(G^2y)-T\,(G^2x),a\|\|G\,(G^2x)-T\,(G^2y),a\|+\|G\,(G^2x)-G\,(G^2y),a\|^3}{\|G\,(G^2x)-G\,(G^2y),a\|^2}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{\|G\,(G^2x)-T\,(G^2x),a\|+\|G\,(G^2y)-T\,(G^2y),a\|}{2}} \emptyset(t)dt$$

$$+\delta \int_0^{\frac{\|G\,(G^2x)-T\,(G^2y),a\|+\|G\,(G^2y)-T\,(G^2x),a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\|G\,(G^2x)-G\,(G^2y),a\|} \emptyset(t)dt$$

$$+\mu \int_0^{max\{\|G\,(G^2x)-T\,(G^2x),a\|\|G\,(G^2x)-T\,(G^2y),a\|,\}} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|Gx-TG(Gx),a\|\|Gy-TG(Gy),a\|}{\|Gx-Gy,a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|Gy-TG(Gy),a\|\|Gy-TG(Gx),a\|\|Gx-TG(Gy),a\|+\|Gx-Gy,a\|^3}{\|Gx-Gy,a\|^2}} \emptyset(t)dt + \gamma$$

$$\int_0^{\frac{\|Gx-TG(Gx),a\|+\|Gy-TG(Gy),a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|Gx-TG(Gy),a\|+\|Gy-TG(Gx),a\|}{2}} \emptyset(t)dt$$

$$+\eta\int_0^{\|Gx-Gy,a\|}\emptyset(t)dt+\mu\int_0^{max\{\|Gx-TG(Gx),a\|\,\|Gx-TG(Gy),a\|,\}}\emptyset(t)dt$$

Taking G(x) = p , G(y) = q , where p≠q

$$\geq\ \alpha\int_0^{\frac{\|p-TG(p),a\|\|q-TG(q),a\|}{\|p-q,a\|}}\emptyset(t)dt$$

$$+\beta\int_0^{\frac{\|q-TG(q),a\|\|q-TG(p),a\|\|p-TG(q),a\|+\|p-q,a\|^3}{\|Gx-Gy,a\|^2}}\emptyset(t)dt+\gamma$$

$$\int_0^{\frac{\|p-TG(p),a\|+\|q-TG(q),a\|}{2}}\emptyset(t)dt+\delta\int_0^{\frac{\|p-TG(q),a\|+\|q-TG(p),a\|}{2}}\emptyset(t)dt$$

$$+\eta\int_0^{\|p-q,a\|}\emptyset(t)dt+\mu\int_0^{max\{\|p-TG(p),a\|\,\|p-TG(q),a\|,\}}\emptyset(t)dt$$

Taking TG = R we get

$$\int_0^{\|R(p)-R(q),a\|}\emptyset(t)dt\geq\ \alpha\int_0^{\frac{\|p-R(p),a\|\|q-R(q),a\|}{\|p-q,a\|}}\emptyset(t)dt$$

$$+\beta\int_0^{\frac{\|q-R(q),a\|\|q-R(p),a\|\|p-R(q),a\|+\|p-q,a\|^3}{\|p-q,a\|^2}}\emptyset(t)dt$$

$$+\gamma\int_0^{\frac{\|p-R(p),a\|+\|q-R(q),a\|}{2}}\emptyset(t)dt+\delta\int_0^{\frac{\|p-R(q),a\|+\|q-R(p),a\|}{2}}\emptyset(t)dt$$

$$+\eta\int_0^{\|p-q,a\|}\emptyset(t)dt+\mu\int_0^{max\{\|p-R(p),a\|\,\|p-R(q),a\|,\}}\emptyset(t)dt$$

It is clear by theorem (1.1) ; that TG = R has at least one fixed point say x_0 in K that is R(x_0) = TG(x_0) = x_0

And so T.(TG) x_0 = T(x_0) or $T^2(Gx_0)=$ T(x_0)

G (x_0) = T (x_0)

Now

$$\int_0^{\|Tx_0-x_0,a\|}\emptyset(t)dt=\int_0^{\|Tx_0-T^2(x_0),a\|}\emptyset(t)dt$$

$$=\int_0^{\|Tx_0-T.T(x_0),a\|}\emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|G(x_0)-T(x_0),a\|\|GT(x_0)-T.T(x_0),a\|}{\|G(x_0)-GT(x_0),a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|G(Tx_0)-T(Tx_0),a\|\|G(Tx_0)-T(x_0),a\|\|G(x_0)-T(Tx_0),a\|+\|G(x_0)-G(Tx_0),a\|^3}{\|G(x_0)-G(Tx_0),a\|^2}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{\|G(x_0)-T(x_0),a\|+\|G(Tx_0)-T(Tx_0),a\|}{2}} \emptyset(t)dt +$$

$$\delta \int_0^{\frac{\|G(x_0)-T(Tx_0),a\|+\|G(Tx_0)-T(x_0),a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|G(x_0)-G(Tx_0),a\|} \emptyset(t)dt$$

$$+\mu \int_0^{max\{\|G(x_0)-T(x_0),a\|\|G(x_0)-T(Tx_0),a\|,\}} \emptyset(t)dt$$

$$= (\beta+\delta+\eta+\mu) \int_0^{\|Tx_0-x_0,a\|} \emptyset(t)dt$$

So $T(x_0) = x_0$ $(\beta+\delta+\eta+\mu > 1)$

That is x_0 is the fixed point of T.

But $T(x_0)$ =G (x_0) so G (x_0) = x_0

Hence x_0 is the fixed point of T and G.

Uniqueness:

If possible let $x_0 \neq y_0$ is another common fixed point of T and G.

$$\int_0^{\|x_0-y_0,a\|} \emptyset(t)dt = \int_0^{\|T^2(x_0)-T^2(y_0),a\|} \emptyset(t)dt$$

$$= \int_0^{\|T(T(x_0))-T(T(y_0)),a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|G(Tx_0)-T(Tx_0),a\|\|GT(y_0)-T.(Ty_0),a\|}{\|G(Tx_0)-G(Ty_0),a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|G(Ty_0)-T(Ty_0),a\|\|G(Ty_0)-T(Tx_0),a\|\|G(Tx_0)-T(Ty_0),a\|+\|G(Tx_0)-G(Ty_0),a\|^3}{\|G(Tx_0)-G(Ty_0),a\|^2}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{\|G(Tx_0)-T(Tx_0),a\|+\|G(Ty_0)-T(Ty_0),a\|}{2}} \emptyset(t)dt$$

$$+\delta \int_0^{\frac{\|G(Tx_0)-T(Ty_0),a\|+\|G(Ty_0)-T(Tx_0),a\|}{2}} \emptyset(t)dt + \eta \int_0^{\|G(Tx_0)-G(Ty_0),a\|} \emptyset(t)dt$$

$$+\mu \int_0^{max\{\|G(Tx_0)-T(Tx_0),a\|\|G(Tx_0)-T(Ty_0),a\|,\}} \emptyset(t)dt$$

$$\geq (\beta + \delta + \eta + \mu)\int_0^{\|x_0 - y_0, a\|} \emptyset(t)dt$$

But $\beta + \delta + \eta + \mu > 1$

So $x_0 = y_0$, so common fixed point in unique.

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